Electronic Journal of Differential Equations, Vol. 2001(2001), No. 54, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ASYMPTOTIC BEHAVIOUR FOR SCHRÖDINGER EQUATIONS WITH A QUADRATIC NONLINEARITY IN ONE-SPACE DIMENSION

NAKAO HAYASHI & PAVEL I. NAUMKIN

Abstract. We consider the Cauchy problem for the Schrödinger equation with a quadratic nonlinearity in one space dimension

$$iu_t + \frac{1}{2}u_{xx} = t^{-\alpha}|u_x|^2, \quad u(0,x) = u_0(x),$$

where $\alpha \in (0,1)$. From the heuristic point of view, solutions to this problem should have a quasilinear character when $\alpha \in (1/2,1)$. We show in this paper that the solutions do not have a quasilinear character for all $\alpha \in (0,1)$ due to the special structure of the nonlinear term. We also prove that for $\alpha \in [1/2,1)$ if the initial data $u_0 \in H^{3,0} \cap H^{2,2}$ are small, then the solution has a slow time decay such as $t^{-\alpha/2}$. For $\alpha \in (0,1/2)$, if we assume that the initial data u_0 are analytic and small, then the same time decay occurs.

1. Introduction

In this paper we consider the Schrödinger equation, with a quadratic derivative term,

$$\begin{aligned} \mathcal{L}u &= t^{-\alpha}|u_x|^2, \quad t,x \in \mathbb{R} \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$, and $\alpha \in (0,1)$. The Cauchy problem for Schrödinger equations with a cubic derivative term was studied in [9]. There the authors considered

$$\begin{aligned} \mathcal{L}u &= t^{1-\delta}F(u,u_x), \quad t,x \in \mathbb{R} \\ u(0,x) &= \epsilon u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where $0 < \delta < 1$, ϵ is a sufficiently small constant, and the nonlinear interaction term F consists of cubic nonlinearities.

$$F(u,u_x) = \lambda_1|u|^2u + i\lambda_2|u|^2u_x + i\lambda_3u^2\bar{u}_x + \lambda_4|u_x|^2u + \lambda_5\bar{u}u_x^2 + i\lambda_6|u_x|^2u_x,$$

where the coefficients $\lambda_1, \lambda_6 \in \mathbb{R}$, $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{C}$, $\lambda_2 - \lambda_3 \in \mathbb{R}$, $\lambda_4 - \lambda_5 \in \mathbb{R}$. In [9], the authors found a time decay estimate for the solutions of this problem,

$$\|u(t)\|_\infty \le C|t|^{-1/2}. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 35Q55, 74G10, 74G25.
Key words and phrases. Schrödinger equation, large time behaviour, quadratic nonlinearity .

Submitted May 22, 2001. Published July 25, 2001.

The same result is also true for the case $\delta > 1$. From the heuristic point of view problem (1.1) corresponds to problem (1.2), when $\delta = \alpha + \frac{1}{2}$. Therefore it is natural to make a conjecture that the solutions of (1.1) also have the decay property (1.3). However, as we will show in the present paper, due to the special oscillating structure of the nonlinear term, for $\alpha \in (0,1)$ the asymptotic behavior of solutions to (1.1) do not obey the estimate (1.3). Our result stated below depends on the structure of nonlinearity which appears in the identity

$$(\mathcal{F}\mathcal{U}(-t)|u_x|^2)(t,\xi) = (2\pi)^{1/2}\int e^{-it\xi\eta}(\overline{\mathcal{F}\mathcal{U}(-t)u_x(t,\eta)})(\mathcal{F}\mathcal{U}(-t)u_x)(t,\xi+\eta)d\eta.$$

In the cases of u_x^2 and $\bar{u}_x^2$ we have

$$\begin{aligned}&(\mathcal{F}\mathcal{U}(-t)u_x^2)(t,\xi)\\ &= (2\pi)^{1/2}e^{\frac{i}{4}t\xi^2}\int e^{-ity^2}(\mathcal{F}\mathcal{U}(-t)u_x)(t,\frac{\xi}{2}-y)(\mathcal{F}\mathcal{U}(-t)u_x)(t,\frac{\xi}{2}+y)dy\end{aligned}$$

and

$$\begin{aligned}&(\mathcal{F}\mathcal{U}(-t)\bar{u}^2)(t,\xi)\\ &= (2\pi)^{1/2}e^{\frac{3i}{4}t\xi^2}\int e^{ity^2}(\overline{\mathcal{F}\mathcal{U}(-t)u_x})(t,\frac{\xi}{2}-y)(\overline{\mathcal{F}\mathcal{U}(-t)u_x})(t,\frac{\xi}{2}+y)dy,\end{aligned}$$

where $\mathcal{U}(t)$ is the linear Schrödinger evolution group

$$\mathcal{U}(t)\phi = \frac{1}{\sqrt{2\pi it}}\int e^{\frac{i}{2t}(x-y)^2}\phi(y)dy = \mathcal{F}^{-1}e^{-\frac{it}{2}\xi^2}\mathcal{F}\phi,$$

