TOTAL MINOR POLYNOMIALS OF ORIENTED HYPERGRAPHS

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Abstract

Concepts of graph theory can be generalized to integer matrices through the use of oriented hypergraphs. An oriented hypergraph is an incidence structure consisting of vertices, edges, and incidences, equipped with three functions: a vertex incidence function, an edge incidence function, and an incidence orientation function. This thesis provides a unifying generalization of Seth Chaiken's All-Minors Matrix-Tree Theorem and Sachs' Coefficient Theorem to all integer adjacency and Laplacian matrices – extending the results of Rusnak, Robinson et. al. – by introducing a polynomial in $|V|^2$ indeterminants indexed by minor order whose monomial coefficients are the minors. The coefficients are determined by embedding the oriented hypergraph into the smallest uniform hypergraph that contains it and summing over a class of sub-monic mappings of paths of length one relative to the original oriented hypergraph. It is known that the non-cancellative mappings associated to each degree-1 monomials are in one-to-one correspondence with Tuttes Matrix-Tree Theorem. This is extended to Tuttes k-arborescence decomposition via the degree-k monomials.

Acknowledgements

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1 Introduction

Matrix-tree type theorems for graphs [13] and signed graphs [3] have been simplified to restrictions of permutations that correspond indirectly to subgraphs. Generalizations of Sachs' characterization of the permanental and determinantal polynomials of the Laplacian and adjacency matrix to signed graphs [2] and to oriented hypergraphs [4] have recently been established using similar techniques. This paper further extends the characteristic polynomial to total minor polynomials through an examination of the contributors of incidence structures.

In Section 2, the definitions and terminology of graphs and signed graphs are examined, along with the methodology of matrices and polynomials used to analyze these graphs. As discussed in Section 2.5, signed graphs are defined as a graph that has a sign $\{+1, -1\}$ on each edge; signed graphs have been used in psychological models [1, 9]. Section 3 further extends these concepts to hypergraphs through the use of the incidence structure. Incidence orientations of signed graphs, that were introduced in [15], were further extended to hypergraphs in [5, 10, 12], and can be used to extend graph theoretic theorems to hypergraphs. Many definitions and results obtained for graphs are easily extended to hypergraphs such as degree of a vertex, spanning tress, Laplacian and adjacency matrices; and theorems such as Matrix-tree Theorem and Sachs' Theorem [2, 3]. The Matrix-tree Theorem and Sachs' Theorem have recently been extended to hypergraphs in [4, 11] and are further extended to the total minor polynomial in Section 3.3 of this paper.

Concepts from signed graph theory can be generalized to integer matrices by the oriented incidence structure. We demonstrate in Section 3.3, that a generalized characteristic polynomial with $|V|^2$ indeterminants provides a generalization that unifies Seth Chaiken's All-minors Matrix-tree Theorem [3] and Sachs' Theorem [2, 6] to oriented hypergraphs by generalizing the hypergraphic polynomial characterizations of [4, 11]. The results are obtained via a sub-object characterization (versus the classical map restriction) utilizing the category theoretic work of [8] and demonstrating the total minor polynomial is related to specific sub-objects within a larger uniform hypergraph.

The coefficients of the total minor polynomials are shown to have natural interpretations as these sub-objects, and the degree-1 monomials always count a reduced class of single-element mappings that are in one-to-one correspondence with the spanning trees in Tutte's Matrix-tree Theorem [11]. Note that the coefficients may count more sub-objects, but only the single-element maps correspond to spanning trees. In Section 4, it is shown that the coefficients of the degree-k monomials always count k-arborescences. The nature of these k-arborescences will be studied in more detail, with particular attention to collections of 2-arborescences called *transpedances* which have been used to reproduce Kirchhoff's Laws using the underlying graph structure. This implies the existence of higher-order Kirchhoff-type Laws that need not be conservative.

2 Background

The definitions and theorems contained in this section are intended to provide background and context for the theory and methods used to establish the total minor polynomials. In Section 2.1, the definition of a graph and the terms associated with graph will be discussed. In Section 2.2 and Section 2.3, methods of analyzing graphs through matrices, permanents, determinants, and polynomials will be presented. Section 2.4 will present the known results of Sachs' Theorem. In Section 2.5, the definitions associated with graphs will be extend to signed graphs and bidirected graphs. Finally, Section 2.6 will discuss known and relevant results regarding the Matrix-tree Theorem and arborescences.

2.1 Graphs

The following definitions provide a structure for graphs that is used in the results of this paper. These definitions are adaptations of the definitions provided by [10, 12]. This section will provide a basic understanding and examples of graphs using an incidence structure.

A *graph* $G = (V, E, I, \varsigma, \omega)$ is a set of vertices V, a set of edges E, and a set of incidences I equipped with two functions $\varsigma : I(G) \to V(G)$ and $\omega : I(G) \to E(G)$, where $|\omega^{-1}(e)| \leq 2$. Graphs and their components have a number of properties. The *degree of a vertex* v is $|\varsigma^{-1}(v)|$. The *size of an edge* is $|\omega^{-1}(e)|$. A vertex and edge are *incident* via incident i if $i \in \varsigma^{-1}(v) \cap \omega^{-1}(e)$.

Connection between edges and vertices on a graph are considered to be paths. A *directed path of length* $n/2$ is a non-repeating sequence

$$\vec{P}_{n/2} = (a_0, i_1, a_1, i_2, a_2, i_3, a_3, ..., a_{n-1}, i_n, a_n)$$

of vertices, edges and incidences, where $\{a_\ell\}$ is an alternating sequence of vertices and edges, and i_h is an incidence between a_{h-1} and a_h. There are a number of different types of paths defined by restrictions on their sequence of edges, incidences, and vertices. A *directed weak walk of* G is the image of an incidence-preserving map of a directed path into G. A *path of* G is a vertex, edge, and incidence-monic directed weak walk. A *backstep of* G is an embedding of $\vec{P}_1$ into G that is neither incidence-monic nor vertex-monic; a *loop of* G is an embedding of $\vec{P}_1$ into G that is incidence-monic but not vertex-monic. A *directed adjacency of* G is an embedding of $\overrightarrow{P}_1$ into G that is incidence-monic. Finally, a *circle of* G is an embedding of $\overrightarrow{P}_n$ into G that is incidence-monic and vertex-monic with the exception of the initial vertex $a_0 = a_n$.

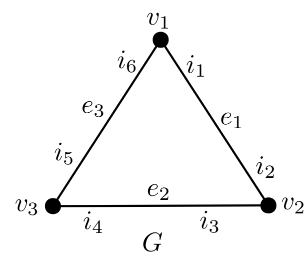


Figure 1: An example of a simple graph.

Consider the graph in Figure 1. There are three vertices labeled v_1, v_2, v_3 and each is connected to two incidences making the degree of each vertex two. Each edge is connected to two incidences and thus each edge has a size of two. The vertices v_1 and v_2 are adjacent since the path $\overrightarrow{P_1} = (v_1, i_1, e_1, i_2, v_2)$ connects them. The vertices v_1, v_2, v_3 form a circle of G. For clarity, incidences may be omitted in future example, however all edges and vertices are bonded by incidences.

Beyond the components that make up a graph and their properties, there are properties of entire graphs or portions of a graph. A *subgraph* of a graph G is a graph formed by a subset of the original graph's vertex, edge, and incidence sets with a restriction of the original mapping functions. Clearly, a graph is a subgraph of itself. A *tree* is a connected acyclic graph. A subgraph H of G is *spanning* if $V(H) = V(G)$, while a *spanning tree* of G is a subgraph of G that is a tree and spanning. A *connected component* of G is a subgraph in which a path exists between all pairs of vertices of the subgraph. All graphs will be assumed to be connected with at least one incidence.

As seen in Figure 2, all the graphs contain all four vertices, while each of the eight spanning trees subgraphs below contain a subset of the original graph's edges and incidences. Moreover, these subgraphs are connected and acyclic.

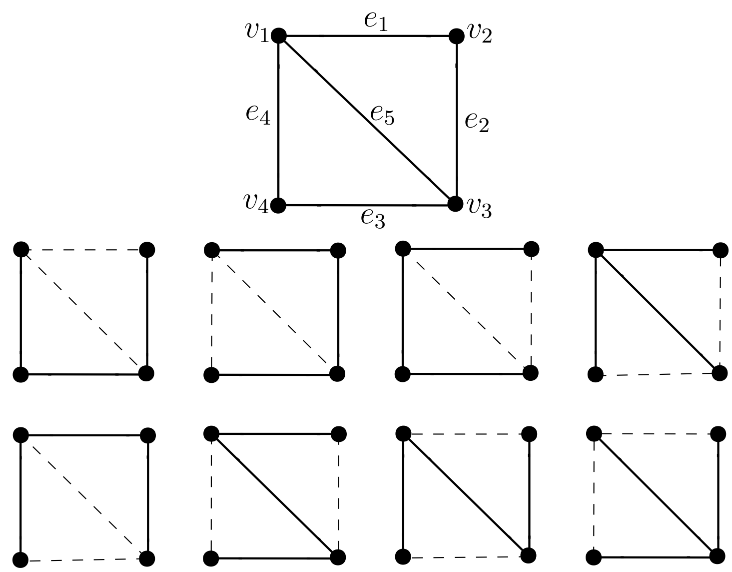


Figure 2: The graph has eight spanning trees, each obtained by deleting two of the original graph's edges (appearing as dashed edges).

