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A new proof for a Rolewicz's type theorem: An evolution semigroup approach *

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Abstract

Let φ be a positive and non-decreasing function defined on the real half-line and $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on a Banach space. We prove that if φ and $\mathcal{U}$ satisfy a certain integral condition (see the relation (2) below) then $\mathcal{U}$ is uniformly exponentially stable. For φ continuous, this result is due to S. Rolewicz.

1 Introduction

Let X be a real or complex Banach space and $L(X)$ the Banach algebra of all linear and bounded operators on X. Let $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$ be a strongly continuous semigroup on X and $\omega_0(\mathbf{T}) = \lim_{t\to\infty} \frac{\ln(\|T(t)\|)}{t}$ be its growth bound. The Datko-Pazy theorem ([1, 2]) states that $\omega_0(\mathbf{T}) < 0$ if and only if for all $x \in X$ the maps $t \longmapsto \|T(t)x\|$ belongs to $L^p(\mathbb{R}_+)$ for some $1 \leq p < \infty$.

A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset L(X)$ is called an *evolution family* of bounded linear operators on X if $U(t,t) = \mathbf{I}$ (the identity operator on X) and $U(t,\tau)U(\tau,s) = U(t,s)$ for all $t \geq \tau \geq s \geq 0$. Such a family is said to be *strongly continuous* if for every $x \in X$, the maps

$$(t,s) \mapsto U(t,s)x : \{(t,s) : t \geq s \geq 0\} \to X$$

are continuous, and *exponentially bounded* if there are $\omega > 0$ and $K_\omega > 0$ such that

$$\|U(t,s)\| \leq K_\omega e^{\omega(t-s)} \quad \text{for all } t \geq s \geq 0. \tag{1}$$

The family $\mathcal{U}$ is called *uniformly exponentially stable* if (1) holds for some negative ω. If $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$ is a strongly continuous semigroup on X, then the family $\{U(t,s) : t \geq s \geq 0\}$ given by $U(t,s) = T(t-s)$ is a strongly continuous and exponentially bounded evolution family on X. Conversely, if $\mathcal{U}$

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is a strongly continuous evolution family on X and $U(t,s) = U(t-s,0)$ then the family $\mathbf{T} = \{T(t) : t \geq 0\}$ given by $T(t) = U(t,0)$ is a strongly continuous semigroup on X.

The Datko-Pazy theorem can be obtained from the following result given by S. Rolewicz ([3], [4]).

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and nondecreasing function such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. If $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset L(X)$ is a strongly continuous and exponentially bounded evolution family on the Banach space X such that

$$\sup_{s\geq 0} \int_s^\infty \varphi(\|U(t,s)x\|)\,dt = M_\varphi < \infty, \text{ for all } x \in X,\ \|x\| \leq 1, \tag{2}$$

then $\mathcal{U}$ is uniformly exponentially stable.

A shorter proof of the Rolewicz theorem was given by Q. Zheng [5] who removed the continuity assumption about φ. Other proofs of (the semigroup case) Rolewicz's theorem were offered by W. Littman [6] and J. van Neervan [7, pp. 81-82]. Some related results have been obtained by K.M. Przyłuski [8], G. Weiss [13] and J. Zabczyk [9].

In this note we prove the following:

Theorem 1 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\varphi(t) > 0$ for all $t > 0$. If $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset L(X)$ is a strongly continuous and exponentially bounded evolution family of operators on X such that (2) holds, then $\mathcal{U}$ is uniformly exponentially stable.*

Our proof of Theorem 1 is very simple. In fact, we apply a result of Neerven (see below) for the evolution semigroup associated to $\mathcal{U}$ on $C_{00}(\mathbb{R}_+, X)$, the space of all continuous, $X-$valued functions defined on $\mathbb{R}_+$ such that $f(0) = \lim_{t\to\infty} f(t) = 0$.

Lemma 1 *Let $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family of operators on X such that*

$$\sup_{s\geq 0} \int_s^\infty \varphi(\|U(t,s)x\|)\,dt = M_\varphi(x) < \infty, \text{ for all } x \in X. \tag{3}$$

Then $\mathcal{U}$ is uniformly bounded, that is,

$$\sup_{t\geq\xi\geq 0} \|U(t,\xi)\| < \infty.$$

Proof of Lemma 1 Let $x \in X$ and $N(x)$ be a positive integer such that $M_\varphi(x) < N(x)$ and let $s \geq 0$, $t \geq s+N$. For each $\tau \in [t-N,t]$, we have

$$\begin{aligned} e^{-\omega N} 1_{[t-N,t]}(u)\|U(t,s)x\| &\leq e^{-\omega(t-\tau)} 1_{[t-N,t]}(u)\|U(t,\tau)U(\tau,s)x\| \quad (4)\\ &\leq K_\omega \|U(u,s)x\|, \end{aligned}$$

for all $u \geq s$. Here K_ω and ω are as in (1) and $\omega > 0$.