$\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\phi(x)dx$ denotes the Fourier transform of the function ϕ. The oscillating function $e^{\pm ity^2}$ yields an additional time decay term through integration by parts. However, the oscillating function $e^{\pm it\xi y}$ does not give an additional time decay uniformly with respect to ξ. This is the main reason why we do not have estimate (1.3) for solutions of (1.1). In [6] we proved (1.3) for solutions of the Cauchy problem

$$\mathcal{L}u = \lambda(\overline{u}_x)^2 + \mu u_x^2, \quad \text{with } \lambda,\mu \in \mathbb{C}.$$

However, the nonlinearity $|u_x|^2$ was out of our scope. In the present paper we intend to fill up this gap studying the case of quadratic nonlinearity $t^{-\alpha}|u_x|^2$. The methods developed for the nonlinear Schrödinger equations with quadratic nonlinearities u_x^2, $|u_x|^2$ and $\overline{u}_x^2$ can be applied also to the study of the large time asymptotic behavior for other quadratic nonlinear equations, such as Benjamin-Ono and Korteweg-de Vries equations (in paper [8], mBO equation was reduced to the cubic nonlinear Schrödinger equation). In paper [2], Cohn used the method of normal forms of Shatah [11] to study the nonlinear Schrödinger equations with quadratic nonlinearity $\overline{u}_x^2$ and showed that the solution exists on $[0,T)$ with T bounded from below by $C\varepsilon^{-6}$, where ε is the size of the data in some Sobolev norm. In paper [10] the nonlinearity u_x^2 was studied by the Hopf-Cole transformation. The L^2-estimate of solutions involving the operator $\mathcal{J} = x + it\partial_x$ plays a crucial role in the large time asymptotic behavior of solutions. However the nonlinearity $\mathcal{N}(u)$ under consideration does not posses a self-conjugate structure $e^{i\omega}\mathcal{N}(u) = \mathcal{N}(e^{i\omega}u)$ for all $\omega \in \mathbb{R}$, therefore we can not use the operator $\mathcal{J} = x + it\partial_x$ directly in (1.1). To overcome these obstacles we use the method developed in [7] and apply systematically the operator $\mathcal{I} = x\partial_x + 2t\partial_t$.

We now state our strategy for the proof. If we put $v = u_x$. Then the problem is written as

$$\mathcal{L}v = t^{-\alpha}\partial_x |v|^2, \quad t, x \in \mathbb{R}.$$

By the identity

$$\partial_x \mathcal{J}|v|^2 = \partial_x(\overline{v}\mathcal{J}v + it\overline{v}_x v) = \overline{v}\mathcal{J}\partial_x v + 2\overline{v}_x \mathcal{J}v - v\overline{\mathcal{J}\partial_x v}$$

we have

$$\mathcal{L}\mathcal{J}v = t^{-\alpha}\mathcal{J}\partial_x|v|^2 = t^{-\alpha}(-|v|^2 + \overline{v}\mathcal{J}\partial_x v + 2\overline{v}_x\mathcal{J}v - v\overline{\mathcal{J}\partial_x v}).$$

Therefore, the operator $\mathcal{J}$ acts on this problem also. Thus global existence in time of small solutions to the problem can be proved for $\alpha \in (1/2, 1)$ and the derivative u_x should have the same asymptotic behaviour as the solutions to the corresponding linear problem (along with time-decay estimate (1.3)). Combining this fact and the identity (1) we prove the time decay of solutions. Roughly speaking, we show there exists a constant c and a positive constant γ such that

$$|u(t, \sqrt{t}) - ct^{-\alpha/2}| \le Ct^{-(\alpha/2)-\gamma}.$$

In the case of $\alpha \in (0, 1/2)$ we use the fact that

$$\partial_x|u|^2 = \frac{1}{it}(\overline{u}\mathcal{J}u - u\overline{\mathcal{J}u})$$

which implies that usual derivative yields an additional time decay, in particular, the fractional derivative $|\partial_x|^\beta$ gives us an additional time decay like $t^{-\beta}$ (see Lemma 2.4 below). However we have the derivative loss on the nonlinear term which requires us to use some analytic function space.

