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Interfering solutions of a nonhomogeneous Hamiltonian system *

Gregory S. Spradlin

Abstract

A Hamiltonian system is studied which has a term approaching a constant at an exponential rate at infinity . A minimax argument is used to show that the equation has a positive homoclinic solution. The proof employs the interaction between translated solutions of the corresponding homogeneous equation. What distinguishes this result from its few predecessors is that the equation has a nonhomogeneous nonlinearity.

1 Introduction

This paper is inspired by a remarkable result of Bahri and Li [3], which is a proof of a result of Bahri and Lions [4] employing a minimax method. The papers treat an elliptic partial differential equation of the form $-\Delta u + u = b(x)u^p$ on $\mathbb{R}^N$, with u^p a superquadratic, subcritical nonlinearity ($1 < p$, and $p < (N+2)/(N-2)$ for $N \geq 3$), and $b(x) \to b_\infty > 0$ as $|x| \to \infty$. One searches for positive solutions u that decay to zero as $|x| \to \infty$. This nonautonomous problem on the noncompact domain $\mathbb{R}^N$ is difficult to solve without assuming symmetry on $b(x)$. Bahri and Li found decaying positive solutions under the assumption that b is positive and continuous, and $(b(x) - b_\infty)_-$ (the negative part of $b(x) - b_\infty$) decays exponentially at a fast enough rate as $|x| \to \infty$ (to be precise, $b(x) - b_\infty = O(\exp(-(2+\delta))$ for some $\delta > 0$). Surprisingly, unlike in other, perturbative results, $b(x) - b_\infty$ may be "large" in just about any other sense, such as L^q norm, $1 \leq q \leq \infty$.

The proof in [3] is variational. A minimax class is formed using sums of translates of a solution of the corresponding autonomous problem $-\Delta u + u = b_\infty u^p$, and exploiting the "interference" between "tails" of that solution. This idea contrasts with many "multibump" results, in which a multibump solution is one that resembles a sum of translates of solutions of an equation [6, 7, 12]. In most of these results, the interference between bumps is a problem that must be managed by separating the translates by a great distance. There seems to have been little work done in exploiting interference of solutions in either multibump

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or simple existence results in elliptic PDE. A notable exception is a singularly perturbed elliptic equation studied by Wei and Xiaosong [13]. They employed interference between bumps to "hold them together" and counteract the lack of compactness in the problem and find multibumps. A related result is [8].

Articles [3, 13, 8] all involve an equation with homogeneous, or power linearity. The new paper [1] proves a result similar to [3] for a problem with nonhomogeneous nonlinearity, but with a symmetry condition on the coefficient function. We seek to avoid any symmetry assumptions. We begin here by studying an ordinary differential equation, with the aim of extending it later to the PDE case.

Here is the theorem:

Theorem 1.1 *Let f and b satisfy*

(f_1) $f \in C([0,\infty),[0,\infty))$

(f_2) $f(0) = 0$, $f(q) > 0$ *for* $q > 0$

(f_3) *there exists* $\mu > 2$ *such that* $f(q)q/\mu \geq F(q) \equiv \int_0^q f(s)\,ds$ *for all* $q \geq 0$, *and*

(f_4) $f(q)/q$ *is an increasing function of* q *for* $q > 0$, *and*

(b_1) $b \in C(\mathbb{R},(0,\infty))$

(b_2) $b(t) \to b_\infty > 0$ *as* $|t| \to \infty$, *and*

(b_3) *there exist* $\delta > 2\mu/(\mu-2)$ *and* $A > 0$ *such that* $b(t) - b_\infty > -Ae^{\delta|t|}$ *for all* $t \in \mathbb{R}$.

Then the Hamiltonian system

$$-u'' + u = b(t)f(u) \tag{1.1}$$

has a positive solution homoclinic to zero, that is, a solution u with $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$.

Hypotheses (f_1)-(f_3) imply that F is "superquadratic," that is, for small q, $F(q) = o(q^2)$ and for large q, $F(q) > O(q^2)$. Condition (f_4) is due to Nehari and has many helpful consequences, as we will see. (f_1) $-$ (f_4) are all satisfied in the canonical case $f(q) = q^p$, $p > 1$.

This paper is organized as follows: Section 2 contains the variational framework of the proof and the beginning of the proof. Section 3 contains the conclusion, which exploits the fact that g decays exponentially, as do homoclinic solutions of the autonomous equation associated with (1.1).