If we choose $x = 0$ in (3), then we get $\varphi(0) = 0$, and thus from (4) we obtain

$$N(x)\varphi\left(\frac{\|U(t,s)x\|}{K_\omega e^{\omega N}}\right) = \int_s^\infty \varphi\left(\frac{1_{[t-N,t]}(u)\|U(t,s)x\|}{K_\omega e^{\omega N}}\right)du \leq \int_s^\infty \varphi(\|U(u,s)x\|)\,du \leq M_\varphi(x). \quad (5)$$

We assume that $\varphi(1) = 1$ (if not, we replace φ be some multiple of itself). Moreover, we may assume that φ is a strictly increasing map. Indeed if $\varphi(1) = 1$ and $a := \int_0^1 \varphi(t)\,dt$, then the function given by

$$\bar{\varphi}(t) = \begin{cases} \int_0^t \varphi(u)\,du, & \text{if } 0 \leq t \leq 1 \\ \dfrac{at}{at+1-a}, & \text{if } t > 1 \end{cases}$$

is strictly increasing and $\bar{\varphi} \leq \varphi$. Now φ can be replaced by some multiple of $\bar{\varphi}$. From (5) it follows that if $t \geq s + N(x)$ and $x \in X$, then

$$\|U(t,s)\| \leq K_\omega e^{\omega N(x)}, \quad \text{for all } x \in X.$$

Using this inequality and the exponential boundedness of the evolution family, we have that

$$\sup_{t \geq \xi \geq 0} \|U(t,\xi)x\| \leq K_\omega e^{\omega N(x)}, \quad \text{for each } x \in X. \quad (6)$$

The conclusion of Lemma 1 follows from (6) and the Uniform Boundedness Theorem. □

Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators on X. We consider the strongly continuous evolution semigroup associated to $\mathcal{U}$ on $C_{00}(\mathbb{R}_+, X)$. This semigroup is defined by

$$(\mathfrak{T}(t)f)(s) := \begin{cases} U(s,s-t)f(s-t), & \text{if } s \geq t \\ 0, & \text{if } 0 \leq s \leq t \end{cases}, \quad t \geq 0 \quad (7)$$

for all $f \in C_{00}(\mathbb{R}_+, X)$. It is known that $\mathfrak{T} = \{\mathfrak{T}(t) : t \geq 0\}$ is a strongly continuous semigroup and in addition $\omega_0(\mathfrak{T}) < 0$ if and only if $\mathcal{U}$ is uniformly exponentially stable ([10], [11], [12]).

Proof of Theorem 1. Let φ be as in Theorem 1. We assume that $\varphi(1) = 1$. Then

$$\Phi(t) := \int_0^t \varphi(u)\,du \leq \varphi(t) \text{ for all } t \in [0,1].$$

Without loss of generality we may assume that

$$\sup_{t \geq 0} \|\mathfrak{T}(t)\| \leq 1,$$

where $\mathfrak{T}$ is the semigroup defined in (7). Then for all $f \in C_{00}(\mathbb{R}_+, X)$ with $\|f\|_\infty \leq 1$, one has

$$\begin{aligned}
&\int_0^\infty \Phi\left(\|\mathfrak{T}(t) f\|_{C_{00}(\mathbb{R}_+,X)}\right) dt \\
&= \int_0^\infty \Phi\left(\sup_{s\geq t} \|U(s, s-t) f(s-t)\|\right) dt \\
&= \int_0^\infty \Phi\left(\sup_{\xi\geq 0} \|U(t+\xi, \xi) f(\xi)\|\right) dt \\
&= \int_0^\infty \left(\int_0^\infty 1_{[0, \sup_{\xi\geq 0}\|U(t+\xi,\xi)f(\xi)\|]}(u)\, \varphi(u)\, du\right) dt \\
&= \sup_{\xi\geq 0} \int_0^\infty \left(\int_0^\infty 1_{[0, \|U(t+\xi,\xi)f(\xi)\|]}(u)\, \varphi(u)\, du\right) dt \\
&= \sup_{\xi\geq 0} \int_0^\infty \Phi(\|U(t+\xi,\xi) f(\xi)\|)\, dt \leq \sup_{\xi\geq 0} \int_0^\infty \varphi(\|U(t+\xi,\xi) f(\xi)\|)\, dt \\
&= \sup_{\xi\geq 0} \int_\xi^\infty \varphi(\|U(\tau,\xi) f(\xi)\|)\, d\tau \leq M_\varphi < \infty,
\end{aligned}$$

where $1_{[0,h]}$ denotes the characteristic function of the interval $[0, h]$, $h > 0$.

Now, from [7, Theorem 3.2.2], it follows that $\omega_0(\mathfrak{T}) < 0$, hence $\mathcal{U}$ is uniformly exponentially stable.

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