To state our results we need some notation. We denote the inverse Fourier transformation by $\mathcal{F}^{-1}\phi = \check{\phi} = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\phi(\xi)d\xi$. We essentially use the estimates of the operators $\mathcal{J} = x + it\partial_x = \mathcal{U}(t)x\mathcal{U}(-t) = itM(t)\partial_x\overline{M}(t)$ and $\mathcal{I} = x\partial_x + 2t\partial_t$, $M = e^{(ix^2)/(2t)}$. Note that the relation $\mathcal{J}\partial_x = \mathcal{I} + 2it\mathcal{L}$ is valid, where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$ and $\mathcal{U}(t) = M(t)\mathcal{D}(t)\mathcal{F}M(t)$, $\mathcal{D}(t)$ is the dilation operator defined by $(\mathcal{D}(t)\psi)(x) = (1/\sqrt{it})\psi(x/t)$. Then since $\mathcal{D}^{-1}(t) = i\mathcal{D}(1/t)$ we have $\mathcal{U}(-t) = \overline{M}\mathcal{F}^{-1}\mathcal{D}^{-1}(t)\overline{M} = i\overline{M}\mathcal{F}^{-1}\mathcal{D}(1/t)\overline{M}$.

We denote the usual Lebesgue space $L^p = \{\phi \in \mathbf{S}'; \|\phi\|_p < \infty\}$, where the norm $\|\phi\|_p = (\int_{\mathbb{R}}|\phi(x)|^p dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\,\{|\phi(x)|; x \in \mathbb{R}\}$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2$. Weighted Sobolev space is

$$H_p^{m,k} = \{\phi \in \mathbf{S}' : \|\phi\|_{m,k,p} \equiv \|\langle x\rangle^k \langle i\partial_x\rangle^m \phi\|_p < \infty\},$$

$m, k \in \mathbb{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1 + x^2}$. The fractional derivative $|\partial_x|^\alpha$, $\alpha \in (0, 1)$ is equal to

$$|\partial_x|^\alpha\phi = \mathcal{F}^{-1}|\xi|^\alpha\mathcal{F}\phi = C\int_{\mathbb{R}}(\phi(x + z) - \phi(x))\frac{dz}{|z|^{1+\alpha}}.$$

We denote also for simplicity $H^{m,k} = H_2^{m,k}$ and the norm $\|\phi\|_{m,k} = \|\phi\|_{m,k,2}$. Different positive constants are denoted by the same letter C. Denote $\Phi(x) = \int e^{-\frac{i}{2}(\xi - x)^2}|\xi|^{\alpha-1}d\xi$.

Now we state the main results of this paper.

Theorem 1.1. *Let $\alpha \in [1/2, 1)$. We assume that the initial data $u_0 \in H^{3,0} \cap H^{2,2}$ and the norm $\|u_0\|_{3,0} + \|u_0\|_{2,2}$ is sufficiently small. Then there exists a unique global solution u of the Cauchy problem (1.1) such that $u \in C(\mathbb{R}; H^{3,0})$. Moreover* *there exist unique constant B and functions P, Q such that $|\xi|^{1-\alpha}P(\xi) \in \mathbf{L}^\infty(\mathbb{R})$, $|\xi|^{1-\alpha}Q(\xi) \in \mathbf{L}^\infty(\mathbb{R})$ and the following asymptotic statement is valid*

$$u(t,x) = Be^{\frac{ix^2}{2t}} t^{-\frac{\alpha}{2}} \Phi(\frac{x}{\sqrt{t}}) + O(t^{-\frac{\alpha}{2}-\gamma}(\langle \frac{x}{\sqrt{t}} \rangle^{\alpha-1} + \langle \frac{x}{\sqrt{t}} \rangle^{-\alpha})) \tag{1.4}$$

for all $t \geq 1$, uniformly in $|x| \leq t^{1-\rho}$, and

$$u(t,x) = t^{-\alpha} P(\frac{x}{t}) + e^{\frac{ix^2}{2t}} \frac{1}{\sqrt{t}} Q(\frac{x}{t}) + O(t^{-\alpha-\gamma} + t^{-\frac{1}{2}-\gamma} \langle \frac{x}{t} \rangle^{-\alpha}) \tag{1.5}$$

for all $t \geq 1$, uniformly in $|x| \geq t^{1-\rho}$, where $\rho, \gamma > 0$ are small.

In the case $\alpha \in (0, 1/2)$ we have to assume that the initial data are analytic. Denote

$$\mathbf{A}_0 = \{\phi \in L^2 : \|\phi\|_{\mathbf{A}_0} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \||\partial_x|^{\frac{1}{2}-\alpha}(x\partial_x)^n \phi\|_{1,0} < \infty\}.$$

Theorem 1.2. *Let $\alpha \in (0, 1/2)$. We assume that the initial data $u_0 \in \mathbf{A}$ and the norm $\|u_0\|_{\mathbf{A}_0}$ is sufficiently small. Then there exists a unique global solution u of the Cauchy problem (1.1) such that $u \in \mathbf{C}(\mathbb{R}; H^{1,0})$. Moreover there exist unique constant B and functions P, Q such that asymptotics (1.4) and (1.5) are valid.*

Remark 1.1. In the region $|x| = t^{1-\rho}$ asymptotics (1.4) coincides with (1.5).