2 The variational argument

Let $E \equiv W^{1,2}(\mathbb{R})$, with the standard inner product and norm. Extend f to 0 on the negative reals, and define the C^2 functional $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} b(t)F(u(t))\,dt.$$

By regularity theory, and the maximum principle, all nonzero critical points of I are positive homoclinic solutions to (1.1), and vice versa. ¿From now on, without loss of generality assume $b_\infty = 1$. Then the functional corresponding to the autonomous equation $-u'' + u = f(u)$ is

$$I_0(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(u(t))\,dt.$$

An analysis of the equation $-u'' + u = f(u)$ in the (u, u') phase plane shows that the equation has a unique nonzero homoclinic solution, modulo translation, which is positive. Let us denote by ω the positive solution satisfying $\omega(0) = \max \omega$. ω is even in t and decays exponentially. I_0 has a unique nonzero critical value, $c_0 = I_0(\omega)$. c_0 is the "mountain pass" value for I_0. That is, defining the set of paths

$$\Phi_0 = \{h \in C([0,1], E) \mid h(0) = 0, I_0(h(1)) < 0\},$$

c_0 is the minimax value

$$c_0 = \inf_{h\in\Phi_0} \max_{\theta\in[0,1]} I_0(h(1)) > 0.$$

Define c, the mountain pass value for I, by defining the set of paths

$$\Phi = \{h \in C([0,1], E) \mid h(0) = 0, I(h(1)) < 0\},$$

and

$$c = \inf_{h\in\Phi} \max_{\theta\in[0,1]} I(h(\theta)) > 0.$$

We will use a concentration compactness type result to describe Palais-Smale sequences of I. Recall that a Palais-Smale sequence of I is a sequence $(u_m) \subset E$ with $I'(u_m) \to 0$ and $(I(u_m))$ convergent. Define the translation operator τ as follows: for $t_0 \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$, let $\tau_{t_0} u$ be u translated t_0 units to the right, that is, $\tau_{t_0} u(t) = u(t - t_0)$ for all $t \in \mathbb{R}$. The proposition below states that a Palais-Smale sequence "splits" into the sum of a critical point of I and critical points of I_0:

Lemma 2.1 *If $(u_m) \subset E$ with $I'(u_m) \to 0$ and $I(u_m) \to a > 0$, there exist $v \in E$, $k \geq 0$, and sequences $(t^i_m)_{1\leq i\leq k \atop m\geq 1} \subset \mathbb{R}$, such that, along a subsequence (also denoted (u_m)),*

(i) $\|u_m - (v + \sum_{i=1}^k \tau_{t^i_m}\omega)\| \to 0$

(ii) $|t_m^i| \to \infty$ *as* $m \to \infty$ *for* $i = 1, \dots, k$

(iii) $t_m^{i+1} - t_m^i \to \infty$ *as* $m \to \infty$ *for* $i = 1, \dots, k-1$

(iv) $kc_0 + I(v) = a$

A proof can be found in [6]. By standard deformation arguments, there exists [11] a Palais-Smale sequence (u_m) for I along which I converges to c. Suppose $c < c_0$. Then by applying Lemma 2.1, and the fact that I has no negative critical values, (iv) implies that the "k" value is zero, so along a subsequence, (u_m) converges to a critical point v of I. Since $I(v) = c > 0$, v is nontrivial.

By (b_2), $c \le c_0$. So from now on, we assume

$$c = c_0.$$

By (f_4), I has the following property (as does I_0): For any $u \in E \setminus \{0\}$, the mapping $s \mapsto I(su)$ is 0 at $s = 0$, increases for small positive s, attains a maximum, then decreases to $-\infty$ (see [6] for proof). Define the Nehari manifold $\mathcal{S}$ for I by

$$\mathcal{S} = \{u \in E \mid u \ne 0,\ I'(u)u = 0\}.$$

Note that a nonzero function u is in $\mathcal{S}$ if and only if $I(u) = \sup_{s>0} I(su)$. Then

$$c = \inf_{\mathcal{S}} I.$$

Define the "location" function $\mathcal{L} : E \setminus \{0\} \to \mathbb{R}$ by

$$\int_{\mathbb{R}} u^2 \tan^{-1}(t - \mathcal{L}(u))\, dt = 0.$$

By the Implicit Function Theorem, $\mathcal{L}$ is a well defined, continuous function. $\mathcal{L}(\omega) = 0$, and $\mathcal{L}(\tau_t u) = \mathcal{L}(u) + t$ for any $u \in E \setminus \{0\}$ and $t \in \mathbb{R}$. Define