2.2 Permanents and Determinants and Polynomials

A *permutation* π is a bijection from a set S to itself. A *cycle of a permutation* is a cyclic sequence obtained by the compositional closure of π. Every permutation can be written as the product of disjoint cycles; one cycles or fixed points are conventionally omitted.

Example 2.2.1 *If* $S = \{1, 2, 3\}$ *then all possible permutation of* S *are*

$$e, (12), (13), (23), (123), (132).$$

The following lemma is well known.

Lemma 2.2.2 *The total number of permutations of a set with* n *elements is* $n!$.

Example 2.2.3 *Consider the permutation* $(12357)(2476)$. *These two cycles are not disjoint given that* 2 *and* 7 *are in both cycles. The disjoint cycle decomposition yields two disjoint cycles.*

$$(12357)(2476) = (124)(3576)$$

Permutations have a number of characteristics. An *inversion* in a permutation occurs for every pair (i, j) with $i < j$, where we have $\pi(i) > \pi(j)$. An *even cycle* is a cycle with an odd number of inversions, while an *odd cycle* is a cycle with an even number of inversions. In Figure 3, the permutation (132) takes two of the three pairs where $i < j$ to pairs where $\pi(i) > \pi(j)$. Therefore, there are two inversions in permutation (132).

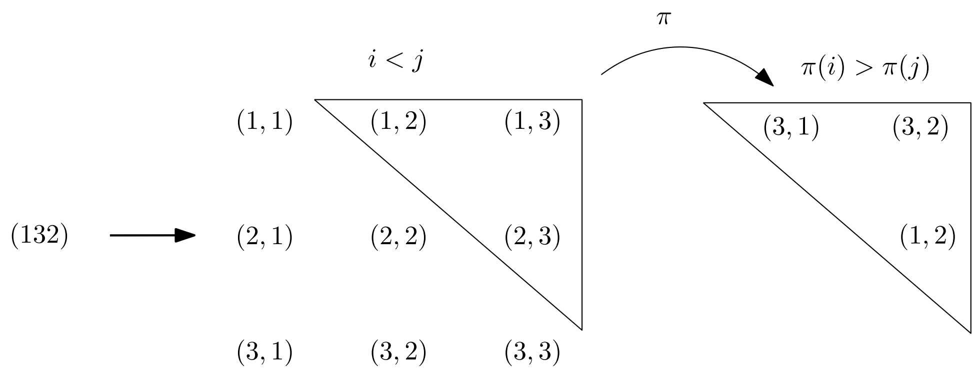


Figure 3: The (i,j) pairs with $i < j$ appear in the triangle. Their image under the permutation $\pi = (132)$ has two inversions, namely, $(3,1)$ and $(3,2)$.

Permutations are important for the calculations of determinants and permanents. Both determinants and permanents are sums over permutations of entries in matrices. Given an $n \times n$ matrix $\mathbf{M}$ and S_n a symmetric group of order n, the *permanent* of $\mathbf{M}$ is

$$\text{perm}(\mathbf{M}) = \sum_{\pi \in S_n} \prod_{i \in [n]} m_{i,\pi(i)},$$

and the *determinant* of $\mathbf{M}$ is

$$\det(\mathbf{M}) = \sum_{\pi \in S_n} \epsilon(\pi) \prod_{i \in [n]} m_{i,\pi(i)}.$$

Where $\epsilon(\pi) = (-1)^{inv(\pi)}$, and where $inv(\pi)$ is the number of inversions of π. It is known that $\epsilon(\pi) = (-1)^{ec(\pi)}$ where $ec(\pi)$ is the number of even cycles in π.

Example 2.2.4 *An example of a determinant calculation is*

$$\begin{aligned}
\det \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} &= (-1)^0 2 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + (-1)^1(-1) \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} + (-1)^2(-1) \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \\
&= 2[(2 \cdot 2) - ((-1) \cdot (-1))] + [(-1)(2) - (-1)(-1)] - [(-1)(-1) - (-1)(2)] \\
&= 2(4-1) + (-2-1) - (1-(-2)) \\
&= 2(3) + (-3) - (3) \\
&= 0
\end{aligned}$$

Note that the determinant is zero.

It will be discussed in Section 2.3 that graphical information can be represented in matrices and it is known that the determinant of one of these matrices is zero for all graphs.

Determinants are used to calculate characteristic polynomials of matrices. The *characteristic polynomial of* $\mathbf{M}$ is $\chi_{\mathbf{M}}(x) = det(x\mathbf{I} - \mathbf{M})$ where $\mathbf{I}$ is the identity matrix.

Example 2.2.5 *The characteristic polynomial for the matrix in Example 2.2.4.*

$$\begin{aligned}
\det\begin{bmatrix} x-2 & 1 & 1 \\ 1 & x-2 & 1 \\ 1 & 1 & x-2 \end{bmatrix} &= (-1)^0(x-2)\begin{bmatrix} x-2 & 1 \\ 1 & x-2 \end{bmatrix} + (-1)^1(1)\begin{bmatrix} 1 & 1 \\ 1 & x-2 \end{bmatrix} + (-1)^2(1)\begin{bmatrix} 1 & x-2 \\ 1 & 1 \end{bmatrix} \\
&= (x-2)[(x-2)(x-2)-1] + (-1)[(x-2)-1] + (1)[1-(x-2)] \\
&= (x-2)(x^2-4x+4-1) + (-1)(x-3) + (-x+3) \\
&= (x^3-4x^2+3x-2x^2+8x-6) + (-x+3) + (-x+3) \\
&= x^3-6x^2+9x
\end{aligned}$$

In the next subsection, we will see the matrix in Example 2.2.4 is the Laplacian of a graph.

2.3 Matrices of Graphs

Information in graphs can be represented through different types of integer matrices. Calculations performed on these matrices yield information about the graphs they represent. Permanents, determinants, and characteristic polynomials are used to analyze the information stored in these integer matrices. In this section, we will define the main matrices that are used with graphs.

The *incidence matrix* of a graph G is the $V \times E$ matrix $\mathbf{H}_G$ where the (v,e)-entry is the number of incidences $i \in I$ such that $\varsigma(i) = v$ and $\omega(i) = e$. Moreover, the entries in each column are signed so that the column sums are zero. The *adjacency matrix* $\mathbf{A}_G$ of a graph G is the $V \times V$ matrix whose (u,w)-entry is the number of adjacencies between vertex u and vertex w. The *degree matrix* of a graph G is the $V \times V$ diagonal matrix whose (v,v)-entry is the number of incidences $i \in I$ such that $\varsigma(i) = v$. The number of incidences at a vertex v is clearly equal to the number of backsteps at v. Using this, the *Laplacian matrix of* G is defined as $\mathbf{L}_G := \mathbf{H}_G\mathbf{H}_G^T = \mathbf{D}_G - \mathbf{A}_G$. See [10] for the result that the Laplacian is the 1-weak-walk matrix.

Example 2.3.1 *For the graph in Figure 1 the degree, adjacency, incidence, and Laplacian matrix are shown below.*

$$\mathbf{D}_G = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{L}_G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \mathbf{H}_G = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The matrix in Example 2.2.5 is the Laplacian matrix for the graph in Figure 1. It is clear that the diagonal entries are the degrees and the off diagonal entries are the adjacencies.

If a graph contains loops then the adjacency matrix would have non-zero entries on the diagonal, changing the values on the diagonal of the Laplacian. The permanent and determinant calculations discussed in Section 2.2 are performed on the adjacency and Laplacian matrices for graphs to find information about the graphs. It is known that for every graph, the determinant of the Laplacian is zero.

2.4 Sachs' Theorem

The coefficients of the characteristic polynomial contain information specific to the matrix on which the calculations are preformed, and are obtained by specific permutation counts. Sachs' Theorem [6] yields a combinatorial count of coefficients of the characteristic polynomial of the adjacency matrix for graphs using subgraphs, and employs the use of elementary figures and basic figures to achieve these counts. An *elementary figure* is a circle on n vertices where $n \geq 2$, a P_1 subgraph, or an isolated vertex. A *basic figure* U is the disjoint union of elementary figures.

Let $\mathscr{U}_k$ be the set of all basic figures in G with exactly k isolated vertices. Let $p(U)$ be the number of elementary figures of U and let $c(U)$ denote the number of circles in U. Given this information for a graph G and the adjacency matrix $\mathbf{A}$ of G, Sachs' Theorem finds the coefficients of the characteristic polynomial of $\mathbf{A}$.