In Section 2 we prove some preliminary estimates. In Section 3 we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2.

2. Preliminaries

First we prove some time decay estimates.

Lemma 2.1. *We have the estimate*

$$\|u_x\|_\infty \leq Ct^{-1/2}\|\mathcal{F}\mathcal{U}(-t)u_x\|_\infty + Ct^{-\frac{1+\beta-\gamma}{2}}(\|u_x\| + \||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x u\|),$$

for all $t > 0$, where $\beta \in (0, \frac{1}{2}]$, $\gamma \in (0, \beta)$.

Proof. Denote $w = \mathcal{U}(-t)u_x$. Then since $\mathcal{U}(t) = M\mathcal{D}\mathcal{F}M$, where $M = e^{\frac{ix^2}{2t}}$, $\mathcal{D}\phi = \frac{1}{\sqrt{it}}\phi(\frac{x}{t})$ is the dilation operator, $\mathcal{J} = x + it\partial_x = \mathcal{U}(t)x\mathcal{U}(-t)$, we get

$$u_x = \mathcal{U}(t)w = M\mathcal{D}\mathcal{F}w + M\mathcal{D}\mathcal{F}(M-1)w$$

and by virtue of the Hölder inequality and Sobolev embedding theorem $\|\phi\|_p \leq C\||\partial_x|^{\frac{1}{2}-\frac{1}{p}}\phi\|$ if $2 \leq p < \infty$, we have

$$\begin{aligned}
&\|M\mathcal{D}\mathcal{F}(M-1)w\|_\infty \\
&\leq Ct^{-1/2}\|\mathcal{F}(M-1)w\|_\infty \leq Ct^{-1/2}\|(M-1)w\|_1 \leq Ct^{-\frac{1+\beta-\gamma}{2}}\||x|^{\beta-\gamma}w\|_1 \\
&\leq Ct^{-\frac{1+\beta-\gamma}{2}}(\|w\| + \|xw\|_{\frac{1}{\beta}}) \leq Ct^{-\frac{1+\beta-\gamma}{2}}(\|w\| + \||\partial_x|^{\frac{1}{2}-\beta}xw\|) \\
&\leq Ct^{-\frac{1+\beta-\gamma}{2}}(\|u_x\| + \||\partial_x|^{\frac{1}{2}-\beta}x\mathcal{U}(-t)u_x\|) \\
&\leq Ct^{-\frac{1+\beta-\gamma}{2}}(\|u_x\| + \||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x u\|),
\end{aligned}$$

therefore the result of the lemma follows. Lemma 2.1 is proved. $\square$

Denote

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} t^{\alpha}\langle t\rangle^{1-2\gamma}\|\partial_t\phi\|_{0,1,\infty} + \sup_{t>0} t^{-\gamma}\||\xi|^{\frac{1}{2}-\beta}\partial_\xi\phi\| + \sup_{t>0}\|\phi\|_{0,1,\infty},$$

where $\beta \in (0, \frac{1}{2}]$, $\gamma > 0$ is small. In the next lemma we obtain the asymptotic representation as $\xi \to 0$ for the integral

$$I = \int_0^t \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}\phi_1(\tau, \xi+\eta)\phi_2(\tau,\eta)d\eta$$

which corresponds to the identity (1).

Lemma 2.2. *If $\phi_l \in \mathbf{Y}$, $l = 1, 2$, then we have*

$$\begin{aligned} I &= \Gamma(1-\alpha)|\xi|^{\alpha-1}(\sin(\frac{\pi\alpha}{2})\int \phi_1(t,\eta)\phi_2(t,\eta)|\xi|^{\alpha-1}d\eta \\ &\quad + i\operatorname{sign}\xi\cos(\frac{\pi\alpha}{2})\int \phi_1(t,\eta)\phi_2(t,\eta)|\eta|^{\alpha-1}\operatorname{sign}\eta\, d\eta) \\ &\quad + O(t^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}). \end{aligned}$$

for all $|\xi| \le t^{-\mu}$, $t \ge 1$, where $\mu = \frac{3\gamma}{\alpha^2}$, $\gamma > 0$ is small.