$$\beta = \inf\{I(u) \mid u \in \mathcal{S},\ \mathcal{L}(u) = 0\}. \tag{2.1}$$

Clearly $\beta \ge c = c_0$. If $\beta = c_0$, then c_0 is a critical value of I: suppose $\beta = c_0$. Take $(u_m) \subset \mathcal{S}$ with $\mathcal{L}(u_m) = 0$ and $I(u_m) \to c_0$. Along a subsequence, $I'|_{\mathcal{S}}(u_m) \to 0$. By (f_4), $I'(u_m) \to 0$ (see [9] for similar minimax arguments on a Nehari manifold). Apply Lemma 2.1. (iv) shows, again, that either $k = 0$, or $k = 1$ with $v = 0$. The latter alternative is impossible because $\mathcal{L}(u_m) = 0$. Therefore (u_m) converges (along a subsequence) to a critical point of I. Thus we assume from now on that

$$\beta > c_0. \tag{2.2}$$

Define $\mathcal{P} : E \setminus \{0\} \to \mathcal{S}$ to be radial projection onto $\mathcal{S}$, that is,

$$\mathcal{P}(u) = tu;\ t > 0,\ tu \in \mathcal{S}.$$

For $R > 0$ define the minimax class

$$\Gamma_R = \{\gamma \in C([0,1], \mathcal{S}) \mid \gamma(0) = \mathcal{P}(\tau_{-R}\omega), \gamma(1) = \mathcal{P}(\tau_R\omega)\}$$

and the minimax value

$$c[R] = \inf_{\gamma \in \Gamma_R} \max_{\theta \in [0,1]} I(\gamma(\theta)).$$

Theorem 1.1 will follow from the following proposition:

Proposition 2.2 *There exists $R_0 = R_0(f, g)$ with the following property: if $R \geq R_0$, then*

(i) $I(\mathcal{P}(\tau_{-R}\omega)) < \beta$ and $I(\mathcal{P}(\tau_R\omega)) < \beta$

(ii) $c[R] \geq \beta$

(i) $c[R] < 2c_0$

To prove Theorem 1.1 from Proposition 2.2, let $R \geq R_0$. By (i)-(ii), there exists a sequence $(u_m) \subset \mathcal{S}$ with $I(u_m) \to c[R]$ and $I'|_{\mathcal{S}}(u_m) \to 0$, and $I'(u_m) \to 0$. Apply Lemma 2.1 to (u_m). Since $c_0 < \beta < c[R] < 2c_0$, Lemma 2.1(iv) implies that $c[R]$ or $c[R] - c_0$ is a critical value of I. Since $0 < c[R] - c_0 < c_0$, assumption (f_4) implies that $c[R] - c_0$ is not a critical value of I. Therefore $c[R]$ is a positive critical value of I. Maximum principle arguments show that (1.1) has a positive homoclinic solution. Theorem 1.1 is proven.

3 Interfering Tails

This section contains the proof of Proposition 2.2. Parts (i) and (ii) are easy. Using (b_2), it is straightforward to show that $I(\mathcal{P}(\tau_{-R}\omega)) \to c_0$ and $I(\mathcal{P}(\tau_R\omega)) \to c_0$ as $R \to \infty$, and we assumed in (2.2) that $c_0 < \beta$. (ii) holds for all $R > 0$, not necessarily large: let $\gamma \in \Gamma_R$. Since $\mathcal{L}(\mathcal{P}(\tau_{-R}\omega)) = -R$, $\mathcal{L}(\mathcal{P}(\tau_R\omega)) = R$, and $\mathcal{L}$ is continuous, there exists $\theta^* \in [0,1]$ with $\mathcal{L}(\gamma(\theta^*)) = 0$, so by the definition of β, $I(\gamma(\theta^*)) \geq \beta$. The remainder of this section is devoted to the proof of (iii), which is more difficult.

We adopt a construction similar to that of [3]. For $R > 0$, define $\gamma_R : [0,1] \to E \setminus \{0\}$ by

$$\gamma_R(\theta) = \max((1-\theta)\tau_{-R}\omega, \theta\tau_R\omega)$$

and $\hat{\gamma}_R \in \Gamma_R$ by

$$\hat{\gamma}_R(\theta) = \mathcal{P}(\gamma_R(\theta))$$

We will show that for large enough R,

$$\max_{\theta \in [0,1]} I(\hat{\gamma}_R(\theta)) < 2c_0, \tag{3.1}$$

proving Proposition 2.2(iii).