Theorem 2.4.1 (Sachs' Theorem) *For a graph G with $n = |V(G)|$,*

$$\chi_G(\mathbf{A}, x) = \sum_{k=1}^{n} \left(\sum_{U \in \mathscr{U}_k} (-1)^{p(U)} (2)^{c(U)} \right) x^k.$$

Example 2.4.2 *Consider the basic figures of Figure 1 shown in Figure 4.*

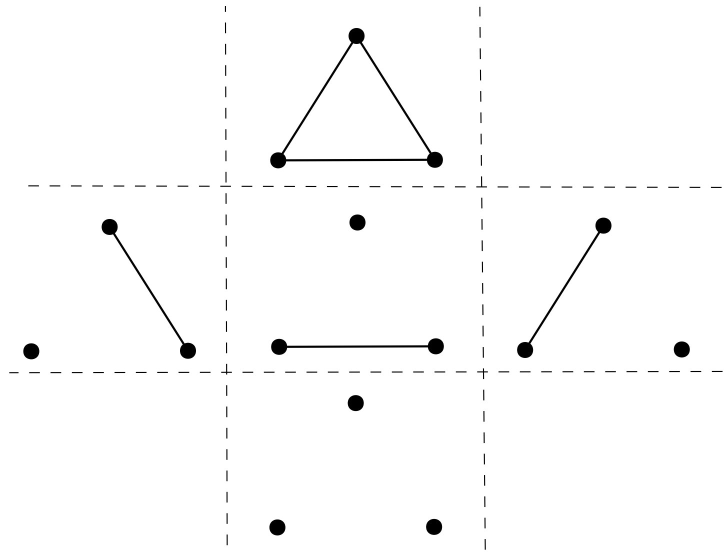


Figure 4: The set of basic figures for the graph in Figure 1.

Observe that the constant term of the determinant of the adjacency matrix $\mathbf{A}$ *for Figure 1 is* -2.

$$\begin{aligned} \det(x\mathbf{I} - \mathbf{A}) &= \det\begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} \\ &= (x)(x^2 - 1) + (-x + -1) - (1 + x) \\ &= x^3 - 3x - 2 \end{aligned}$$

From the set of basic figures in Figure 4, there is only one where no vertices are isolated and it is an elementary figure containing one circle. Thus Sachs' Theorem calculates the coefficient of x^0 *to be* $(-1)^1(2)^1 = -2$ *which is the constant term of the characteristic polynomial of the adjacency matrix of the graph.*

2.5 Bidirected and Signed Graphs

The definition of graph can be extended to bidirected graphs and signed graphs by adding signing functions to the incidences and edges respectively. A *signed graph* is a graph in which each edge is assigned a sign $\{+1, -1\}$. A *bidirected graph* is a graph in which each incidence is assigned a direction of entering $(+1)$ or exiting (-1) the vertex to which it is mapped. The relative orientation of the two arrows on an edge determines the edge sign. As seen in Figure 5, if the two arrows point to the same vertex, the edge is positive, and if the two arrows point to different vertices, the edge is negative.

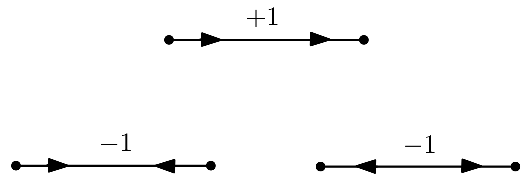


Figure 5: Different orientations of bidirected edges correspond to positively and negatively signed edges in a signed graph.

Thus, bidirected graphs can be regarded as incidence oriented signed graphs as studied in [7, 14, 15]. This correspondence between bidirections and edge signs was generalized to the sign of a weak walk in [10]. The *sign of a weak walk* W is

$$sgn(W) = (-1)^{\lfloor n/2 \rfloor} \prod_{h=1}^{n} \sigma(i_h),$$

This is equivalent to taking the product of the signed adjacencies if W is a vertex walk.

A graph can be converted to an oriented graph by placing directional arrows onto edges, and is used to build the traditional incidence matrix, as seen in Figure 6.

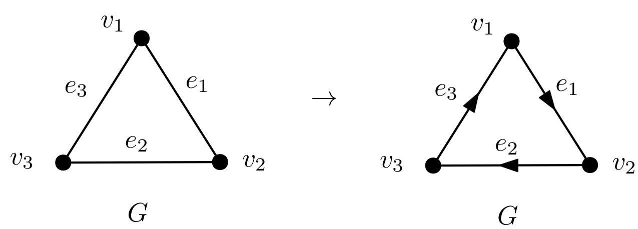


Figure 6: Graph with an orientation assigned to the edges.

While graphs can be converted to oriented graphs by assigning directions to edges so to can graphs be translated to signed graphs by assigning each edge a positive or negative. Comparing the graph in Figure 1 to the graph with positive or negatives assigned to each edge in Figure 7, it can be seen that the signing function is the only difference. Transitioning the edge orientations demonstrated in Figure 5 to the incidences, it can be seen that bidirected graphs are orientations of signed graphs.

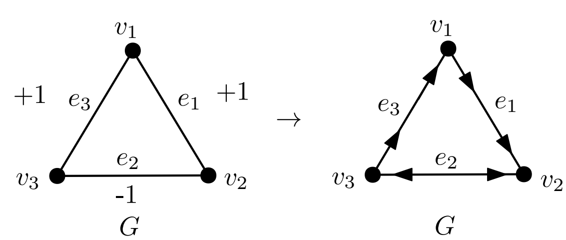


Figure 7: A Signed Graph can be represented as a Bidirected Graph.

The matrices that are used to to analyze graphs can also be used for bidirected graphs. The incidence matrix $\mathbf{H}_G$ has (v,e)-entries now calculated by the sum of the signs of all $i \in I$ where $\varsigma(i) = v$ and $\omega(i) = e$. Thus the sum of the columns may no longer be zero. For the bidirected graph in Figure 7, we have the following four matrices.

$$\mathbf{D}_G = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \mathbf{L}_G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad \mathbf{H}_G = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Observe the difference in the signs of some entries in the adjacency, Laplacian, and incidence matrices. This will result in changes in the values determined by the determinant, permanent and characteristic polynomials.

Example 2.5.1 *The characteristic polynomial of the Laplacian of the bidirected graph in Figure 7 is as follows.*

$$\det \begin{bmatrix} x-2 & -1 & 1 \\ -1 & x-2 & 1 \\ 1 & 1 & x-2 \end{bmatrix} = x^3 - 6x^2 + 9x - 4$$

The difference between this polynomial and the characteristic polynomial obtained in Example 2.2.5 is the constant term.

It is known that the constant term for the characteristic polynomials of graphs is zero. For signed graphs and bidirected graphs containing negative circles, the constant term is not zero as determined in [4, 11]. While in the previous example only the constant term has changed, it need not be the case in larger signed graph that only the constant term is changed.

2.6 Matrix-tree Theorem and Arborescences

The Matrix-tree Theorem calculates the number of spanning trees of a graph by examining the first minors of the Laplacian of a graph. A *root* of a tree of G is a vertex from which all paths are regarded as emanating from. An *k-arborescence* is a set of k disjoint, rooted trees of G whose union spans G. Observe that a 1-arborescence is a rooted spanning tree. The *tree number of* G is the number of spanning trees of G.

The $\mathbf{L}_{ij}$ minor of a $n \times n$ matrix is a $(n-1) \times (n-1)$ matrix obtained by removing row i and column j of the original matrix.

Example 2.6.1 *Consider the* $(1,2)$*-entry, indicated in bold, minor of the matrix* $\mathbf{L}$

$$\mathbf{L} = \begin{bmatrix} 3 & \mathbf{2} & -1 & 5 \\ -1 & 1 & -2 & -1 \\ -3 & 1 & 0 & 1 \\ 0 & 4 & -1 & 3 \end{bmatrix}, \quad \mathbf{L}_{12} = \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}.$$

The first row and second column of the matrix L are removed to obtain the L_{ij} minor.

Theorem 2.6.2 (Tutte's Matrix-Tree Theorem [13]) *If v is a vertex of a graph G, with Laplacian matrix $\mathbf{L}(G)$ then*

$$\det(\mathbf{L}_v(G)) = \sum_T \prod_{e \in E(T)} wt(e)$$

Where the sum is over all spanning trees T, rooted at v, and $wt(e)$ is the weight of edge e.

In [3], Seth Chaiken generalized the Matrix-tree Theorem to all minors for signed graphs.