Proof. We write $I = \sum_{l=1}^4 I_l$, where

$$\begin{aligned} I_1 &= \int_0^{t^\nu/|\xi|} \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}\phi_1(t,\eta)\phi_2(t,\eta)d\eta, \\ I_2 &= \int_{t^\nu/|\xi|}^t \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}\phi_1(\tau,\xi+\eta)\phi_2(\tau,\eta)d\eta, \\ I_3 &= \int_0^{t^\nu/|\xi|} \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}(\phi_1(\tau,\eta)\phi_2(\tau,\eta) - \phi_1(t,\eta)\phi_2(t,\eta))d\eta \\ I_4 &= \int_0^{t^\nu/|\xi|} \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}(\phi_1(\tau,\xi+\eta) - \phi_1(\tau,\eta))\phi_2(\tau,\eta)d\eta, \end{aligned}$$

where $\nu = 2\gamma/\alpha$. If $\tau|\xi| \ge 1$, we integrate by parts with respect to η to obtain

$$\begin{aligned} &|\int e^{-i\tau\xi\eta}\phi_1(t,x+\eta)\phi_2(t,\eta)d\eta| \\ &\le \langle\tau\xi\rangle^{-1}|\int e^{-i\tau\xi\eta}\partial_\eta(\phi_1(t,x+\eta)\phi_2(t,\eta))d\eta| \\ &\le C\langle\tau\xi\rangle^{-1}t^\gamma\sum_{l=1}^2\|\phi_{3-l}\|_\infty \sup_{t>0} t^{-\gamma}\||\xi|^{\frac{1}{2}-\gamma}\partial_\xi\phi_l\| \le C\langle\tau\xi\rangle^{-1}t^\gamma\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}, \end{aligned}$$

hence changing $\tau|\xi| = z$ we obtain

$$\begin{aligned} &|\int_{t^\nu/|\xi|}^\infty \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}\phi_1(t,x+\eta)\phi_2(t,\eta)d\eta| \\ &\le Ct^\gamma\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}\int_{t^\nu/|\xi|}^\infty \langle\tau\xi\rangle^{-1}\tau^{-\alpha}d\tau \le Ct^\gamma|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}\int_{t^\nu}^\infty z^{-\alpha-1}dz \\ &\le C|\xi|^{\alpha-1}t^{\gamma-\alpha\nu}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}} \le Ct^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}. \end{aligned}$$

Since

$$\begin{aligned}\int_0^\infty \tau^{-\alpha} e^{i\tau\xi\eta} d\tau &= \int_0^\infty \tau^{-\alpha}\cos(\tau\xi\eta)d\tau + i\int_0^\infty \tau^{-\alpha}\sin(\tau\xi\eta)d\tau \\ &= \Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})|\xi\eta|^{\alpha-1} \\ &\quad + i\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})|\xi\eta|^{\alpha-1}\operatorname{sign}(\xi\eta)\end{aligned}$$

(see [1]), we find

$$\begin{aligned} I_1 &= \int_0^\infty \tau^{-\alpha} d\tau \int e^{-i\tau\xi\eta}\phi_1(t,\eta)\phi_2(t,\eta)d\eta \\ &\quad - \int_{t^\nu/|\xi|}^\infty \tau^{-\alpha} d\tau \int e^{-i\tau\xi\eta}\phi_1(t,\eta)\phi_2(t,\eta)d\eta \\ &= \Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})|\xi|^{\alpha-1}\int \phi_1(t,\eta)\phi_2(t,\eta)|\eta|^{\alpha-1}d\eta \\ &\quad + i\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})|\xi|^{\alpha-1}\int \operatorname{sign}(\xi\eta)\phi_1(t,\eta)\phi_2(t,\eta)|\eta|^{\alpha-1}d\eta \\ &\quad + O(t^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}).\end{aligned}$$

In the same manner we obtain

$$\begin{aligned} &\Big|\int_{t^\nu/|\xi|}^t \tau^{-\alpha} d\tau \int e^{-i\tau\xi\eta}\phi_1(\tau,x+\eta)\phi_2(\tau,\eta)d\eta\Big| \\ &\le Ct^\gamma\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}\int_{t^\nu/|\xi|}^t \langle\tau\xi\rangle^{-1}\tau^{-\alpha}d\tau \le Ct^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}},\end{aligned}$$

hence

$$|I_2| \le Ct^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}.$$

To estimate I_3 we note that

$$\|\phi_l(t,\xi)-\phi_l(\tau,\xi)\|_{0,1,\infty} = \|\int_\tau^t \partial_\tau\phi_l(\tau,\xi)d\tau\|_{0,1,\infty} = O(\tau^{2\gamma-\alpha}\|\phi_l\|_{\mathbf{Y}})$$

which implies

$$\begin{aligned} |I_3| &= \Big|\int_0^{t^\nu/|\xi|}\tau^{-\alpha}d\tau\int e^{-i\tau\xi\eta}(\phi_1(\tau,\eta)\phi_2(\tau,\eta)-\phi_1(t,\eta)\phi_2(t,\eta))d\eta\Big| \\ &\le C\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}\Big|\int_0^{t^\nu/|\xi|}\tau^{2\gamma-2\alpha}d\tau\Big| \le Ct^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}\end{aligned}$$

since $\mu\alpha \ge \gamma+\nu$ and $|\xi| \le t^{-\mu}$. Now using the estimate