To avoid the complications in working with the manifold $\mathcal{S}$, we have the following lemma. “For large enough R” means there exists $R_0 = R_0(f, g)$ such that if $R \geq R_0$, etc., as in Proposition 2.2.

Lemma 3.1 *There exists $T = T(f, g)$ such that for large enough R, and all $\theta \in [0,1]$,*

$$I(T\gamma_R(\theta)) < 0.$$

Proof: let $T > 0$ be large enough so that $I_0(T\omega) < -c_0 - 2$. Then for all $\theta \in [0,1]$, $I_0(\theta T\omega) + I_0((1-\theta)T\omega) < -2$. Let $\tilde{\omega} \in E$ have compact support in $\mathbb{R}$ and be close enough to ω that $I_0(\theta T\tilde{\omega}) + I_0((1-\theta)T\tilde{\omega}) < -1$ for all $\theta \in [0,1]$. I is Lipschitz on bounded subsets of E (see [11] for a proof in a similar setting), so for large enough R, and all $\theta \in [0,1]$, $I(T\gamma_R(\theta)) < -1 + 1/2 < 0$.

By Lemma 3.1, in order to prove (3.1), it suffices to show that for large enough R, all $\theta \in [0,1]$ and $s \in [0,T]$,

$$I(s\gamma_R(\theta)) < 2c_0. \tag{3.2}$$

We will treat separately the case where θ is close to 0 or 1:

Lemma 3.2 *For large enough R, all $s \in [0,T]$ and all $\theta \in [0,1/4] \cup [3/4,1]$,*

$$I(s\gamma_R(\theta)) < 2c_0. \tag{3.3}$$

Proof: without loss of generality let $\theta \in [0,1/4]$. If $I_0(s\theta\omega) < I_0(\omega/2)$, then $I_0(s\theta\omega) + I_0((1-\theta)s\omega) < c_0 + I_0(\omega/2)$. If $I_0(s\theta\omega) \geq I_0(\omega/2)$, then $s\theta \geq 1/2$, so $(1-\theta)s = (\frac{1}{\theta} - 1)(s\theta) \geq 3(\frac{1}{2}) \geq 3/2$, so $I_0((1-\theta)s\omega) \leq I_0(\frac{3}{2}\omega)$, $I_0(s\theta\omega) + I_0((1-\theta)s\omega) < c_0 + I_0(\frac{3}{2}\omega)$. Assume that R is large enough so that for all $s \in [0,T]$,

$$|I(s\gamma_R(\theta) - (I_0(s(1-\theta)\omega) + I_0(s\theta\omega))| < \frac{1}{2}\min(c_0 - I_0(\omega/2), c_0 - I_0(3\omega/2)).$$

Then the triangle inequality gives (3.3).

Now we must prove (3.2) for $\theta \in [1/4,3/4]$. For all $R > 0$ and $s \geq 0$, $I_0(s\tau_{-R}\omega) \leq c_0$ and $I_0(s\tau_R\omega) \leq c_0$. So it suffices to show that for large enough R, $s \in [0,T]$, and $\theta \in (1/4,3/4)$,

$$\begin{aligned} &2c_0 - I(s\gamma_R(\theta)) \\ &\quad\geq \Big[(I_0((1-\theta)s\tau_{-R}\omega) + I_0(\theta s\tau_R\omega)) - (I((1-\theta)s\tau_{-R}\omega) + I(\theta s\tau_R\omega))\Big] \\ &\qquad + \big[(I((1-\theta)s\tau_{-R}\omega) + I(\theta s\tau_R\omega)) - I(s\gamma_R(\theta))\big] \\ &\quad\equiv A(R,\theta,s) + B(R,\theta,s) > 0\,. \end{aligned} \tag{3.4}$$

We begin with $B(R,\theta,s)$. To analyze $I(s\gamma_R(\theta))$, we have the following lemma:

Lemma 3.3 *For large enough R, and $\theta \in (1/4,3/4)$, there exists a unique $t \in \mathbb{R}$ with $(1-\theta)\tau_{-R}\omega(t) = \theta\tau_R\omega(t)$. Calling this value of t, $t_{R,\theta}$,*

$$t_{R,\theta}/R \to 0 \ \text{as}\ R \to \infty.$$

Proof: let $\epsilon \in (0,1)$. Let R be large enough so that $\omega(t+2R) < \omega(t)/4$ for all $t \geq -(1-\epsilon)R$. This is possible because ω decays exponentially. Now $(1-\theta)\tau_{-R}\omega \equiv (1-\theta)\omega(\cdot + R)$ is decreasing on $[-\epsilon R, \epsilon R]$ and $\theta\tau_R\omega$ is increasing on $[-\epsilon R, \epsilon R]$. Thus, to prove the existence and uniqueness of $t_{R,\theta}$, it now suffices to prove that for large R and $\theta \in (1/4,3/4)$,

(i) $(1-\theta)\tau_{-R}\omega > \theta\tau_R\omega$ on $(-\infty, -\epsilon R]$, and

(ii) $\theta\tau_R\omega > (1-\theta)\tau_{-R}\omega$ on $[\epsilon R, \infty)$.

By the symmetry of the problem, the proof of (i) and (ii) are practically the same, so we prove (ii). Let $t \geq \epsilon R$. Then $t - R \geq -(1-\epsilon)R$, so $\omega(T+R) < \omega(t-R)/4$, and

$$(1-\theta)\tau_{-R}\omega(t) \equiv (1-\theta)\omega(t+R) < \frac{1}{4}(1-\theta)\omega(t-R) < \frac{1}{4}\omega(t-R) < \theta\tau_R\omega(t).$$

For $U \subset \mathbb{R}$, define $\|u\|_U = \|u\|_{W^{1,2}(U)}$. For large R and $s \in [0,T]$,

$$\begin{aligned} B(R,\theta,s) &= I((1-\theta)s\tau_{-R}\omega) + I(\theta s\tau_R\omega) - I(s\gamma_R(\theta)) \\ &= \frac{1}{2}(1-\theta)^2 s^2 \|\tau_{-R}\omega\|^2_{[t_{R,\theta},\infty)} + \frac{1}{2}\theta^2 s^2 \|\tau_R\omega\|^2_{(-\infty,t_{R,\theta}]} \\ &- \int_{t_{R,\theta}}^{\infty} b(t)F(s(1-\theta)\omega)\tau_{-R}\omega - \int_{-\infty}^{t_{R,\theta}} b(t)F(s\theta\omega)\tau_R\omega. \end{aligned} \tag{3.5}$$

Assume that R is large enough so that $|t_{R,\theta}| < R/2$ and that for all $\theta \in (1/4, 3/4)$, $s \in [0,T]$ and $t \geq R/2$,

$$b(t)F(s(1-\theta)\omega(t)) \leq \frac{1}{4}s^2(1-\theta)^2\omega(t)^2. \tag{3.6}$$

Since $t_{R,\theta} > -R/2$, $t + R > R/2$, so (3.6) gives, for all $t \geq t_{R,\theta}$,

$$b(t)F(s(1-\theta)\tau_{-R}\omega(t)) \leq \frac{1}{4}s^2(1-\theta)^2\tau_{-R}\omega(t)^2. \tag{3.7}$$

Similarly, for all $t \leq t_{R,\theta}$,

$$b(t)F(s\theta\tau_R\omega(t)) \leq \frac{1}{4}s^2\theta^2\tau_R\omega(t)^2, \tag{3.8}$$

so (3.5), (3.7), (3.8) and the Sobolev inequality $\|u\|_{L^\infty(0,\infty)} \leq \|u\|_{W^{1,2}(0,\infty)}$ give

$$\begin{aligned} B(R,\theta,s) &\geq \frac{1}{4}s^2\big[(1-\theta)^2\|\tau_{-R}\omega\|^2_{[t_{R,\theta},\infty)} + \theta^2\|\tau_R\omega\|^2_{(-\infty,t_{R,\theta}]}\big] \\ &\geq s^2(\omega(R+t_{R,\theta})^2 + \omega(R-t_{R,\theta})^2)/16. \end{aligned} \tag{3.9}$$