Theorem 2.6.3 (Seth Chaiken's All Minors Matrix-tree Theorem [3]) *Let G be a signed graph with Laplacian matrix $\mathbf{L}$. For $U, W \subseteq V$ with $|U| = |W|$, let $\mathbf{L}_{U,W}$ be (U,W) minor of $\mathbf{L}$ then*

$$\det(\mathbf{L}_{U,W}) = \epsilon(\bar{U}, V)\epsilon(\bar{W}, V) \sum_F \epsilon(\pi^*)(-1)^{np(F)} 4^{nc(F)} a_F$$

Where the sum is over all edge sets F, subset of E, such that

1. *F contains $|U|$ components that are trees.*

2. *Each tree from 1 contains exactly one vertex from U and one vertex from W.*

3. *Each tree from 1 is rooted at its vertex in U and contains exactly one vertex of W. This defines a linking $\pi* : W \to U$. $\epsilon(\pi*)$ is negative one to the number of inversions of $\pi*$, and $np(F)$ is the number of negative paths in $\pi*$.*

4. *Each of the remaining components of F contains exclusively a backstep or exactly one negative circle. $nc(F)$ is the number of negative circles.*

5. $\epsilon(\bar{U}, V) = (-1)^{|\{(i,j)|i<j, i\in U, j\in\bar{U}\}|}$

Seth Chaiken's All Minor Matrix-tree Theorem for all positive edges and first minors is equivalent to Tutte's Matrix-tree Theorem with edge weights one. An example of this is Theorem 2.6.4.

Theorem 2.6.4 (Matrix-tree Theorem) *Let G be a connected graph with Laplacian $\mathbf{L}$ and ij-minor of $\mathbf{L}_{ij}$.*

$$\det(\mathbf{L}_{ij}) = (-1)^{i+j}T(G)$$

Example 2.6.5 *For the graph in Figure 1 consider the $\mathbf{L}_{11}$ minor.*

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \mathbf{L}_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The determinant of the $\mathbf{L}_{11}$ minor is $(2)(2) - (-1)(-1) = 4 - 1 = 3$. From Figure 8 we show all three spanning trees.

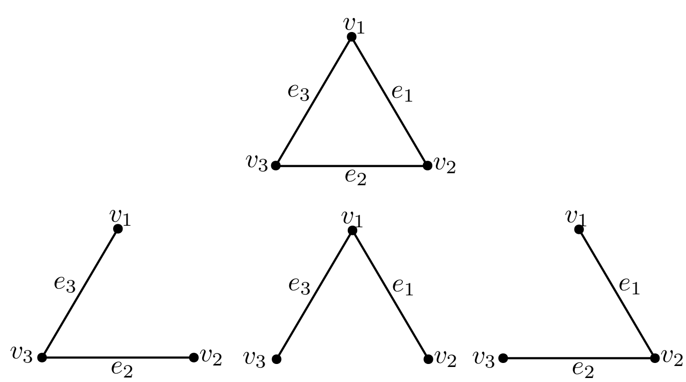


Figure 8: There are three spanning trees each with three possible roots.

The tree number of the graph is three and $i + j = 1 + 1 = 2$ and $(-1)^{i+j}T(G) = (-1)^2(3) = 3$. So by the Matrix-tree Theorem, the determinant of the $\mathbf{L}_{11}$ minor is equal to the tree number with the appropriate sign adjustment.

In Example 2.2.5 it can be seen that the coefficient of the x^1 term is equal to the number of spanning trees for each root. Thus there are three trees and three possible roots giving the nine total possibilities. In Section 3 and Section 4, this paper will look further into arborescence and spanning trees and their relationship to contributors and the total minor polynomials.

3 Incidence Hypergraphs

Graphs, signed graphs, and bidirected graphs are a category of graphs that do not allow for hyperedges. Theorems from graph theory can be extended to hypergraph theory through the use of the incidence structure. In Section 3.1 we will discuss the basics of incidence hypergraphs and oriented hypergraphs. In Section 3.2 we will examine known results of graph theory that have been extended to hypergraphs and in Section 3.3 we will establish the new result of the total minor polynomials.

3.1 Introduction and Background

We adapt the definition of an incidence hypergraph from [8] to examine oriented hypergraphs. An *incidence hypergraph* G is a tuple $G = (V, E, I, \varsigma, \omega)$ where V, E and I are disjoint, finite sets of vertices, edges, incidences respectively, $\varsigma : I \to V$, and $\omega : I \to E$. The concepts of graph theory are easily extended through their locally graphic incidence structure. Hypergraphs can be represented by matrices similar to those used for graphs, signed graphs, and bidirected graphs. These matrices can be analyzed using the same techniques discussed in the previous Section 2.2, and we demonstrate a generalization of Seth Chaiken's All Minors Matrix-tree Theorem via an extension of the characteristic polynomial [2, 3, 4, 6].

As discussed in Section 2.2, the determinants and permanents of a matrix are calculated using permutation. We introduce a refinement of the concept of a permutation via an embedding of path maps into a given incidence hypergraph G. A *contributor of* G is an incidence preserving map from a disjoint union of $\overrightarrow{P}_1$'s into G defined by $c : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ such that $c(t_v) = v$ and $\{c(h_v) \mid v \in V\} = V$. Let $\mathcal{C}(G)$ denote the set of contributors. A *strong contributor of* G is a incidence-monic contributor. Observe strong contributors are orientations of basic figures. Let $\mathcal{S}(G)$ denote the set of strong contributors. Let $U, W \subseteq V$, then a (U, W)*-restricted contributor of* G is an incidence preserving map from a disjoint union of $\overrightarrow{P}_1$'s into G defined by $c : \coprod_{u \in U} \overrightarrow{P}_1 \to G$ such that $c(t_u) = u$ and $\{c(h_u) \mid u \in U\} = W$. Observe that each contributor that only contains backsteps corresponds to the identity permutation.

Example 3.1.1 *Consider the graph in Figure 1. There are sixteen different contributors of the graph.*

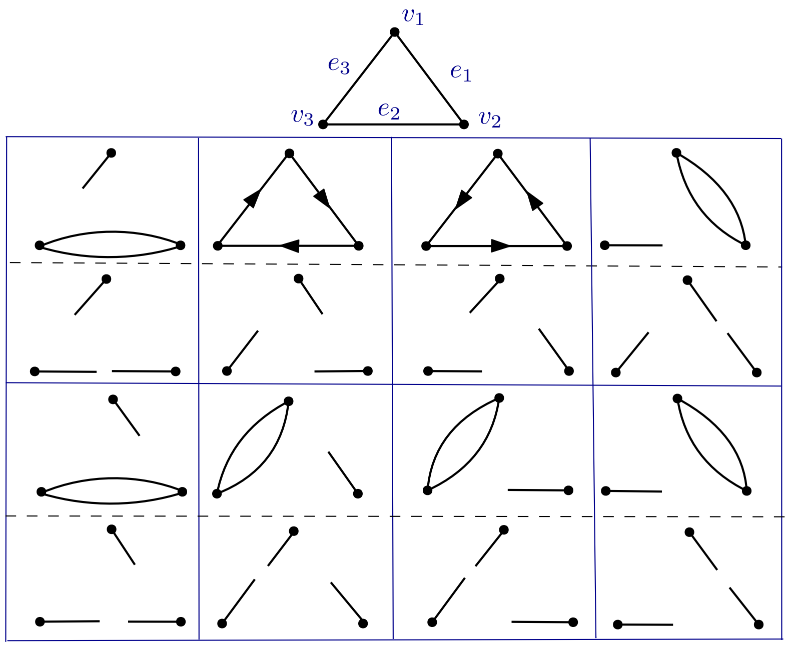


Figure 9: The contributors of a three edge, three vertex graph.

There are two strong contributors of Figure 1 and eight identity contributors.

Given an incidence hypergraph $G = (V, E, I, \varsigma, \omega)$ define the *loading of* G, denoted $L(G)$, as $(V, E, I \cup I_0, \varsigma_L, \omega_L)$ where I_0 is a set of new incidences of the form (v, e) if $\varsigma^{-1}(v) \cap \omega^{-1}(e) = \varnothing$, $\varsigma_L\big|_I = \varsigma$ and $\varsigma_L\big|_{I_0} : (v, e) \mapsto v$, and $\omega_L\big|_I = \omega$ and $\omega_L\big|_{I_0} : (v, e) \mapsto e$.

Example 3.1.2 *Given the graph in Figure 1, the loading of the graph is shown in Figure 10.*

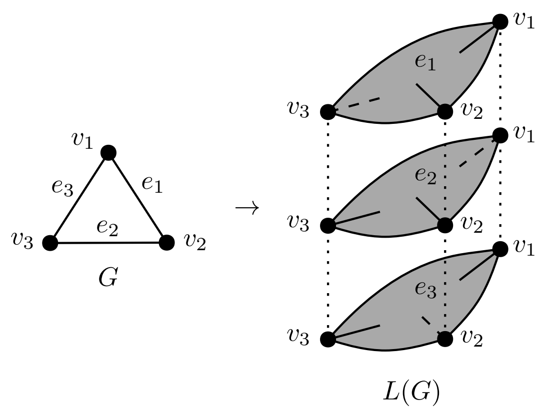


Figure 10: The incidence loading of K_3 to produce a uniform hypergraph.

New incidences appear dashed within each hyperedge, and the vertices are identified along the dashed vertical lines.