$$\begin{aligned}\|\langle\eta\rangle^{-1}(\phi(t,\xi+\eta)-\phi(t,\eta))\|_1 &= \|\langle\eta\rangle^{-1}\int_0^\xi \partial_y\phi(t,y+\eta)dy\|_1 \\ &\le C|\xi|\||\xi|^{\frac12-\beta}\partial_\xi\phi\| \le Ct^\gamma|\xi|\|\phi\|_{\mathbf{Y}}\end{aligned}$$

for all $|\xi| \le 1$, we get

$$|I_4| \le C\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}|\xi|\int_0^{t^\nu/|\xi|}\tau^{\gamma-\alpha}d\tau \le Ct^{-\gamma}|\xi|^{\alpha-1}\|\phi_1\|_{\mathbf{Y}}\|\phi_2\|_{\mathbf{Y}}$$

since $\mu(1-\gamma) \ge \gamma+\nu$ and $|\xi| \le t^{-\mu}$. Lemma 2.2 is proved. □

In the next lemma we consider the asymptotic behaviour of the integral

$$I(t,x) = \int e^{-\frac{it}{2}(\xi - \frac{x}{t})^2} f(t,\xi) d\xi$$

as $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$. Define $\Phi(x) = \int e^{-\frac{i}{2}(\xi - x)^2} |\xi|^{\alpha-1} d\xi$. Note that

$$\Phi(x) = O(\langle x \rangle^{-\alpha} + \langle x \rangle^{\alpha-1})$$

as $|x| \to \infty$. Let γ be a small positive number and

$$\beta = \min(1/2, \alpha) - \gamma, \quad \mu = 3\gamma/\alpha^2,$$
$$\rho = \frac{5\gamma}{\alpha^2(1-\alpha)}, \quad \theta = \frac{6\gamma}{\alpha^2(1-\alpha)^2}, \quad \delta = \theta + \gamma.$$

Lemma 2.3. *Let $\partial_\xi f(t,\xi) = O(|\xi|^{\alpha-2})$ and $f(t,\xi) = t^{1-\alpha}\Psi(t\xi) + O(t^{1-\alpha-\delta})$ for all $|\xi| \le t^{\theta-1}$, $\partial_\xi f(t,\xi) = (\alpha-1)B|\xi|^{\alpha-1}\xi^{-1} + O(t^{-\gamma}|\xi|^{\alpha-2})$ for all $t^{\theta-1} \le |\xi| \le t^{-\mu}$ and $\||\xi|^{\frac{1}{2}-\beta}\xi\partial_\xi f(t,\xi)\| \le Ct^\gamma$, then we have the asymptotic formula*

$$I(t,x) = Bt^{-\frac{\alpha}{2}}\Phi(xt^{-\frac{1}{2}}) + O(t^{-\frac{\alpha}{2}-\gamma}(\langle xt^{-\frac{1}{2}}\rangle^{-\alpha} + \langle xt^{-\frac{1}{2}}\rangle^{\alpha-1}))$$

for all $t \ge 1$ uniformly in $|x| \le t^{1-\rho}$ and

$$I(t,x) = \sqrt{2\pi}t^{-\alpha}e^{-\frac{ix^2}{2t}}\check{\Psi}(\frac{x}{t}) + \frac{\sqrt{\pi}}{\sqrt{it}}f(t,\frac{x}{t}) + O(t^{-\alpha-\gamma} + t^{-\frac{1}{2}-\gamma}\langle xt^{-1}\rangle^{-\alpha})$$

for all $t \ge 1$ uniformly in $|x| \ge t^{1-\rho}$.

Proof. For $x > 0$, we have

$$\begin{aligned}
f(t,\xi) &= f(t,1) + \int_{t^{-\mu}}^{\xi} \partial_\eta f(t,\eta)d\eta + \int_1^{t^{-\mu}} \partial_\eta f(t,\eta)d\eta \\
&= f(t,1) + (\alpha-1)B\int_{t^{-\mu}}^{\xi} |\eta|^{\alpha-2}d\eta + O(t^{-\gamma}\int_{t^{-\mu}}^{\xi} |\eta|^{\alpha-2}d\eta) \\
&\quad + O(\||\xi|^{\frac{1}{2}-\beta}\xi\partial_\xi f(t,\xi)\|(\int_1^{t^{-\mu}} |\xi|^{2\beta-3}d\xi)^{1/2}) \\
&= B|\xi|^{\alpha-1} + O(1 + t^{-\gamma}|\xi|^{\alpha-1} + t^{\mu(1-\beta)+\gamma}) \\
&= B|\xi|^{\alpha-1} + O(t^{-\gamma}|\xi|^{\alpha-1})
\end{aligned}$$

