Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 53, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Positive solutions of singular elliptic equations outside the unit disk *

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Abstract

We study the existence and the asymptotic behaviour of positive solutions for the nonlinear singular elliptic equation $\Delta u + \varphi(., u) = 0$ in the outside of the unit disk in $\mathbb{R}^2$, with homogeneous Dirichlet boundary condition. The aim is to prove some existence results for the above equation in a general setting by using a potential theory approach.

1 Introduction

The singular semi-linear elliptic equation

$$\Delta u + q(x)u^{-\gamma} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad \gamma > 0, \tag{1.1}$$

has been extensively studied for both bounded and unbounded domain Ω (see for example [4, 5, 6, 7] and the references therein).

For $0 < \gamma < 1$, Edelson [4] proved the existence of an entire positive solution $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^2)$ of (1.1), having logarithmic growth as $|x| \to \infty$, provided that $q \in C^{\alpha}_{\rm loc}(\mathbb{R}^2)$, $0 < \alpha < 1$, $q(x) > 0$ for $|x| > 0$ and

$$\int_e^\infty t(Logt)^{-\gamma}(\max_{|x|=t} q(x))dt < \infty.$$

Lazer and Mckenna [7] considered (1.1) in the case where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary. They proved the existence and the uniqueness of a positive solution $u \in C^{2+\alpha}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ with homogeneous Dirichlet boundary condition, provided that $q \in C^{\alpha}(\overline{\Omega})$ and $q(x) > 0$ for all $x \in \overline{\Omega}$.

Kusano and Swanson [5] considered the generalized equation

$$\Delta u + f(x, u) = 0, \; x \in \Omega, \tag{1.2}$$

where Ω is an exterior domain of $\mathbb{R}^n$, $n \geq 2$. For $n = 2$, they proved the existence of an exterior domain $\Omega_T = \{x \in \mathbb{R}^2 : |x| > T > 1\}$ and a positive

* *Mathematics Subject Classifications:* 34B16, 34B27, 35J65.
Key words: Singular elliptic equation, Green function, Schauder fixed point theorem, maximum principle, superharmonic function.

Submitted February 13, 2001. Published July 24, 2001.

solution u on Ω_T such that $u(x)/Log|x|$ is bounded and bounded away from zero provided that the following conditions are satisfied

C1) $f \in C^{\alpha}_{\rm loc}(\Omega \times (0,\infty))$.

C2) There exist two functions ψ and $\phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ of class $C^{\alpha}_{\rm loc}((0,\infty)\times(0,\infty))$, such that $\psi(t,u)$ and $\phi(t,u)$ are non-increasing functions of u for each fixed $t > 0$, and

$$\psi(|x|,u) \le f(x,u) \le \phi(|x|,u), \text{ for all } (x,u) \in \Omega \times (0,\infty).$$

C3) $\int^{\infty} \phi(t, cLogt)dt < \infty$, for some positive constant c.

Kusano and Swanson showed also for $n = 2$, the existence of a bounded positive solution of (1.2) in some exterior domain Ω_T, T sufficiently large, provided that ϕ satisfies C1 and C2, and $\int^{\infty} t\phi(t,c)Logtdt < \infty$, for some constant $c > 0$.

In this article, we improve the results of [4] by letting the exponent γ be unbounded. More precisely, we are concerned with the following problem

$$\begin{gathered}\Delta u + \varphi(x,u) = 0, \quad \text{in } D, \text{ (in the weak sense)} \\ u\,|_{\partial D} = 0,\end{gathered} \tag{1.3}$$

where $D = \{x \in \mathbb{R}^2 : |x| > 1\}$ and φ is a nonnegative Borel measurable function in $D \times (0,\infty)$ that belongs to a convex cone which contains, in particular, all functions

$$\varphi(x,t) = q(x)t^{-\gamma}, \quad \gamma > 0$$

with nonnegative Borel function q. Under appropriate conditions on φ, we show that (1.3) has infinitely many positive solutions continuous on $\overline{D}$. More precisely, for each $\mu > 0$, there exists a positive solution $u \in C(\overline{D})$ such that $\lim_{|x|\to\infty} u(x)/Log|x| = \mu$. Under additional conditions on φ, we prove that (1.3) has a bounded positive solution continuous on $\overline{D}$.

This paper is organized as follows. In section 2, we recall and establish some properties of functions belonging to the Kato class introduced in [9]. In section 3, we prove the existence of many positive solutions of (1.3) which are continuous on $\overline{D}$. In the last section, we give some estimates on the solutions of (1.3). We point out that for some functions φ, we get better estimates on the solutions; namely for each $x \in \overline{D}$, we have

$$\mu Log|x| \le u(x) \le C Log|x|, \quad \text{if } \lim_{|x|\to\infty} \frac{u(x)}{Log|x|} = \mu > 0$$

and

$$\frac{1}{C}\big(1 - \frac{1}{|x|}\big) \le u(x) \le C\big(1 - \frac{1}{|x|}\big), \quad \text{if } u \text{ is bounded},$$

where C is a positive constant.

As usual let $B(D)$ be the set of Borel measurable functions in D and let $B^+(D)$ be the subset of the nonnegative functions.

We recall that the potential kernel V associated to Δ is defined on $B^+(D)$ by

$$V\Psi(x) = \int_D G(x,y)\Psi(y)dy, \quad \text{for } x \in D,$$

where G is the Green's function of the Laplacian in D. Hence, for any $\Psi \in B^+(D)$ such that $\Psi \in L^1_{\text{loc}}(D)$ and $V\Psi \in L^1_{\text{loc}}(D)$, we have (in the distributional sense)

$$\Delta(V\Psi) = -\Psi \text{ in } D. \tag{1.4}$$

We note that for any $\Psi \in B^+(D)$ such that $V\Psi \neq \infty$, we have $V\Psi \in L^1_{\text{loc}}(D)$ (see [2, p.51]). Let us recall that V satisfies the complete maximum principle [10, p.175], i.e for each $f \in B^+(D)$ and v a nonnegative superharmonic function on D such that $Vf \leq v$ in $\{f > 0\}$ we have $Vf \leq v$ in D.

Throughout this paper, the function φ is required to satisfy combinations of the following hypotheses

H1) φ is continuous and non-increasing with respect to the second variable.

H2) $\varphi(.,c) \in K^\infty(D)$ for every $c > 0$.

H3) $V\varphi(.,c) > 0$ for every $c > 0$.

Finally we mention that the letter C will denote a generic positive constant which may vary from line to line.

2 The Kato class $K^\infty(D)$

Throughout this paper, let $D = \{x \in \mathbb{R}^2, |x| > 1\}$, $\overline{D} = \{x \in \mathbb{R}^2, |x| \geq 1\}$, and $G(x,y) = \frac{1}{2\pi} Log(1 + \dfrac{(|x|^2-1)(|y|^2-1)}{|x-y|^2})$ be the Green's function of Δ in D.