3.1.1 Incidence Orientations

Let $G = (V, E, I, \varsigma, \omega)$ be an incidence hypergraph. An *orientation of an incidence hypergraph* G is a signing function $\sigma : I \to \{+1, -1\}$. The *sign of a weak walk* W is

$$sgn(W) = (-1)^{\lfloor n/2 \rfloor} \prod_{h=1}^{n} \sigma(i_h),$$

which is equivalent to taking the product of the signed adjacencies if W is a vertex walk; see [14, 15, 7] for bidirected graphs as orientations of signed graphs. An oriented hypergraph in which there are exactly 2 incidences per edge can be represented as a *bidirected graph* as shown in [7],[14],[15].

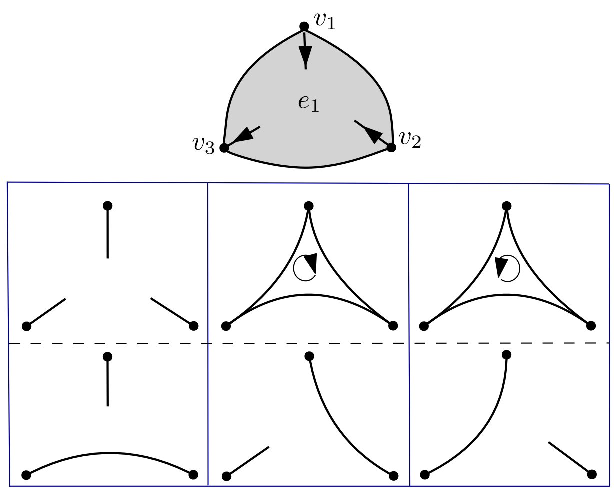


Figure 11: An orientation of a hypergraph consisting of one edge and three vertices and its six contributors.

Consider Figure 11. Like graphs, hypergraphs can be analyzed by examining their contributors. There are only six contributors and one identity contributor for this graph as seen in Figure 11.

3.1.2 Matrices

Just as graphs can be represented by integer matrices, so too can hypergraphs. The same four matrices, the adjacency matrix, Laplacian matrix, incidence matrix, and degree matrix, are created the same way for hypergraphs as they are for bidirected graphs. The sign of adjacencies in hypergraphs can be viewed by examining the direction of two incidences along an edge the same way signs of edges are represented on bidirected graphs.

The matrices associated with Figure 11 are shown below. Each vertex has degree one because there is one incidence mapped to each vertex. All adjacencies are signed positive or negative, as are all incidences.

$$\mathbf{D}_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_G = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{L}_G = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{H}_G = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Determinant and permanent operations performed on these matrices yield information about the hypergraphs associated with them.

3.2 Known Results

There has been prior work done by a number of mathematicians on this topic. Tutte formalized the Matrix-tree Theorem where determinants are calculated through spanning trees [13] and Sachs' used basic figures to characterize the characteristic polynomial of the adjacency matrix for graphs [6]. Zaslavsky generalized the concept of algebraic graph theoretic matrices to signed graphs in [16], which was further generalized to hypergraphs by Reff and Rusnak in [10]. Belardo and Simic generalized Sachs' Theorem to signed graphs in [2]. Rusnak et. al. in [4] generalizes [2] to provide a unified interpretation of integer matrix Laplacian minors and extended Sachs' Theorem to oriented hypergraphs. In [11] Rusnak et. al. streamlines the results of the previous matrix tree type theorems by specializing the techniques from [4] to signed graphs. The generalizations of the Matrix-tree Theorem by [4, 11] are stated explicitly in Theorem 3.2.1.

Theorem 3.2.1 *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

1. $\operatorname{perm}(\mathbf{L}_G) = \sum_{c \in \mathcal{C}_{\geq 0}(G)} (-1)^{oc(c)+nc(c)}$,
2. $\det(\mathbf{L}_G) = \sum_{c \in \mathcal{C}_{\geq 0}(G)} (-1)^{pc(c)}$,
3. $\operatorname{perm}(\mathbf{A}_G) = \sum_{c \in \mathcal{C}_{=0}(G)} (-1)^{nc(c)}$,
4. $\det(\mathbf{A}_G) = \sum_{c \in \mathcal{C}_{=0}(G)} (-1)^{ec(c)+nc(c)}$.

Calculating the minors from Theorem 3.2.1 a generalization of Sachs' Theorem in Theorem 3.2.2 was done by [4].

Theorem 3.2.2 *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

1. $\chi^P(\mathbf{A}_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \hat{\mathcal{C}}_{=k}(G)} (-1)^{oc(c)+nc(c)} \right) x^k,$

2. $\chi^D(\mathbf{A}_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \hat{\mathcal{C}}_{=k}(G)} (-1)^{pc(c)} \right) x^k,$

3. $\chi^P(\mathbf{L}_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \hat{\mathcal{C}}_{\geq k}(G)} (-1)^{nc(c)+bs(c)} \right) x^k,$

4. $\chi^D(\mathbf{L}_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \hat{\mathcal{C}}_{\geq k}(G)} (-1)^{ec(c)+nc(c)+bs(c)} \right) x^k.$

For the graph in Figure 1 the characteristic polynomial can be calculated by this theorem and will yield the polynomial in Example 2.2.5. To further distinguish the x's in the polynomial they can be given subscripts.

Example 3.2.3 *Given the graph in Figure 1 the characteristic polynomial can be calculated with x's distinguished by subscript along the diagonal.*

$$
\begin{aligned}
\det \begin{bmatrix} x_{11}-2 & 1 & 1 \\ 1 & x_{22}-2 & 1 \\ 1 & 1 & x_{33}-2 \end{bmatrix} &= x_{11}x_{22}x_{33} \\
&- 2x_{11}x_{22} - 2x_{11}x_{33} - 2x_{22}x_{33} \\
&+ 3x_{11} + 3x_{22} + 3x_{33}
\end{aligned}
$$

The nine that makes up the coefficient of x^1 in 2.2.5 can now be seen to be made up of three groups of three. In Figure 8 we see that there are three possible spanning trees each with three possible roots and this breakup of the nine into three groups of three illustrates this fact. The total minor polynomials yield further breakup of coefficients of characteristic polynomials.

3.3 Total Minor Polynomials and Zero Loading

In order to further extend the known results and examine how the coefficients of the characteristic polynomial relate to contributor, the characteristic polynomial can be extended to the total minor polynomial. The *total minor polynomial* is the polynomial obtained by calculating $det(\mathbf{X}-\mathbf{M})$ where $\mathbf{X}$ is a $V \times V$ matrix such that each entry is x_{ij} where ij is the column, row location in the matrix and $\mathbf{M}$ is either the adjacency matrix $\mathbf{A}$ or the Laplacian matrix $\mathbf{L}$. Let $\chi^D(\mathbf{M},\mathbf{x}) := \det(\mathbf{X}-\mathbf{M})$ be the determinant-based multivariable characteristic polynomial and $\chi^P(\mathbf{M},\mathbf{x}) := \operatorname{perm}(\mathbf{X}-\mathbf{M})$ be the permanent-based multivariable characteristic polynomial.

Example 3.3.1 *Consider the graph in Figure 11. Then the determinant of its Laplacian subtracted from* $\mathbf{X}$ *yields the following polynomial.*

$$\begin{aligned}
\det(\mathbf{X}-\mathbf{L}) &= \det\begin{bmatrix} x_{11}-1 & x_{12}-1 & x_{13}+1 \\ x_{21}-1 & x_{22}-1 & x_{23}+1 \\ x_{31}+1 & x_{32}+1 & x_{33}-1 \end{bmatrix} \\
&= x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{13}x_{22}x_{31} - x_{12}x_{21}x_{33} \\
&+ x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} \\
&- x_{11}x_{22} - x_{11}x_{23} - x_{11}x_{32} - x_{11}x_{33} - x_{13}x_{22} - x_{22}x_{31} \\
&- x_{22}x_{33} + x_{12}x_{21} + x_{13}x_{21} + x_{12}x_{23} + x_{12}x_{31} + x_{13}x_{31} \\
&- x_{23}x_{31} - x_{13}x_{32} + x_{21}x_{32} + x_{23}x_{32} \\
&+ x_{12}x_{33} + x_{12}x_{33} \\
&+ 0x_{11} + 0x_{22} + 0x_{33}
\end{aligned}$$

For clarity only the x_{ij} terms along the diagonal are listed as all the single x terms have a coefficient of zero.

The *zero loading* of a oriented hypergraph G is the loading of G where the incidence signing function assigns all incidences in I_0 zero. That is $\varsigma_{\mathbf{L}}(I) = \varsigma$ and $\varsigma_{\mathbf{L}}(I_0) = 0$. The zero loading of Figure 1 would be the loading shown in Figure 10 with the dashed incidences assigned a value of zero.