for all $t^{\mu-1} \le |\xi| \le 2t^{-\rho}$ since $\mu(1-\beta) + 2\gamma \le \rho(1-\alpha)$. We make a change of variable of integration $\xi = zt^{-1/2}$, then we have

$$I(t,x) = t^{-1/2}\int e^{-\frac{i}{2}(z-b)^2} f(t, zt^{-1/2})dz,$$

where $b = a\sqrt{t} = x/\sqrt{t}$. First consider the case $|x| \le t^{1-\rho}$, i.e. $b \le t^{\frac{1}{2}-\rho}$. We represent

$$I = Bt^{-\frac{\alpha}{2}}\Phi(b) + R_1 + R_2,$$

where the remainder terms are

$$R_j = t^{-1/2}\int e^{-\frac{i}{2}(z-b)^2}(f(t,zt^{-1/2}) - Bt^{\frac{1-\alpha}{2}}|z|^{\alpha-1})\varphi_j(z)dz,$$

the function $\varphi_1(z) \in \mathbf{C}^1(\mathbb{R}) : \varphi_1(z) = 1$ if $z < b/3$ and $\varphi_1(z) = 0$ if $z > 2b/3$, $\varphi_2(z) = 1 - \varphi_1(z)$. In the remainder term R_1 we integrate by parts via the identity

$$e^{-\frac{i}{2}(z-b)^2} = \frac{1}{1 - iz(z-b)} \frac{d}{dz}(ze^{-\frac{i}{2}(z-b)^2}) \tag{2.1}$$

to get

$$\begin{aligned} |R_1| &\leq Ct^{-\frac{\alpha}{2}-\gamma} \int |z|^{\alpha-1} \langle zb \rangle^{-1} (|\varphi_1| + |z\varphi_1'|) dz \\ &\quad + Ct^{-\frac{\alpha}{2}} \int_{|z| \leq t^{\mu - \frac{1}{2}}} |z|^{\alpha-1} \langle zb \rangle^{-1} dz \\ &\leq Ct^{-\frac{\alpha}{2}-\gamma} \langle b \rangle^{-\alpha} \leq Ct^{-\frac{\alpha}{2}-\gamma} \langle a\sqrt{t} \rangle^{-\alpha}. \end{aligned} \tag{2.2}$$

In the remainder term R_2 we use the identity

$$e^{-\frac{i}{2}(z-b)^2} = \frac{1}{1 - i(z-b)^2} \frac{d}{dz}((z-b)e^{-\frac{i}{2}(z-b)^2}) \tag{2.3}$$

to find

$$\begin{aligned} |R_2| &\leq Ct^{-\frac{\alpha}{2}-\gamma} \int |z|^{\alpha-1} \langle z-b \rangle^{-2} (|\varphi_2| + |z\varphi_2'|) dz \\ &\quad + Ct^{-\frac{\alpha}{2}} \int_{|z| \leq t^{\mu - \frac{1}{2}}} |z|^{\alpha-1} \langle z-b \rangle^{-2} dz \\ &\quad + Ct^{-1/2} \int_{|z| > 2t^{\frac{1}{2}-\rho}} \langle z-b \rangle^{-2} |zt^{-1/2}| |f'(t, zt^{-1/2})| dz \\ &= O(t^{-\frac{\alpha}{2}-\gamma} \langle b \rangle^{\alpha-1}) = O(t^{-\frac{\alpha}{2}-\gamma} \langle a\sqrt{t} \rangle^{\alpha-1}), \end{aligned} \tag{2.4}$$

since

$$\begin{aligned} &\int_{|z| > 2t^{\frac{1}{2}-\rho}} \langle z-b \rangle^{-2} |zt^{-1/2}| |f'(t, zt^{-1/2})| dz \\ &\leq C \| |\xi|^{\frac{1}{2}-\beta} \xi f'(t,\xi) \| \Big(\int_{|z| > 2t^{\frac{1}{2}-\rho}} |zt^{-1/2}|^{2\beta-1} \langle z-b \rangle^{-4} dz \Big)^{1/2} \\ &\leq Ct^{\frac{1-2\beta}{4}} \langle b \rangle^{-1} \| |\xi|^{\frac{1}{2}-\beta} \xi f'(t,\xi) \| \Big(\int_{|z| > 2t^{\frac{1}{2}-\rho}} z^{2\beta-3} dz \Big)^{1/2} \\ &\leq Ct^{-\frac{1}{4}+\rho(1-\beta)} \langle b \rangle^{-1} \| |\xi|^{\frac{1}{2}-\beta} \xi f'(t,\xi) \| \leq Ct^{-\gamma} \langle b \rangle^{-1}. \end{aligned}$$