Definition A Borel measurable function q in D belongs to the Kato class $K^\infty(D)$ if q satisfies the following conditions

$$\lim_{\alpha\to 0} \sup_{x\in D} \int_{(|x-y|\leq\alpha)\cap D} \frac{|y|-1}{|y|} \frac{|x|}{|x|-1} G(x,y)|q(y)|dy = 0 \tag{2.1}$$

$$\lim_{M\to\infty} \sup_{x\in D} \int_{(|y|\geq M)} \frac{|y|-1}{|y|} \frac{|x|}{|x|-1} G(x,y)|q(y)|dy = 0. \tag{2.2}$$

Lemma 2.1 *For each x,y in D,*

$$\frac{1}{2\pi}(1 - \frac{1}{|x|})(1 - \frac{1}{|y|}) \leq G(x,y).$$

Proof By the definition of G, we have

$$\begin{aligned} G(x,y) &= \frac{1}{2\pi} Log\left(1+\frac{(|x|^2-1)(|y|^2-1)}{|x-y|^2}\right) \\ &= \frac{1}{2\pi}\frac{(|x|-1)}{|x|}\frac{(|y|-1)}{|y|}\int_0^1 \frac{|x||y|(1+|x|)(1+|y|)}{|x-y|^2+t(|x|^2-1)(|y|^2-1)}dt. \end{aligned}$$

For every $t\in[0,1]$ and x,y in D, we have

$$\begin{aligned} \frac{|x-y|^2+t(|x|^2-1)(|y|^2-1)}{|x||y|(1+|x|)(1+|y|)} &\le \frac{(|x|+|y|)^2+(|x|^2-1)(|y|^2-1)}{|x||y|(1+|x|)(1+|y|)} \\ &= \frac{(|x||y|+1)^2}{|x||y|(1+|x|)(1+|y|)} \le 1. \end{aligned}$$

Hence $G(x,y) \ge \frac{1}{2\pi}(1-\frac{1}{|x|})(1-\frac{1}{|y|})$.

Proposition 2.2 *Let q be a function in the class $K^\infty(D)$.Then the function $y \to (1-\frac{1}{|y|})^2 q(y)$ is in $L^1(D)$. In particular $q\in L^1_{\rm loc}(D)$.*

Proof. Let $q\in K^\infty(D)$. Then by (2.1) and (2.2), there exist $\alpha>0$ and $M>1$ such that

$$\sup_{x\in D}\int_{(|x-y|\le\alpha)} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \le 1$$

and

$$\sup_{x\in D}\int_{(|y|\ge M)} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \le 1.$$

Let $x_1,x_2,\dots,x_n$ in D such that $\overline{D}\cap\overline{B}(0,M)\subset\bigcup_{1\le i\le n}B(x_i,\alpha)$. By using Lemma 2.1, we get

$$\begin{aligned} &\int_D (1-\frac{1}{|y|})^2|q(y)|dy \\ &\le \int_{(|y|\ge M)}(1-\frac{1}{|y|})^2|q(y)|dy+\int_{(1\le|y|\le M)\cap D}(1-\frac{1}{|y|})^2|q(y)|dy \\ &\le 2\pi\sup_{x\in D}\int_{(|y|\ge M)}\frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \\ &\quad +\sum_{i=1}^n\int_{B(x_i,\alpha)\cap D}(1-\frac{1}{|y|})^2|q(y)|dy \\ &\le 2\pi+2\pi\sum_{i=1}^n\int_{B(x_i,\alpha)\cap D}\frac{|y|-1}{|y|}\frac{|x_i|}{|x_i|-1}G(x_i,y)|q(y)|dy \\ &\le 2\pi+2\pi n\sup_{x\in D}\int_{B(x,\alpha)\cap D}\frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \\ &\le 2\pi(1+n)<\infty. \end{aligned}$$

Lemma 2.3 *Let $M > 1$ and $r > 0$. Then there exists a constant $C > 0$ such that for each $x \in D$ and $y \in D$ satisfying $|x - y| \geq r$ and $|y| \leq M$,*

$$G(x, y) \leq C(1 - \frac{1}{|x|})(1 - \frac{1}{|y|}).$$

Proof. We have for $|x - y| \geq r$ and $|y| \leq M$,

$$\begin{aligned}
\frac{|x|}{|x|-1}\frac{|y|-1}{|y|}G(x,y) &\leq \frac{1}{2\pi}\frac{|x|(|y|-1)}{(|x|-1)|y|}\frac{(|x|^2-1)(|y|^2-1)}{|x-y|^2} \\
&= \frac{1}{2\pi}\frac{(|y|-1)^2(|y|+1)}{|y|}\frac{|x|(|x|+1)}{|x-y|^2} \\
&\leq \frac{1}{2\pi}(1-\frac{1}{|y|})^2 M(M+1)\frac{|x|(|x|+1)}{((|x|-M)\vee r)^2},
\end{aligned}$$

where $(|x| - M) \vee r = \max(|x| - M, r)$. Since the function $t \to \frac{t(t+1)}{((t-M)\vee r)^2}$ is continuous and positive on $[1, \infty)$ and $\lim_{t\to+\infty} \frac{t(t+1)}{((t-M)\vee r)^2} = 1$, then there exists $C > 0$ such that

$$\frac{|x|}{|x|-1}\frac{|y|-1}{|y|}G(x,y) \leq C(1-\frac{1}{|y|})^2. \quad \diamondsuit$$

In the sequel , we use the notation

$$\|q\|_D = \sup_{x\in D}\int_D \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy.$$

Proposition 2.4 *If $q \in K^\infty(D)$, then $\|q\|_D < \infty$.*

Proof. Let $\alpha > 0$ and $M > 1$. Then

$$\begin{aligned}
&\int_D \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \\
&\leq \int_{(|x-y|\leq\alpha)\cap D} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \\
&\quad + \int_{(|y|\geq M)} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \\
&\quad + \int_{(|x-y|\geq\alpha)\cap(|y|\leq M)\cap D} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy.
\end{aligned}$$

By Lemma 2.3,

$$\int_{(|x-y|\geq\alpha)\cap(|y|\leq M)\cap D} \frac{|y|-1}{|y|}\frac{|x|}{|x|-1}G(x,y)|q(y)|dy \leq C\int_D (1-\frac{1}{|y|})^2|q(y)|dy.$$

Thus, the result follows from (2.1) , (2.2) and Proposition 2.2. $\diamondsuit$

The following result of Selmi [11], will be needed in the sequel.