Consider χ^P to be the total minor polynomial taken with a permanent and χ^D to be the total minor polynomial obtained by taking the determinant. Let oc, nc, ec, and bs represent the number of odd circles, negative components, even circles, and backsteps respectively. Let $\mathcal{S}$ be the set of contributors in the loading of the graph and s a member of the set and s' a subcontributor of s.

Theorem 3.3.2 *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

1. $\chi^P(\mathbf{A}_G,\mathbf{x}) = \sum_{s\in\mathcal{S}(L^0(G))} \sum_{\substack{s'\subseteq s \\ sgn(s')\neq 0}} \left((-1)^{oc(s')+nc(s')} \prod_{u\in T(\bar{s}')} x_{u,s(u)} \right),$

2. $\chi^D(\mathbf{A}_G,\mathbf{x}) = \sum_{s\in\mathcal{S}(L^0(G))} \sum_{\substack{s'\subseteq s\\ sgn(s')\neq 0}} \left((-1)^{ec(s)+oc(s')+nc(s')} \prod_{u\in T(\bar{s}')} x_{u,s(u)}\right),$

3. $\chi^P(\mathbf{L}_G,\mathbf{x}) = \sum_{c\in\mathcal{C}(L^0(G))} \sum_{\substack{c'\subseteq c\\ sgn(c')\neq 0}} \left((-1)^{nc(c')+bs(c')} \prod_{u\in T(\bar{c}')} x_{u,c(u)}\right),$

4. $\chi^D(\mathbf{L}_G,\mathbf{x}) = \sum_{c\in\mathcal{C}(L^0(G))} \sum_{\substack{c'\subseteq c\\ sgn(c')\neq 0}} \left((-1)^{ec(c)+nc(c')+bs(c')} \prod_{u\in T(\bar{c}')} x_{u,c(u)}\right),$

Proof. The first half of the proof is an adaptation of [4], then utilizes the injective closure of the incidence hypergraph.

Let $p : \overrightarrow{P}_1 \to G$, and let q denote and incidence-monic maps from $\overrightarrow{P}_1 \to G$. For a given permutation $\pi \in S_V$, let $\mathcal{P}_\pi = \{p \mid p(t) = v \text{ and } p(h) = \pi(v)\}$, and $\mathcal{Q}_\pi$ be defined similarly for incidence-monic maps.

Proof of 1. For a given permutation π and vertex v let $\alpha : v \to \left\{x_{v,\pi(v)}, -\sum_{q\in\mathcal{Q}_\pi} sgn(q(\overrightarrow{P}_1))\right\}$ be the function that chooses either the variable or the value at coordinate $(v,\pi(v))$. Let $\mathcal{A}_\pi$ be the set of all α for a given π.

Thus, $\chi^P(\mathbf{A}_G,\mathbf{x})$ can be written as

$$\begin{aligned}\chi^P(\mathbf{A}_G,\mathbf{x}) &= \operatorname{perm}(\mathbf{X}-\mathbf{A}_G)\\ &= \sum_{\pi\in S_V}\prod_{v\in V}\sum_{\alpha\in\mathcal{A}_\pi}\alpha(v).\end{aligned}$$

Distributing we get

$$= \sum_{\pi\in S_V}\sum_{\beta\in\mathcal{B}_\pi}\prod_{v\in V}\beta(v),$$

where $\mathcal{B}_\pi$ is the set of all functions $\beta : V \to \left\{x_{v,\pi(v)}, -\sum_{q\in\mathcal{Q}_\pi} sgn(q(\overrightarrow{P}_1))\right\}$.

Now partition $\mathcal{B}_\pi$ into $\bigcup_{k=0}^{|V|}\mathcal{B}_{k,\pi}$, where $\mathcal{B}_{k,\pi}$ is the set of all $\beta\in\mathcal{B}_\pi$ with exactly k variables in its image. For each $\beta\in\mathcal{B}_{k,\pi}$ let $U_\beta\subseteq V$ be the set of vertices mapped to an $x_{v,\pi(v)}$. This gives:

$$= \sum_{\pi\in S_V}\sum_{k=0}^{|V|}\sum_{\beta\in\mathcal{B}_{k,\pi}}\left[\left(\prod_{u\in\overline{U}_\beta}\beta(v)\right)\prod_{u\in U_\beta}x_{u,\pi(u)}\right].$$

Evaluating $\beta(v)$ we have:

$$= \sum_{\pi\in S_V}\sum_{k=0}^{|V|}\sum_{\beta\in\mathcal{B}_{k,\pi}}\left[\left(\prod_{u\in\overline{U}_\beta}\sum_{q\in\mathcal{Q}_\pi(G|\overline{U}_\beta)} -sgn(q(\overrightarrow{P}_1))\right)\prod_{u\in U_\beta}x_{u,\pi(u)}\right].$$

Where $\mathcal{Q}_\pi(G|\overline{U}_\beta)$ is the set of maps q whose tail-set is $\overline{U}_\beta$ and head-set is $\pi(\overline{U}_\beta)$. Distributing produces:

$$= \sum_{\pi \in S_V} \sum_{k=0}^{|V|} \sum_{\substack{U \subseteq V \\ |U|=k}} \sum_{s \in \mathcal{S}_\pi(G|\overline{U}))} \left[\left(\prod_{u \in \overline{U}} \sigma(s(i_v)) \sigma(s((j_v)) \right) \prod_{u \in U} x_{u,\pi(u)} \right],$$

where $\mathcal{S}_\pi(G|\overline{U})$ is the restricted set of strong contributors that correspond to permutation π with tails at $\overline{U}$. Now pass to the injective envelope of the underlying incidence hypergraph and extend the incidence orientation function σ to σ_L such that $\sigma_L(i) = \sigma(i)$ for all $i \in I(G)$ and the new incidence orientations are assigned arbitrary. Using the G-subobject indicator δ_G the sum can be rewritten as:

$$= \sum_{\pi \in S_V} \sum_{k=0}^{|V|} \sum_{\substack{U \subseteq V \\ |U|=k}} \sum_{s \in \mathcal{S}_\pi(L(G))} \left[\delta_G(s|\overline{U}) \left(\prod_{u \in \overline{U}} \sigma_L(s(i_v)) \sigma_L(s((j_v)) \right) \prod_{u \in U} x_{u,\pi(u)} \right].$$

The product of signs is evaluated by first factoring out a negative for each adjacency producing a value of $(-1)^{oc(s)}$, and then factoring out a negative for each negative adjacency producing a value of $(-1)^{nc(s)}$ — leaving behind only +1's for all adjacencies, and reducing to a count of subcontributors of the underlying incidence hypergraph.

$$= \sum_{\pi \in S_V} \sum_{k=0}^{|V|} \sum_{\substack{U \subseteq V \\ |U|=k}} \sum_{s \in \mathcal{S}_\pi(L(G))} \left[\delta_G(s|\overline{U}) \cdot (-1)^{oc(s)+nc(s)} \prod_{u \in U} x_{u,\pi(u)} \right].$$

Combining the subset sums and reordering we get:

$$= \sum_{U \subseteq V} \sum_{\pi \in S_V} \sum_{s \in \mathcal{S}_\pi(L(G))} \left[\delta_G(s|\overline{U}) \cdot (-1)^{oc(s)+nc(s)} \prod_{u \in U} x_{u,\pi(u)} \right].$$

Resolving δ_G we pass to the 0-loading $L^0(G)$ of the oriented hypergraph.

$$= \sum_{s \in \mathcal{S}(L^0(G))} \sum_{\substack{s' \subseteq s \\ sgn(s') \neq 0}} \left[(-1)^{oc(s')+nc(s')} \prod_{u \in T(\bar{s}')} x_{u,s(u)} \right].$$

Where $T(\bar{s}') = T(s) \smallsetminus T(s')$ is the set of tail-vertices of s not in s'. Completing the proof of part 1.

Proof of 2. Proceeding as in part 1 with the inclusion of the sign of the permutation we get

$$\chi^D(\mathbf{A}_G, \mathbf{x}) \quad = \quad \det(\mathbf{X} - \mathbf{A}_G)$$

$$= \sum_{U \subseteq V} \sum_{\pi \in S_V} \epsilon(\pi) \sum_{s \in \mathcal{S}_\pi(L(G))} \left[\delta_G(s|\overline{U}) \cdot (-1)^{oc(s)+nc(s)} \prod_{u \in U} x_{u,\pi(u)} \right].$$

Using the facts that the sign of a permutation is equal to $(-1)^{ec(\pi)}$, where $ec(\pi)$ is the number of even algebraic cycles in π, and each contributor is associated to a unique permutation we have

$$= \sum_{U \subseteq V} \sum_{\pi \in S_V} \sum_{s \in \mathcal{S}_\pi(L(G))} (-1)^{ec(s)} \cdot \left[\delta_G(s|\overline{U}) \cdot (-1)^{oc(s)+nc(s)} \prod_{u \in U} x_{u,\pi(u)} \right].$$

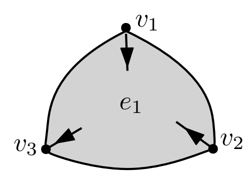
Again, resolving δ_G, but this time observing that the value $(-1)^{ec(s)}$ is unchanged as they are determined by algebraic cycles yields

$$= \sum_{s \in \mathcal{S}(L^0(G))} \sum_{\substack{s' \subseteq s \\ sgn(s') \neq 0}} \left[(-1)^{ec(s)+oc(s')+nc(s')} \prod_{u \in T(\bar{s}')} x_{u,s(u)} \right].$$

Proofs of 3. and 4. The proofs for the Laplacian are similar with the following modifications: (1) switch from incidence-monic maps $\mathcal{Q}_\pi$ to arbitrary maps $\mathcal{P}_\pi$ to allow backsteps and sum over contributors instead of strong contributors; (2) since $\mathbf{L}_G = \mathbf{D}_G - \mathbf{A}_G$ there is no need to factor out a -1 for each adjacency, and instead factor out a -1 for each backstep. ■

This theorem can be used to calculate the coefficient of any term of the total minor polynomials given a graph.