Since $\delta > 2\mu/(\mu-2)$ ((b_3)), we may choose

$$d \in (\frac{2}{\delta}, 1 - \frac{2}{\mu}).$$

Since $(1-d)\mu > 2$ and $d\delta > 2$, we may choose $\epsilon_1 \in (0,1)$ and small enough so that

$$2(1+\epsilon_1)^2 < \min((1-\epsilon_1)\mu(1-d),\ \delta d). \tag{3.10}$$

By the maximum principle, and the superlinear growth of f near 0, there exists $l > 0$ such that

$$\omega(t) > le^{-(1+\epsilon_1)|t|}$$

for all $t \in \mathbb{R}$. Assume R is large enough (Lemma 3.3) so

$$|t_{R,\theta}| < \epsilon_1 R\,.$$

Then

$$\tau_{-R}\omega(t_{R,\theta}) = \omega(R + t_{R,\theta}) \geq \omega((1+\epsilon_1)R) > le^{-(1+\epsilon_1)^2 R},$$

so by (3.9),

$$B(R,\theta,s) \geq s^2 l^2 e^{-2(1+\epsilon_1)R}/8. \tag{3.11}$$

Next, let us estimate $A(R,\theta,s)$ from (3.4), still assuming $\theta \in (1/4, 3/4)$ and $s \in [0,T]$. ¿From now on, C will denote a large positive constant depending only on f, b, and the already chosen T. The value of C may change from line to line. By (f_3), for all $q \in [0, T\max\omega]$,

$$F(q) \leq Cq^\mu.$$

By the form of $A(R,\theta,s)$, all of the "$\|\ \|^2$" terms in $A(R,\theta,s)$ cancel out. Since $f \geq 0$, it is easy to show, by comparing ω to a solution v of $-v'' + v = 0$, that $\omega(t) \leq Ce^{-|t|}$ for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} A(R,\theta,s) &= \int_{\mathbb{R}} (b(t)-1)F((1-\theta)s\tau_{-R}\omega) + \int_{\mathbb{R}} (b(t)-1)F(\theta s\tau_R\omega) \\ &\geq -C\big[\int_{\mathbb{R}} e^{-\delta|t|}F((1-\theta)s\omega(t+R))\,dt + \int_{\mathbb{R}} e^{-\delta|t|}F(\theta s\omega(t-R))\,dt\big] \\ &\geq -Cs^\mu\big[\int_{\mathbb{R}} e^{-\delta|t|}e^{-\mu|t-R|}\,dt + \int_{\mathbb{R}} e^{-\delta|t|}e^{-\mu|t+R|}\,dt\big] \\ &\geq -Cs^2\int_{\mathbb{R}} e^{-\delta|t|}e^{-\mu|t-R|}\,dt. \end{aligned}$$

The integral can be estimated by

$$\begin{aligned} \int_{\mathbb{R}} e^{-\delta|t|}e^{-\mu|t-R|}\,dt &\leq \int_{-\infty}^{dR} e^{-\mu|t-R|}\,dt + \int_{dR}^{\infty} e^{-\delta|t|}\,dt \\ &= \frac{e^{\mu(t-R)}}{\mu}\Big|_{t=-\infty}^{t=dR} + \frac{e^{-\delta t}}{-\delta}\Big|_{t=dR}^{t=\infty} \leq C\big[e^{-\mu(1-d)R} + e^{-d\delta R}\big]. \end{aligned}$$

By (3.11) and (3), for large R, $s \in [0,T]$, and $\theta \in (1/4, 3/4)$,

$$\begin{gathered} A(R,\theta,s) \geq -Cs^2[e^{-\mu(1-d)(1-\epsilon_1)R} + e^{-d\delta R}] \quad \text{and} \\ B(R,\theta,s) \geq l^2 s^2 e^{-2(1+\epsilon_1)^2 R}/8. \end{gathered}$$

By (3.10), $A(R,\theta,s) + B(R,\theta,s) > 0$ for large R. By (3.4), Lemma 3.2 is proven, from which follow Proposition 2.2 and Theorem 1.1.

The proof of Theorem 1.1 was indirect, and we can not say much about the positive homoclinic solution of (1.1). We do know that if $c < c_0$, then I has a critical point at critical level c. If the alternative $c = c_0$ holds, then if $\beta = c_0$ (see (2.1)), c is a critical level of I. If $c = c_0$ and $\beta > c_0$, then for large enough R, $c[R]$ ($\in (c_0, 2c_0)$) is a critical level of I (assuming otherwise, then a deformation argument yields a contradiction to the definition of $c[R]$).

The proof of Theorem 1.1 suggests that, under additional conditions on b, (1.1) may have "two-bump" solutions, with parts resembling translations of ω to the left and to the right of zero.

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Gregory S. Spradlin
Department of Computing and Mathematics
Embry-Riddle Aeronautical University
Daytona Beach, Florida 32114-3900 USA
e-mail: spradlig@db.erau.edu