Theorem 2.5 *There exists a constant $C_0 > 0$ depending only on D such that for all x, y and z in D, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0 \left[\frac{|z|-1}{|z|} \frac{|x|}{|x|-1} G(x,z) + \frac{|z|-1}{|z|} \frac{|y|}{|y|-1} G(z,y) \right]. \quad (2.3)$$

By using the above theorem we have the following

Proposition 2.6 *There exists a constant $C_D > 0$ depending only on D such that for any function q belonging to $K^\infty(D)$, any nonnegative superharmonic function h in D and all $x \in D$*

$$\int_D G(x,y)h(y)|q(y)|dy \leq C_D \|q\|_D h(x). \quad (2.4)$$

Proof. Let h be a nonnegative superharmonic function in D, then there exists a sequence $(f_n)_n$ of nonnegative measurable functions in D such that

$$h(y) = \sup_n \int_D G(y,z) f_n(z)dz \ , \ \forall y \in D.$$

Hence, we need only to verify (2.4) for $h(y) = G(y,z)$ for all $z \in D$. By using (2.3), we obtain

$$\frac{1}{G(x,z)} \int_D G(x,y)G(y,z)|q(y)|dy \leq 2C_0 \|q\|_D. \quad \diamondsuit$$

If we take $h = 1$ in Proposition 2.6, we obtain the following statement.

Corollary 2.7 *Let q be a function in $K^\infty(D)$. Then*

$$\sup_{x \in D} \int_D G(x,y)|q(y)|dy < \infty. \quad (2.5)$$

Corollary 2.8 *Let q be a function in the class $K^\infty(D)$. Then the function $y \to (1 - \frac{1}{|y|})q(y)$ is in $L^1(D)$.*

Proof. For each x, y in D, by Lemma 2.1 we have

$$\frac{1}{2\pi}(1 - \frac{1}{|x|})(1 - \frac{1}{|y|}) \leq G(x,y).$$

Hence $\int_D (1 - \frac{1}{|y|})|q(y)|dy \leq 2\pi \frac{|x|}{|x|-1} \int_D G(x,y)|q(y)|dy$. The result follows from Corollary 2.7. $\diamondsuit$

In the next proposition we prove that for q radial , $q \in K^\infty(D)$ if and only if (2.5) is satisfied.

Proposition 2.9 *Let q be a radial function in D. Then $q \in K^\infty(D)$ if and only if*

$$\int_1^{+\infty} rLog(r)|q(r)|dr < \infty. \quad (2.6)$$

Proof. By elementary calculus, we have

$$\int_D G(x,y)|q(y)|dy = \int_1^{+\infty} rLog(r\wedge R)|q(r)|dr\,,$$

where $R=|x|$ and $r\wedge R=\min(r,R)$. Hence by (2.5), we deduce that if $q\in K^\infty(D)$ then (2.6) is satisfied. The proof of the converse is found in [9, Prop.2]. $\diamondsuit$

Using the same argument as in the proof of Proposition 2.6, we establish the following lemma (see also [9]).

Lemma 2.10 *Let $x_0\in\overline{D}$. Then for any function q belonging to $K^\infty(D)$ and any positive superharmonic function h in D, we have*

$$\lim_{r\to 0}\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G(x,y)h(y)|q(y)|dy = 0$$

and

$$\lim_{M\to+\infty}\sup_{x\in D}\frac{1}{h(x)}\int_{(|y|\geq M)} G(x,y)h(y)|q(y)|dy = 0.$$

Proposition 2.11 *Let q be a function in $K^\infty(D)$. Then $Vq\in C(D)$ and $\lim_{x\to\partial D} Vq(x)=0$.*

Proof. Without loss of generality, assume that q is nonnegative. Let $x_0\in D$ and $\varepsilon>0$. By Lemma 2.10, there exist $r>0$ and $M>1$ such that

$$\sup_{z\in D}\int_{B(x_0,2r)\cap D} G(z,y)q(y)dy\leq\frac{\varepsilon}{4}$$

and

$$\sup_{z\in D}\int_{(|y|\geq M)} G(z,y)q(y)dy\leq\frac{\varepsilon}{4}.$$

Let $x,x'\in B(x_0,r)\cap D$, then we have

$$\begin{aligned}
&|Vq(x)-Vq(x')|\\
&\leq\; 2\sup_{z\in D}\int_{B(x_0,2r)\cap D} G(z,y)q(y)dy + 2\sup_{z\in D}\int_{(|y|\geq M)} G(z,y)q(y)dy\\
&\quad+\int_{(|x_0-y|\geq 2r)\cap(1<|y|\leq M)} |G(x,y)-G(x',y)|q(y)dy\\
&\leq\; \varepsilon+\int_{(|x_0-y|\geq 2r)\cap(1<|y|\leq M)} |G(x,y)-G(x',y)|q(y)dy.
\end{aligned}$$

For every $y\in(|x_0-y|\geq 2r)\cap(1<|y|\leq M)$ and $x,x'\in B(x_0,r)\cap D$, using Lemma 2.3 we obtain

$$|G(x,y)-G(x',y)|\leq G(x,y)+G(x',y)\leq C\big(1-\frac{1}{|y|}\big).$$

Now since G is continuous out the diagonal, we deduce by Corollary 2.8 and the Lebesgue's theorem that

$$\int_{(|x_0-y|\geq 2r)\cap(1<|y|\leq M)} |G(x,y)-G(x',y)|q(y)dy \to 0 \; as \; |x-x'|\to 0.$$

Hence $Vq\in C(D)$. Next, we consider $x_0\in\partial D$ and $\varepsilon>0$. By Lemma 2.10, there exist $r>0$ and $M>1$ such that

$$\sup_{z\in D}\int_{B(x_0,2r)\cap D} G(z,y)q(y)dy \leq \frac{\varepsilon}{4}$$

and

$$\sup_{z\in D}\int_{(|y|\geq M)} G(z,y)q(y)dy \leq \frac{\varepsilon}{4}.$$

Let $x\in B(x_0,r)\cap D$, then we have

$$\begin{aligned} Vq(x) &= \int_D G(x,y)q(y)dy \\ &= \int_{B(x_0,2r)\cap D} G(x,y)q(y)dy + \int_{(|y|\geq M)} G(x,y)q(y)dy \\ &\quad + \int_{B^c(x_0,2r)\cap(1\leq|y|\leq M)} G(x,y)q(y)dy \\ &\leq \frac{\varepsilon}{2} + \int_{B^c(x_0,2r)\cap(1<|y|\leq M)} G(x,y)q(y)dy. \end{aligned}$$

For every $y\in B^c(x_0,2r)\cap D\cap\overline{B}(0,M)$ and $x\in B(x_0,r)$, we get by using Lemma 2.3

$$G(x,y)q(y)\leq C(1-\frac{1}{|y|})q(y).$$

Now, since for all $y\in D, \lim_{x\to\partial D} G(x,y)=0$, then as in the above argument, we get $\lim_{x\to x_0} Vq(x)=0$. This achieves the proof of the proposition.

3 Positive solutions of $\Delta u+\varphi(.,u)=0$

In this section, we study the existence of positive solutions for the nonlinear singular elliptic boundary value problem (1.3).

Lemma 3.1 *Let $h\in B^+(D)$ and v be a nonnegative superharmonic function on D. Then for all $w\in B(D)$ such that $V(h|w|)<\infty$ and $w+V(hw)=v$, we have $0\leq w\leq v$.*

Proof. Let $w^+ = \max(w, 0)$ and $w^- = \max(-w, 0)$. Since $V(h|w|) < \infty$, then

$$w^+ + V(hw^+) = v + w^- + V(hw^-).$$

Hence

$$V(hw^+) \leq v + V(hw^-) \quad \text{in } \left\{w^+ > 0\right\}.$$

Since $v + V(hw^-)$ is a nonnegative superharmonic function in D, we have as consequence of the complete maximum principle

$$V(hw^+) \leq v + V(hw^-) \quad \text{in } D,$$

that is $V(hw) \leq v = w + V(hw)$. This implies that $0 \leq w \leq v$.