Example 3.3.3 *The graph in Figure 11 has the following set of contributors.*



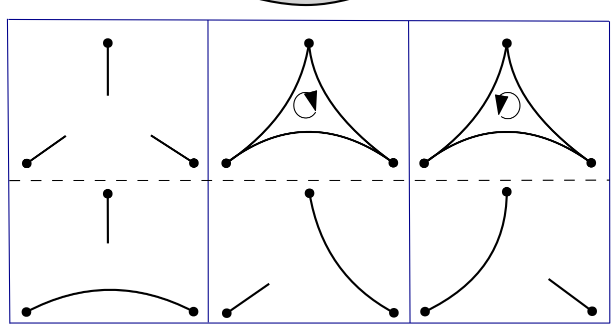


Figure 12: There are six contributors for the hypergraph in Figure 11.

To find the coefficient of x_{11} examine the two contributors with the backstep on vertex one. Once this backstep is removed the sign of the contributor can be calculated. The contributor with all *backsteps shows up as a positive contributor, since it is negative one squared, and the contributor without any backsteps, once the backstep on vertex one is removed, has an even circle and will show up as negative in the calculation. The alternating signs on the contributors account for why in 3.3.1 the single x terms are zero.*

While the contributors for all the single x terms cancel in the hypergraph example this is not always the case. To demonstrate this, consider the graph in Figure 1 and the coefficient of three on the x_{11} term as seen in Example 3.2.3. The number of spanning trees rooted at vertex one is also three. So the sum of the contributors with x_{11} should be three.

Example 3.3.4 *Consider the subset of contributors of Figure 1 where there exists a backstep on vertex one. Only contributors that contain a backstep on vertex one will be counted in the calculation of the coefficient of the x_{11} term and thus all other contributors are irrelevant to the calculation.*

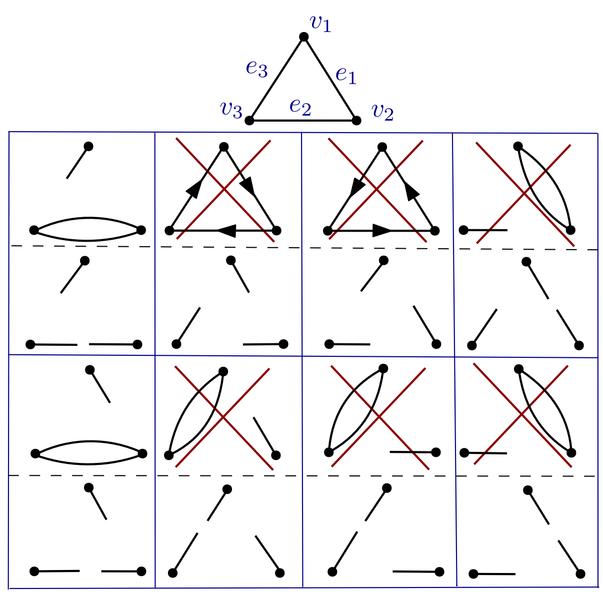


Figure 13: Of the sixteen contributors of Figure 1 there are ten with a backstep on vertex one.

This set of contributors can be grouped into five pairs of contributors with the backstep on vertex one deleted. Of the ten contributors that exist once the backstep on vertex one is deleted, six of them can be group in three pairs of two identical contributors. These pairs of identical contributors only count once in the calculation. This indicates that the maximum count for the contributors is now seven. The five remaining contributors with two backsteps contribute a positive one to the calculation and the two contributors with even cycles contribute a negative one yielding the expected result of three.

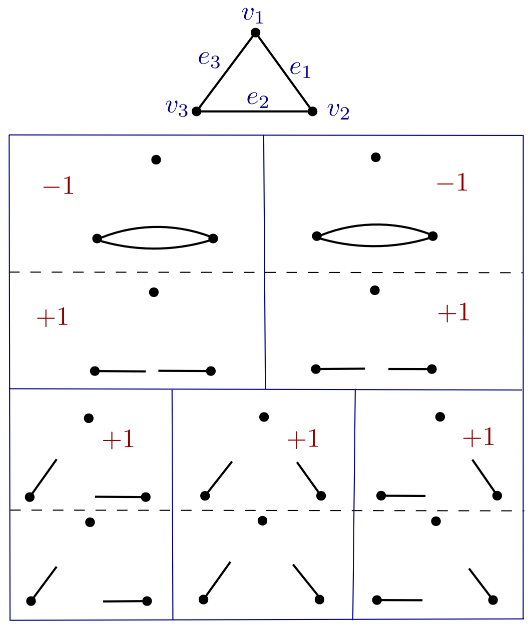


Figure 14: Each of the seven remaining contributors is assigned a positive or negative one

The total minor polynomial enables the calculation of coefficients of any term through an examination of the contributors containing the path maps indicated by the subscripts of the x terms.

Example 3.3.5 *Consider the two strong contributors of Figure 1. Given these two contributors, the produced sets of subcontributors differ only in orientation of the edges as shown in Figure 15.*

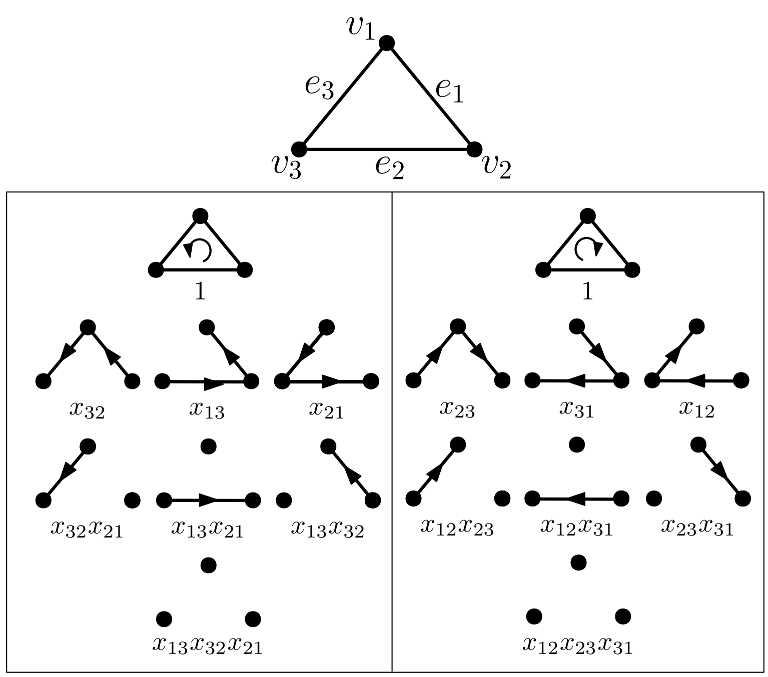


Figure 15: A sub-contributor of a strong contributor can be counted in multiple expressions.

Subcontributors of a graph will produce monomials that are counted in multiple terms of the total minor polynomials. Observe that the contributor labeled $x_{13}x_{32}x_{21}$ and $x_{12}x_{23}x_{31}$ in Figure 15 are identical but will be counted in both terms.

4 Applications and Future Work

The higher order minors of the Matrix-tree Theorem for graphs have been shown to correspond to specific collections of k-arborescence, moreover, this is included in property 1 of Theorem 2.6.3. Subsets of second minors of graphs have been used to reclaim Kirchhoff's Laws by providing an edge labeling of the graph. In this section we will discuss a generalization of the Matrix-tree Theorem to k-arborescences by specializing Theorem 3.3.2 to graphs. Future work will include linking this to Seth Chaiken's signed graphic families and generalizations of transpedances to have Kirchhoff-like laws.