Proposition 3.2 *Let $\varphi : D \times (0, \infty) \to [0, \infty)$ be a measurable function satisfying H1 and $\lambda_1, \lambda_2, \mu_1, \mu_2$ be real numbers such that $0 \leq \lambda_1 \leq \lambda_2$ and $0 \leq \mu_1 \leq \mu_2$. If u_1 and u_2 are two positive functions continuous on $\overline{D}$ satisfying for each $x \in D$*

$$u_1(x) = \lambda_1 + \mu_1 Log|x| + V(\varphi(\cdot, u_1))(x)$$

and

$$u_2(x) = \lambda_2 + \mu_2 Log|x| + V(\varphi(\cdot, u_2))(x).$$

Then we have

$$0 \leq u_2(x) - u_1(x) \leq \lambda_2 - \lambda_1 + (\mu_2 - \mu_1)Log|x| \ , \ \forall x \in \overline{D}.$$

Proof. Let h be the function defined on D by

$$h(x) = \begin{cases} \dfrac{\varphi(x, u_1(x)) - \varphi(x, u_2(x))}{u_2(x) - u_1(x)} & \text{if } u_2(x) \neq u_1(x) \\ 0, & \text{if } u_2(x) = u_1(x). \end{cases}$$

Then $h \in B^+(D)$ and we have

$$u_2 - u_1 + V(h(u_2 - u_1)) = \lambda_2 - \lambda_1 + (\mu_2 - \mu_1)Log|\cdot|.$$

Furthermore, we have

$$V(h|u_2 - u_1|) \leq V(\varphi(\cdot, u_2)) + V(\varphi(\cdot, u_1)) \leq u_2 + u_1 < \infty.$$

Hence we deduce the result from Lemma 3.1.

Theorem 3.3 *Let $\lambda > 0$, $\mu > 0$ and $\varphi : D \times (0, \infty) \to [0, \infty)$ be a Borel measurable function satisfying H1 and H2. Then the problem*

$$\begin{gathered} \Delta u + \varphi(., u) = 0, \quad \textit{in } D \textit{ (in the weak sense)}, \\ u\,|_{\partial D} = \lambda, \quad \lim_{|x| \to \infty} \frac{u(x)}{Log|x|} = \mu, \end{gathered} \tag{3.1}$$

has a unique positive solution $u_\lambda \in C(\overline{D})$.

Proof. Let $\lambda > 0$. Then by hypothesis H2, the function $\varphi(.,\lambda) \in K^\infty(D)$ and by Corollary 2.7, we deduce that $\|V\varphi(.,\lambda)\|_\infty < \infty$. To apply a fixed point argument, we consider the convex set

$$F = \Big\{\omega \in C(\overline{D} \cup \{\infty\}) : \lambda \le \omega(x) \le \lambda + \frac{\lambda \|V\varphi(\cdot,\lambda)\|_\infty}{\lambda + \mu Log|x|},\ \forall x \in \overline{D}\Big\}$$

and define the operator T on F by

$$T\omega(x) = \lambda + \frac{\lambda}{\lambda + \mu Log|x|} \int_D G(x,y)\varphi\Big(y, \omega(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big) dy\,,\ x \in \overline{D}.$$

Since for all $\omega \in F$ and $y \in D, \varphi\left(y, \omega(y)(1 + \frac{\mu}{\lambda} Log|y|)\right) \le \varphi(y,\lambda)$, then for each $\omega \in F, \lambda \le T\omega \le \lambda + \dfrac{\lambda\|V\varphi(\cdot,\lambda)\|_\infty}{\lambda + \mu Log|x|}$ and as in the proof of Proposition 2.11, we show that the family TF is equicontinuous in $\overline{D} \cup \{\infty\}$. In particular, for all $\omega \in F, T\omega \in F$. Moreover, the family $\{T\omega(x), \omega \in F\}$ is uniformly bounded in $\overline{D} \cup \{\infty\}$. It follows by Ascoli's theorem that TF is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Next, we prove the continuity of T in F. Consider a sequence $(\omega_k)_{k\in\mathbb{N}}$ in F which converges uniformly to a function $\omega \in F$. Then

$$\begin{aligned} |T\omega_k(x) - T\omega(x)| \le{} & \frac{\lambda}{\lambda + \mu Log|x|} \int_D G(x,y)\Big|\varphi\Big(y, \omega_k(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big) \\ & -\varphi\Big(y, \omega(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big)\Big| dy. \end{aligned}$$

Now by the monotonocity of φ, we have

$$\Big|\varphi\Big(y, \omega_k(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big) - \varphi\Big(y, \omega(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big)\Big| \le 2\varphi(y,\lambda),$$

and since φ is continuous with respect to the second variable, we deduce by the dominated convergence theorem and Corollary 2.7, that

$$\forall\, x \in \overline{D},\ T\omega_k(x) \to T\omega(x) \quad \text{as } k \to \infty.$$

Since TF is relatively compact in $C(\overline{D} \cup \{\infty\})$, then $T\omega_k$ converges uniformly to $T\omega$ as $k \to \infty$. Thus we have proved that T is a compact mapping from F to itself. Hence by the Schauder's fixed point theorem, there exists $\omega_\lambda \in F$ such that for each $x \in D$,

$$\omega_\lambda(x) = \lambda + \frac{\lambda}{\lambda + \mu Log|x|} \int_D G(x,y)\varphi\Big(y, \omega_\lambda(y)(1 + \frac{\mu}{\lambda} Log|y|)\Big) dy.$$

Put $u_\lambda(x) = \omega_\lambda(x)(1 + \frac{\mu}{\lambda} Log|x|)$, for $x \in \overline{D}$. Then we have

$$u_\lambda(x) = \lambda + \mu Log|x| + \int_D G(x,y)\varphi(y, u_\lambda(y))dy. \tag{3.2}$$

In addition, since for each $y \in D$, $\varphi(y, u_\lambda(y)) \le \varphi(y, \lambda)$, we deduce by hypothesis H2 and Proposition 2.2 that the map $y \to \varphi(y, u_\lambda(y)) \in L^1_{\rm loc}(D)$. On the other hand, using Proposition 2.11, it follows that $V(\varphi(\cdot, u_\lambda)) \in C(\overline{D})$ and $\lim_{x\to\partial D} V(\varphi(\cdot, u_\lambda))(x) = 0$. So we can apply Δ to the equation (3.2) to obtain $\Delta u_\lambda + \varphi(\cdot, u_\lambda) = 0$ (in the weak sense). Furthermore, for every $x \in D$, we have

$$\mu + \frac{\lambda}{Log|x|} \le \frac{u_\lambda(x)}{Log|x|} \le \mu + \frac{\lambda + \|V\varphi(\cdot, \lambda)\|_\infty}{Log|x|}.$$

Thus $\lim_{|x|\to\infty} \dfrac{u_\lambda(x)}{Log|x|} = \mu$, and by (3.2), we have $u_\lambda/_{\partial D} = \lambda$. This shows that u_λ is a positive continuous solution of (3.1).

Finally, we show the uniqueness of the solution. Let u be a positive continuous solution of the problem in Theorem 3.3. Clearly u is a superharmonic function with boundary value λ and $\lim_{|x|\to\infty}(u(x) - \lambda) \ge 0$. So, we have by the maximum principle [3, p.465] that $u \ge \lambda$ on D. Which together with the monotonicity of φ imply that $\varphi(\cdot, \lambda) \ge \varphi(\cdot, u) \in K^\infty(D)$. So, we deduce by Proposition 2.2 and Proposition 2.11 that the functions $\varphi(\cdot, u)$ and $V\varphi(\cdot, u)$ are in $L^1_{\rm loc}(D)$ and $C(\overline{D})$ respectively with $\lim_{x\to\partial D} V\varphi(\cdot, u)(x) = 0$. Hence u satisfies $\Delta(u - V\varphi(\cdot, u)) = 0$ (in the weak sense). It follows that the function $h = u - V\varphi(., u) - \mu Log|x| - \lambda$ is harmonic in D satisfying $h/_{\partial D} = 0$ and $\lim_{|x|\to\infty} \dfrac{h(x)}{Log|x|} = 0$. Thus by [3, p.419], we have $h = 0$. So u satisfies (3.2), which yields with Proposition 3.2 to the uniqueness of u_λ. $\diamondsuit$

Lemma 3.4 *If $u \in C(\overline{D})$ is a nonnegative solution of the problem*

$$\begin{gathered} \Delta u + \varphi(., u) = 0, \quad \text{in } D \text{ (in the weak sense)} \\ u\,|_{\partial D} = 0, \quad \lim_{|x|\to\infty} \tfrac{u(x)}{Log|x|} = \mu \ge 0, \end{gathered} \tag{3.3}$$

then for each $x \in D$,

$$\mu Log|x| \le u(x) \le \mu Log|x| + V\left(\varphi(\cdot, u)\right)(x). \tag{3.4}$$

Proof. We assume that $V\varphi(\cdot, u) \ne \infty$, otherwise the upper inequality is satisfied. Let $\varepsilon > 0$. Since $\lim_{|x|\to\infty} \dfrac{u(x)}{Log|x|} = \mu$, there exists $M > 1$ such that

$$(\mu - \varepsilon) Log|x| \le u(x) \le (\mu + \varepsilon) Log|x|, \quad \text{for } |x| \ge M.$$

The functions defined on D by $v_\varepsilon(x) = u(x) + (\varepsilon - \mu) Log|x|$ and $w_\varepsilon(x) = V\varphi(., u)(x) - u(x) + (\mu + \varepsilon) Log|x|$ satisfy the following properties:

$$\begin{gathered} v_\varepsilon \in C(\overline{D}), \quad \Delta v_\varepsilon = \Delta u \le 0 \quad \text{in } D, \\ v_\varepsilon = 0 \quad \text{in } \partial D, \quad \liminf_{|x|\to\infty} v_\varepsilon(x) \ge 0 \end{gathered}$$

The function w_ε is lower semi-continuous on D,

$$\begin{gathered}\Delta w_\varepsilon = -\varphi(.,u) - \Delta u = 0 \quad \text{in } D,\\ w_\varepsilon \geq 0 \quad \text{in } \partial D, \quad \liminf_{|x|\to\infty} w_\varepsilon(x) \geq 0.\end{gathered}$$

Hence by [3, p.465], we get

$$(\mu-\varepsilon)Log|x| \leq u(x) \leq (\mu+\varepsilon)Log|x| + V\varphi(.,u) \ \text{ in } D.$$

Since ε is arbitrary, we obtain (3.4). ♢

Now we are ready to prove one of the main results of this section.

Theorem 3.5 *Let $\varphi : D\times(0,\infty) \to [0,\infty)$ be a measurable function satisfying H1 and H2, and $\mu > 0$. Then the problem (3.3) has a unique positive solution $u \in C(\overline{D})$ satisfying $V\varphi(\cdot,u) \neq \infty$. If we suppose further that $\varphi \in C^\alpha_{\text{loc}}(D\times(0,\infty))$, $(0<\alpha<1)$, then the solution $u \in C^{2+\alpha}_{loc}(D)\cap C(\overline{D})$.*

Proof. Let $(\lambda_n)_{n\geq 0}$ be a sequence of real numbers that decreases to zero. For each $n \in \mathbb{N}$, we denote by u_n the unique positive solution of problem (3.1) given in Theorem 3.3 for $\lambda = \lambda_n$. Then by Proposition 3.2, the sequence $(u_n)_{n\geq 0}$ decreases to a function u and so by (3.2), the sequence $(u_n - \lambda_n)_{n\geq 0}$ increases to u. Due to the monotonicity of φ, we have for each $x \in D$

$$\begin{aligned} u(x) &\geq u_n(x) - \lambda_n = \mu Log|x| + \int_D G(x,y)\varphi(y,u_n(y))dy\\ &\geq \mu Log|x| > 0. \end{aligned}$$

Hence, applying the monotone convergence theorem, we get

$$u(x) = \mu Log|x| + \int_D G(x,y)\varphi(y,u(y))dy\ ,\ \forall x \in D. \tag{3.5}$$

Since $u = \sup_n(u_n - \lambda_n) = \inf_n(u_n)$ and for each $n \in \mathbb{N}$ the function u_n is continuous on D, then u is a positive continuous function on D. Which together with (3.5) imply that $V(\varphi(\cdot,u)) \in L^1_{\text{loc}}(D)$. So using hypothesis H2 and Proposition 2.2, we deduce that the map $y \to \varphi(y,u(y)) \in L^1_{\text{loc}}(D)$. Applying Δ on both sides of equality (3.5) we obtain

$$\Delta u + \varphi(\cdot,u) = 0, \qquad \text{in } D \text{ (in the weak sense)}.$$

Now, since for each $x \in D$ and $n \in \mathbb{N}$, we have $0 \leq u_n(x) - \lambda_n \leq u(x) \leq u_n(x)$ and $\lim_{|x|\to\infty} \dfrac{u_n(x)}{Log|x|} = \mu$, then

$$\lim_{x\to\partial D} u(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \frac{u(x)}{Log|x|} = \mu.$$