We build on the work of [11] and include the following definitions. A *pre-contributor of G* is an incidence preserving function $p : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ with $p(t_v) = v$, meaning that the disjoint union of $|V|$ copies of $\overrightarrow{P}_1$ into G such that every tail-vertex labeled by v is mapped to v. For a pre-contributor p with $p(t_v) \neq p(h_v)$ for vertex $v \in V$, define *packing a directed adjacency of a pre-contributor p into a backstep at vertex v* to be a pre-contributor p_v such that $p_v = p$ for all $u \in V \smallsetminus v$, and for vertex v

$$\begin{aligned} p((\overrightarrow{P}_1)_v) &= (v, i, e, j, w),\ i \neq j, \\ \text{and } p_v((\overrightarrow{P}_1)_v) &= (v, i, e, i, v). \end{aligned}$$

Thus, the head-incidence and head-vertex of adjacency $p((\overrightarrow{P}_1)_v)$ are identified to the tail-incidence and tail-vertex. *Unpacking a backstep of a pre-contributor p into an adjacency out of vertex v* is a pre-contributor p^v is defined analogously but for vertex v, the head-incidence and head-vertex of backstep $p((\overrightarrow{P}_1)_v)$ are identified to the incidence and vertex that would complete the adjacency in bidirected graph G. *Activating a circle of contributor c* is a minimal sequence of unpackings that results in a new contributor, and define the *activation partial order $\leq_a$* where $c \leq_a d$ if d is formed by a sequence of activations starting with c. This induces the *activation equivalence relation $\sim_a$* where $c \sim_a d$ if $c \leq_a d$ or $d \leq_a c$, and the elements of $\mathcal{C}(G)/\sim_a$ are called the *activation classes of G*.

Lemma 4.0.1 ([11]) *For a bidirected graph G, all activation classes of G are Boolean lattices.*

Let $U, W \subseteq V$ such that $|U| = |W|$, and let $\mathbf{u}, \mathbf{w}$ be a vector representing a total ordering of the elements of U and W, respectively. Two contributors c and d are said to be $(\mathbf{u}, \mathbf{w})$-*equivalent*, denoted $c \sim_{\mathbf{uw}} d$, if $c(h_{u_i}) = d(h_{u_i}) = w_i$, where u_i and w_i are the i^{th} coordinate of $\mathbf{u}$ and $\mathbf{w}$, respectively. Using the notation established in [11], let $\mathcal{A}(\mathbf{u};\mathbf{w};G)$ denote the $(\mathbf{u},\mathbf{w})$-equivalent elements in activation class $\mathcal{A}$.

Lemma 4.0.2 ([11]) *The elements of $\mathcal{A}(\mathbf{u};\mathbf{w};G)$ form a sub-Boolean lattice of $\mathcal{A}$ determined by sequential order ideals.*

Let $\hat{\mathcal{A}}(\mathbf{u};\mathbf{w};G)$ be the elements of $\mathcal{A}(\mathbf{u};\mathbf{w};G)$ with the adjacency or backstep from u_i to w_i is removed for each i.

Lemma 4.0.3 ([11]) *If G is a bidirected graph, then the set of elements in all single-element $\hat{\mathcal{A}}_{\neq 0}(u;w;G')$ is activation equivalent to the set of spanning trees of G.*

The total minor polynomials can be used to extend the results of Lemma 4.0.3.

Theorem 4.0.4 *In a bidirected graph G the set of all elements in a single-element $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ is activation equivalent to k-arborescences.*

Proof. Let $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ contain a single element contributor, call it c. If c contains a circle, then there would be a $(\mathbf{u},\mathbf{w})$-equivalent contributor d with $d <_a c$ such that there is a sequence of unpackings that activates into c, and $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ would contain more than one element. Moreover, c cannot have any circle that can be activated, or there would be $(\mathbf{u},\mathbf{w})$-equivalent contributor d' with $c <_a d'$, and $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ would contain more than one element.

Additionally, since the single-element of $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ is a non-zero contributor in $L(G)$, the corresponding totally unpacked pre-contributor p exists in G. Thus, p is circle-free with exactly $|V|$ vertices and $|V| - k$ edges, so it is a k-arborescence. ∎

Example 4.0.5 *Consider the three contributors in Figure 14 that are counted counted in the coefficient of x_{11}.*

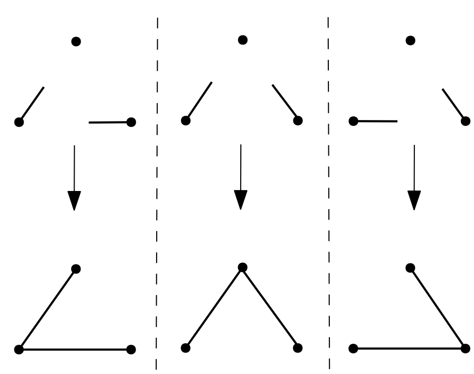


Figure 16: The contributors that contribute to the coefficient of x_{11} unpack into spanning trees.

These three contributors are counted in the x_{11} coefficient since they are contributors where there existed a backstep on vertex one before it was removed and once unpacked they contain no cycles are thus will not cancel with any other contributors. These contributors unpack into the three spanning trees of Figure 1. The spanning trees of a graph are 1-arborescences and thus the monomials of the total minor polynomial provide an arborescence count.

While first minors unpack into spanning trees, kth minors create k-arborescences. We continue using the graph in Figure 1 to demonstrate this. Consider the contributors in Figure 9 that contain a backstep on vertex one and an adjacency between vertex two and vertex three. There are two such contributors as seen in Figure 17, and the contributor obtained from a second minor, specifically the $x_{11}x_{23}$ minor, yields a 2-arborescence.

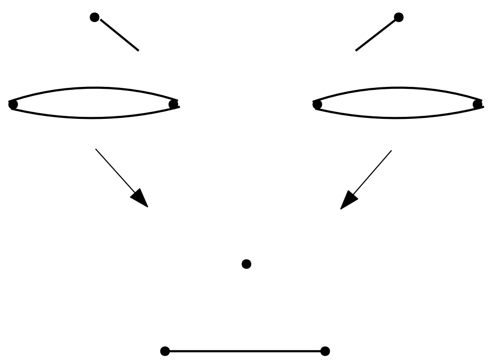


Figure 17: The two contributors may form a common sub-contributor once the given x's are chosen.

Example 4.0.6 *Consider the set of contributor of the graph with four vertices and five edges in Figure 2. Consider the contributor for the permutation* $(12)(34)$ *and its Boolean activation class as seen in Figure 18 where we examine the contributor counted in the coefficient of* $x_{11}x_{34}$.

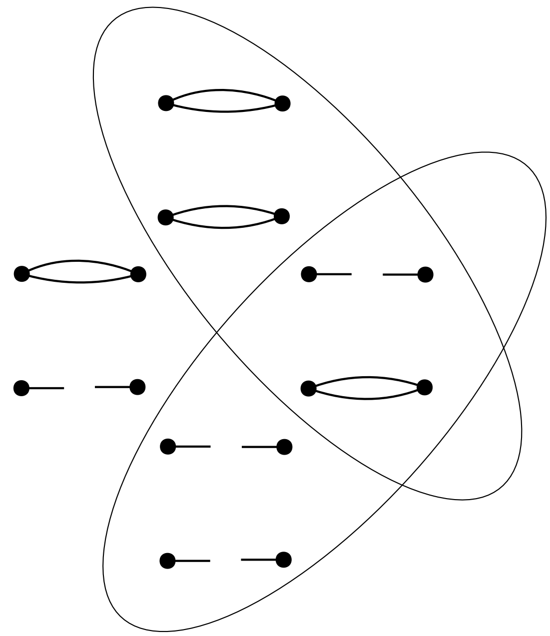


Figure 18: The two cuts on the Boolean lattice yield a single contributor.

The cuts on the Boolean lattice of contributors yields a single contributor that once unpacked will create a 2-arborescence. Note that the second minor terms are two cuts on lattices and will thus yield two arborescences. Furthermore k-minors result in k cuts and k-arborescences in the same manner.

Example 4.0.5, Figure 18 and Example 4.0.6 show that the monomials of the total minor polynomials contain arbor counts. A degree one monomial will be a count of 1-arborescences or spanning trees, while a degree two monomial will provide a 2-arborescence count. Thus a degree k monomial provides a count of k-arborescences.

Future work includes examining the rooted nature of k-arborescences and how they relate to Seth Chaiken's Matrix-Tree Theorem. The nature of unpacking contributors into arborescences creates a relationship between the deleted edges and the roots of the created trees. Future work will also include investigating generalizations of Kirchhoff's Laws via Tutte's transpedance decomposition. A *transpedance* of a graph G, denoted $[st, ab]$ where $s, t, a, b \in V$ is equal to the coefficients of $x_{sa}x_{tb}$ and $x_{sb}x_{ta}$. Observe that Tutte's definition of transpedances are 2-arborescences that correspond to the coefficients of the total minor polynomials and, moreover are consistent with part one of Seth Chaiken's Matrix-tree Theorem. A signed graphic generalization of transpedances should provide non-conservative laws that could be used in social network modeling. Furthermore the total minor polynomial enables us to investigate higher order transpedance-like objects.

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