Thus $u \in C(\overline{D})$ and u is a positive solution of the problem (3.3). Finally, we intend to show the uniqueness of the solution. Let u be a positive continuous solution of the problem (3.3) such that $V(\varphi(\cdot,u)) \neq \infty$. Then the functions $\varphi(\cdot,u)$

and $V\varphi(\cdot,u)$ are in $L^1_{\rm loc}(D)$. We deduce by [2, p.52] that $\Delta(V\varphi(\cdot,u)) = -\varphi(\cdot,u)$, in D (in the weak sense) and consequently $\Delta\left(V\varphi(\cdot,u) + \mu Log|\cdot| - u\right) = 0$ in D (in the weak sense). Hence there exists a harmonic function h in D such that

$$h(x) + u(x) - \mu Log|x| = V\varphi(\cdot,u)(x) \quad a.e \text{ on } D.$$

Since u and $V\varphi(.,u)$ are superharmonic functions in D, we get by [10, p.134] that

$$h(x) + u(x) - \mu Log|x| = V\varphi(\cdot,u)(x) \text{ on } D.$$

Now using (3.4), we get $0 \le h \le V\varphi(\cdot,u) < \infty$. Hence by [10, p.158], we deduce that $h = 0$. The function u satisfies (3.5) and the uniqueness follows by Proposition 3.2. $\diamondsuit$

Corollary 3.6 *Let φ_1, φ_2 be nonnegative measurable functions in $D \times (0,\infty)$ satisfying the hypotheses H1 and H2, and $\mu_1, \mu_2 \in \mathbb{R}_+$ such that $0 \le \varphi_1 \le \varphi_2$ and $0 < \mu_1 \le \mu_2$. If we denote by $u_j \in C(\overline{D})$ the unique positive solution of the problem (3.3) with $\varphi = \varphi_j$ and $\mu = \mu_j$, $j \in \{1,2\}$, given in Theorem 3.5, then we have*

$$0 \le u_2 - u_1 \le (\mu_2 - \mu_1) Log|\cdot| + V\left(\varphi_2(\cdot,u_2) - \varphi_1(\cdot,u_2)\right) \text{ in } D.$$

Proof. It follows by Theorem 3.5, that

$$u_1 = \mu_1 Log|\cdot| + V\varphi_1(\cdot,u_1) \text{ and } u_2 = \mu_2 Log|\cdot| + V\varphi_2(\cdot,u_2).$$

Let h be the nonnegative measurable function defined on D by

$$h(x) = \begin{cases} \dfrac{\varphi_1(x,u_2(x)) - \varphi_1(x,u_1(x))}{u_1(x) - u_2(x)}, & \text{if } u_1(x) \ne u_2(x) \\ 0, & \text{if } u_1(x) = u_2(x). \end{cases}$$

Then $h \in B^+(D)$ and we have

$$u_2 - u_1 + V(h(u_2 - u_1)) = (\mu_2 - \mu_1) Log|\cdot| + V\big(\varphi_2(\cdot,u_2) - \varphi_1(\cdot,u_2)\big).$$

Now, since

$$V(h|u_2 - u_1|) \le V\varphi_1(\cdot,u_2) + V\varphi_1(\cdot,u_1) \le V\varphi_2(\cdot,u_2) + V\varphi_1(\cdot,u_1) \le u_1 + u_2 < \infty$$

and $(\mu_2 - \mu_1) Log|\cdot| + V(\varphi_2(\cdot,u_2) - \varphi_1(\cdot,u_2))$ is a nonnegative superharmonic function on D, we deduce the result from Lemma 3.1.

Theorem 3.7 *Let $\varphi : D \times (0,\infty) \to [0,\infty)$ be a measurable function satisfying H1-H3. Then the problem (1.3) has a unique positive bounded solution $u \in C(\overline{D})$ satisfying $V\varphi(\cdot,u) \ne \infty$.*

Proof. Let $\lambda > 0$ and (μ_n) be a sequence of real numbers that decreases to zero. For each $n \in \mathbb{N}$, we denote by u_{λ,μ_n} the unique positive continuous solution of the problem (3.1) and by v_n the unique positive continuous solution of the problem (3.3), given in Theorem 3.5 for $\mu = \mu_n$. Then by Corollary 3.6, the sequence $(v_n)_{n\in\mathbb{N}}$ decreases to a function u and so by (3.5), the sequence $(v_n - \mu_n Log|\cdot|)_n$ increases to u. Due to the monotonicity of φ and by (3.2), we have for each $x \in D$

$$\begin{aligned} &\lambda + \|V\varphi(\cdot,\lambda)\|_\infty + \mu_n Log|x| \\ &\quad\geq\ u_{\lambda,\mu_n}(x) \geq v_n(x) \\ &\quad\geq\ u(x) \geq v_n(x) - \mu_n Log|x| \\ &\quad\geq\ \int_D G(x,y)\varphi\left(y, \mu_n Log|y| + \lambda + \|V\varphi(\cdot,\lambda)\|_\infty\right) dy. \end{aligned}$$

Letting n tends to infinity, we get

$$\lambda + \|V\varphi(\cdot,\lambda)\|_\infty \geq u(x) \geq V\varphi\left(\cdot, \lambda + \|V\varphi(\cdot,\lambda)\|_\infty\right)(x)\ ,\ \ \forall\, x \in D.$$

By H2, H3 and Corollary 2.7, u is a positive bounded function in D. Since, for each $n \in \mathbb{N}$ and $x \in D$

$$v_n - \mu_n Log|x| = \int_D G(x,y)\varphi(y, v_n(y))\, dy,$$

we obtain, as $n \to \infty$, that

$$u(x) = \int_D G(x,y)\varphi(y,u(y))dy\ ,\ \forall\, x \in D. \tag{3.6}$$

As in the proof of Theorem 3.5, we show that $u \in C(\overline{D})$ and u is a positive bounded solution of (1.3). Using (3.4), we establish the uniqueness of such a solution. $\diamondsuit$

Corollary 3.8 *Let $\varphi : D \times (0,\infty) \to [0,\infty)$ be a measurable function satisfying H1 and H2. Then for each $\mu > 0$ and f a nonnegative continuous function on ∂D, the following nonlinear problem*

$$\begin{aligned} &\Delta u + \varphi(\cdot,u) = 0, \quad \text{in } D \text{ (in the weak sense)} \\ &u_{/\partial D} = f, \quad \lim_{|x|\to\infty} \frac{u(x)}{Log|x|} = \mu, \end{aligned} \tag{3.7}$$

has a unique positive solution $u \in C(\overline{D})$ satisfying $V\varphi(\cdot,u) \neq \infty$.

Proof. Let H_f^D denotes the unique bounded solution of the following Dirichlet problem

$$\begin{aligned} &\Delta\omega = 0, \quad \text{in } D, \\ &\omega_{/\partial D} = f. \end{aligned}$$

We note that if u is a continuous solution of (3.7), then as φ is a nonnegative function, we deduce that $u-H_f^D$ is superharmonic such that $u-H_f^D=0$ on ∂D and $\lim_{|x|\to\infty}\Big(u(x)-H_f^D(x)\Big)=+\infty$. We conclude by the maximum principle, that

$$u\geq H_f^D \quad \text{in } D.$$

Let Ψ be the function defined on $D\times(0,\infty)$ by $\Psi(x,t)=\varphi(x,t+H_f^D(x))$. It is clear to verify that Ψ satisfies the same hypotheses H1-H2 as φ. Hence by Theorem 3.5, the problem

$$\begin{gathered}\Delta v+\Psi(\cdot,v)=0, \quad \text{in } D,\\ v_{/\partial D}=0, \quad \lim_{|x|\to\infty}\frac{v(x)}{Log|x|}=\mu,\end{gathered}$$

has a unique positive solution $v\in C(\overline{D})$ satisfying $V\Psi(\cdot,v)\neq\infty$. Moreover, u is a solution of (3.7) if and only if $u=v+H_f^D$. This completes the proof. ♢

Corollary 3.9 *Let $\varphi:D\times(0,\infty)\to[0,\infty)$ be a measurable function satisfying H1-H3 and f be a nonnegative continuous function on ∂D. Then the nonlinear Dirichlet problem*

$$\begin{gathered}\Delta u+\varphi(.,u)=0, \quad \textit{in } D \textit{ (in the weak sense)}\\ u_{/\partial D}=f,\end{gathered} \tag{3.8}$$

has a unique positive bounded solution $u\in C(\overline{D})$ satisfying $V\varphi(\cdot,u)\neq\infty$.

4 Estimates on solutions

In this section, we give some estimates on the solutions of (1.3) given by (3.5) and (3.6). We denote by $\beta=\inf_{\lambda>0}\{\lambda+\|V\varphi(\cdot,\lambda)\|_\infty\}$ and we remark that if H3 is satisfied then $\beta>0$.

Theorem 4.1 *Let $\mu>0$ and u be the solution of problem (3.3) given by (3.5). Then for each $x\in\overline{D}$, we have*

$$\mu Log|x|\leq u(x)\leq\mu Log|x|+\min\Big(\beta,V\varphi\left(.,\mu Log|\cdot|\right)(x)\Big).$$

Proof. For each $\lambda>0$ and each $x\in\overline{D}$, we have

$$\mu Log|x|\leq u(x)\leq u_\lambda(x)\leq\mu Log|x|+\lambda+\|V\varphi(\cdot,\lambda)\|_\infty,$$

where u_λ is the solution of the problem (3.3). Thus

$$\mu Log|x|\leq u(x)\leq\mu Log|x|+\beta\ ,\ \forall\, x\in\overline{D}.$$

Since φ is non-increasing with respect to the second variable, from (3.5) we obtain

$$\mu Log|x| \le u(x) \le \mu Log|x| + \int_D G(x,y)\varphi(y, \mu Log|y|)dy\,,\ \forall x \in \overline{D}.$$

Which completes the proof. $\diamondsuit$

Remark 4.1 Let $\varepsilon > 0$, sufficiently small, $D_\varepsilon = \{x \in \mathbb{R}^2, 1 < |x| \le 1+\varepsilon\}$ and u be the solution of (3.3) given by (3.5). If φ satisfies

$$\sup_{x\in D_\varepsilon} \int_D \frac{1}{|x-y|}\varphi(y, \mu Log|y|)dy < \infty,$$

then there exists a constant $c > 0$ such that for each $x \in \overline{D}$,

$$\mu Log|x| \le u(x) \le (\mu + c)Log|x|.$$

Indeed, there exists $C > 0$ such that for every x, y in D, we have

$$G(x,y) \le C\frac{(|x|-1)\wedge(|y|-1)}{|x-y|}.$$

Hence, for each $x \in D_\varepsilon$

$$\begin{aligned} u(x) &\le \mu Log|x| + V(\varphi(\cdot, \mu Log|\cdot|))(x) \\ &\le \mu Log|x| + C\Big(\int_D \frac{\varphi(y, \mu Log|y|)}{|x-y|}dy\Big)(|x|-1) \\ &\le \mu Log|x| + C\Big(\sup_{z\in D_\varepsilon}\int_D \frac{\varphi(y, \mu Log|y|)}{|z-y|}\Big)(1+\varepsilon)Log|x| \\ &= \mu Log|x| + C_1 Log|x|. \end{aligned}$$

Moreover,

$$\forall\, x \in \overline{D}\backslash D_\varepsilon,\ \ u(x) \le \mu Log|x| + \beta \le \mu Log|x| + \frac{\beta}{Log(1+\varepsilon)}Log|x|.$$

Consequently, for each $x \in \overline{D}$

$$\mu Log|x| \le u(x) \le \mu Log|x| + \max(C_1, \frac{\beta}{Log(1+\varepsilon)})Log|x| = (\mu + c)Log|x|.$$

Example 4.1 Let u be the positive solution of (3.3) given by (3.5). For $r \in [1, \infty)$, we denote by $\phi(r, \cdot) = \sup_{|x|=r}\varphi(x, \cdot)$. If

$$\int_1^\infty r\phi(r, \mu Log r)dr < \infty,$$

then there exists $c > 0$ such that for every $x \in \overline{D}$,

$$\mu Log|x| \leq u(x) \leq (\mu + c)Log|x|.$$

Indeed, by Theorem 4.1, we have for each $x \in \overline{D}$,

$$\begin{aligned}
\mu Log|x| \leq u(x) &\leq \mu Log|x| + \int_D G(x,y)\varphi(y, \mu Log|y|)dy \\
&\leq \mu Log|x| + \int_1^\infty rLog(|x| \wedge r)\phi(r, \mu Logr)dr \\
&\leq \mu Log|x| + (\int_1^\infty r\phi(r, \mu Logr)dr)Log|x| \\
&= (\mu + c)Log|x|.
\end{aligned}$$

Theorem 4.2 *Let $u \in C(\overline{D})$ be the unique positive bounded solution of (1.3). Then there exists $c > 0$ such that*

$$c(1 - \frac{1}{|x|}) \leq u(x) \leq \min\Big(\beta, V\varphi\Big(., c(1 - \frac{1}{|\cdot|})\Big)(x)\Big)\ ,\ \forall\, x \in \overline{D}.$$

Proof. As it can be seen in the proof of Theorem 3.7, we have

$$V\varphi(\cdot, \beta)(x) \leq u(x) \leq \beta\ ,\ \forall\, x \in D.$$

On the other hand, from Lemma 2.1, we have

$$\frac{1}{2\pi}(1 - \frac{1}{|x|})\Big(\int_D (1 - \frac{1}{|y|})\varphi(y, \beta)dy\Big) \leq V\varphi(\cdot, \beta)(x)\ ,\ \forall\, x \in D.$$

Hence, the lower bound inequality follows from H2 and Corollary 2.8, with

$$c = \frac{1}{2\pi}\int_D (1 - \frac{1}{|y|})\varphi(y, \beta)dy.$$

Now, since φ is non-increasing with respect to the second variable, we get by using (3.6) that

$$u(x) \leq \int_D G(x,y)\varphi\Big(y, c(1 - \frac{1}{|y|})\Big)\, dy.$$

This completes the proof.

Remark 4.2 Let $\varepsilon > 0$, sufficiently small, $D_\varepsilon = \{x \in \mathbb{R}^2, 1 < |x| \leq 1 + \varepsilon\}$ and u be the unique positive bounded solution of (1.3). If φ satisfies

$$\sup_{z \in D_\varepsilon} \int_D \frac{1}{|z - y|}\varphi\Big(y, c(1 - \frac{1}{|y|})\Big)dy < \infty\ ,\ \forall\, c > 0,$$

then there exists a constant $C > 0$ such that for each $x \in \overline{D}$,

$$\frac{1}{C}\big(1 - \frac{1}{|x|}\big) \leq u(x) \leq C\big(1 - \frac{1}{|x|}\big).$$

Example 4.2 Let u be the positive bounded solution of (1.3) given by (3.6) and ϕ defined as in Example 4.1. If

$$\forall c>0,\quad \int_1^\infty r\phi\Big(r,c(1-\frac{1}{r})\Big)\,dr<\infty,$$

then there exists $C>0$ such that

$$\frac{1}{C}\big(1-\frac{1}{|x|}\big)\le u(x)\le C\big(1-\frac{1}{|x|}\big)\ ,\ \forall x\in\overline{D}.$$

Indeed, for $1\le|x|\le 2$, we have

$$\begin{aligned} c\big(1-\frac{1}{|x|}\big)\le u(x) &\le \int_D G(x,y)\varphi\Big(y,c(1-\frac{1}{|y|})\Big)\,dy\\ &\le \int_1^\infty rLog(|x|\wedge r)\phi\Big(r,c(1-\frac{1}{r})\Big)\,dr\\ &\le Log|x|\int_1^\infty r\phi\Big(r,c(1-\frac{1}{r})\Big)\,dr\\ &\le \big(1-\frac{1}{|x|}\big)\Big(2\int_1^\infty r\phi\big(r,c(1-\frac{1}{r})\big)\,dr\Big)\end{aligned}$$

Moreover, for $|x|>2$,

$$c\big(1-\frac{1}{|x|}\big)\le u(x)\le\beta\le 2\beta\big(1-\frac{1}{|x|}\big).$$

This gives the desired estimates.

We close this paper by giving an other comparison result for the solutions of the problem (1.3), in the case of the special nonlinearity $\varphi(x,t)=q(x)f(t)$. The following hypotheses on q and f are adopted.

- $f:(0,\infty)\to(0,\infty)$ is a continuously differentiable non-increasing function.
- $q\in C^\alpha_{\rm loc}(D)\cap K^\infty(D)$, $0<\alpha<1$, is a nontrivial nonnegative function in D.

We define the function F in $[0,\infty)$ by $F(t)=\int_0^t 1/f(s)ds$. From the hypotheses on f, we note that the function F is a bijection from $[0,\infty)$ to itself. Then, we have the following statement.

Theorem 4.3 *Let $u\in C(\overline{D})$ be the positive solution of the problem*

$$\begin{gathered}\Delta u+q(x)f(u)=0\quad \text{in } D \text{ (in the weak sense)}\\ u\,|_{\partial D}=0,\quad \lim_{|x|\to\infty}\tfrac{u(x)}{Log|x|}=\mu>0,\end{gathered}\tag{4.1}$$

such that $V\big(qf(u)\big)\neq\infty$. Then

$$V\big(qf(\beta+\mu Log|\cdot|)\big)(x)+\mu Log|x|\le u(x)\le F^{-1}(Vq(x))+\mu Log|x|\ ,\ \forall\, x\in\overline{D}.$$

Proof. Since $u \leq \beta + \mu Log|\cdot|$ in D and f is nonincreasing with respect to the second variable, we deduce that for each x in D,

$$V(qf(\beta + \mu Log|\cdot|))(x) + \mu Log|x| \leq u(x) = \mu Log|x| + \int_D G(x,y)q(y)f(u(y))dy.$$

To show the upper estimate, we consider $\varepsilon > 0$ and define the function v_ε in D by $v_\varepsilon(x) = F\big(u(x) - \mu Log|x|\big) - Vq(x) - \varepsilon Log|x|$. Then, $v_\varepsilon \in C^2(D)$ and

$$\begin{aligned} \Delta v_\varepsilon(x) &= \frac{1}{f\big(u(x) - \mu Log|x|\big)}\Delta u(x) + q(x) \\ &\quad - \frac{f'\big(u(x) - \mu Log|x|\big)}{f^2\big(u(x) - \mu Log|x|\big)}\|\nabla(u - \mu Log|\cdot|)(x)\|^2. \end{aligned}$$

Thus, $\Delta v_\varepsilon \geq 0$. Moreover, since $0 \leq u - \mu Log|.| \leq \beta$, Vq is bounded in D and $\lim_{x\to\partial D} Vq(x) = 0$, we get $v_{\varepsilon/\partial D} = 0$ and $\lim_{|x|\to\infty} v_\varepsilon(x) \leq 0$. Hence, by [3, p.465] we deduce that $v_\varepsilon \leq 0$ in D. Since ε is arbitrary, we get the upper inequality. ◇

Using the same arguments as in the proof above, we can prove the following theorem.

Theorem 4.4 *Let $u \in C(\overline{D})$ be the positive bounded solution of the problem(1.3) with $\varphi(.,u) = q(x)f(u)$, such that $V\big(qf(u)\big) \neq \infty$. Then we have*

$$f(\beta)Vq(x) \leq u(x) \leq F^{-1}\big(Vq(x)\big)\,,\ \forall x \in \overline{D}.$$

Corollary 4.5 *Let $\varphi : D \times (0,\infty) \to [0,\infty)$ be a measurable function satisfying H1. Further, assume that φ satisfies*

$$\varphi(x,t) \leq q(x)f(t)\,,\ \forall\ (x,t) \in D \times (0,\infty).$$

Let $\mu > 0$ and u be the solution of the problem (3.3). Then u satisfies

$$\mu Log|x| \leq u(x) \leq \mu Log|x| + F^{-1}(Vq(x))\,,\ \forall\ x \in \overline{D}. \tag{4.2}$$

Proof. Let v be the solution of the problem (4.1). Then, by Corollary 3.8, we deduce that $u \leq v$ in D. Which together with Theorems 4.1 and 4.3, give (4.2).

Example 4.3 Let $\gamma > 0$. Then the problem

$$\begin{gathered} \Delta u + q(x)u^{-\gamma} = 0, \quad \text{in } D \\ u\,|_{\partial D} = 0\,, \quad \lim_{|x|\to\infty} \frac{u(x)}{Log|x|} = \mu \geq 0 \end{gathered}$$

has a positive solution $u \in C(\overline{D})$ satisfying

$$V\big((\beta + \mu Log|\cdot|)^{-\gamma}q\big)(x) \leq u(x) - \mu Log|x| \leq \big[(\gamma+1)Vq(x)\big]^{\frac{1}{1+\gamma}}\,,\ \forall x \in \overline{D}.$$

Acknowledgements. I want to thank Professor Habib Maâgli, my adviser, for his valuable help. I also want to thank the anonymous referee for his/her suggestions.

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