We consider now the case $|x| > t^{1-\rho}$, i.e. $b > t^{\frac{1}{2}-\rho}$. Then we represent I in the form

$$I = t^{-1/2} \int_{|z| \leq t^{\theta - \frac{1}{2}}} e^{-\frac{i}{2}(z-b)^2} f(t, zt^{-1/2}) dz + \sqrt{\frac{\pi}{it}} f(t,a) + R_3 + R_4,$$

where the remainder terms are

$$R_3 = t^{-1/2} \int_{|z| > t^{\theta - \frac{1}{2}}} e^{-\frac{i}{2}(z-b)^2} f(t, zt^{-1/2}) \varphi_1(z) dz$$

$$R_4 = t^{-1/2} \int e^{-\frac{i}{2}(z-b)^2} (f(t, zt^{-1/2}) - f(t,a)) \varphi_2(z) dz.$$

Consider the integral

$$\begin{aligned} t^{-1/2}\int_{|z|\le t^{\theta-\frac12}} e^{-\frac{i}{2}(z-b)^2} f(t,zt^{-1/2})dz &= \int_{|\xi|\le t^{\theta-1}} e^{-\frac{i}{2}t(\xi-a)^2} f(t,\xi)d\xi \\ &= t^{1-\alpha}\int_{|\xi|\le t^{\theta-1}} e^{-\frac{i}{2}t(\xi-a)^2}\Psi(t\xi)d\xi + O(t^{-\alpha-\gamma}) \\ &= t^{-\alpha}e^{-\frac{ix^2}{2t}}\int_{|y|\le t^{\theta}} e^{iya}\Psi(y)dy + O(t^{-\alpha-\gamma}) \\ &= \sqrt{2\pi}t^{-\alpha}e^{-\frac{ix^2}{2t}}\widehat{\Psi}(a) + O(t^{-\alpha-\gamma}). \end{aligned}$$

In the remainder term R_3 above we integrate by parts via identity (2.1) to get

$$\begin{aligned} |R_3| &\le Ct^{-\frac{\alpha}{2}}\int_{|z|\ge t^{\theta-\frac12}} |z|^{\alpha-1}\langle zb\rangle^{-1}(|\varphi_1| + |z\varphi_1'|)dz \\ &\quad + Ct^{-1/2}\int_{|z|>2t^{\frac12-\rho}} \langle zb\rangle^{-1}|zt^{-1/2}||f'(t,zt^{-1/2})|dz \\ &\le Ct^{-\alpha+\rho-\theta(1-\alpha)} + Ct^{-\alpha-\gamma} \le Ct^{-\alpha-\gamma} \end{aligned} \tag{2.5}$$

since $\theta(1-\alpha)-\rho\ge\gamma$. In the remainder term R_4 we integrate by parts via (2.3) to find

$$\begin{aligned} |R_4| &\le Ct^{-1/2}\int_{b/3}^{\infty}|f(t,zt^{-1/2}) - f(t,a)|\langle z-b\rangle^{-4}dz \\ &\quad + t^{-1}\int_{b/3}^{\infty}|f'(t,zt^{-\frac12})|\langle z-b\rangle^{-1}dz \\ &\le C|a|^{-1}t^{\gamma-\frac{1+\beta}{2}} \le C\langle a\rangle^{-1}t^{-\frac12-\gamma}, \end{aligned} \tag{2.6}$$

since

$$\begin{aligned} |f(t,zt^{-1/2}) - f(t,a)| &= |\int_{zt^{-1/2}}^{a}\partial_\xi f(t,\xi)d\xi| \le \int_{zt^{-1/2}}^{a}|\xi|^{\beta-\frac32}|\xi|^{\frac32-\beta}|\partial_\xi f(t,\xi)|d\xi \\ &\le C\||\xi|^{\frac12-\beta}\xi\partial_\xi f(t,\xi)\|(\int_{zt^{-1/2}}^{a}|\xi|^{2\beta-3}d\xi)^{1/2} \\ &\le C|a|^{-1}t^{\gamma-\frac{\beta}{2}}|z-b|^{\beta}. \end{aligned}$$

Collecting estimates (2.2), (2.4)-(2.6) we get the asymptotic statement needed and Lemma 2.3 is proved. □

In the next lemma we obtain time-decay estimate via additional derivative for the nonlinear term. We will use this estimate in the proof of Theorem 1.2.

Lemma 2.4. *We have the estimate*

$$\begin{aligned} &\||\partial_x|^{\frac12-\beta}(u_x\overline{v_x})\|_{1,0} \\ &\le Ct^{\beta-1}\||\partial_x|^{\frac12-\beta}u\|_{1,0}\||\partial_x|^{\frac12-\beta}v\|_{1,0} \\ &\quad + Ct^{-1}(t^{\beta}\|\mathcal{FU}(-t)u_x\|_\infty + \||\partial_x|^{\frac12-\beta}u\|_{1,0})\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x v\|_{1,0} \\ &\quad + Ct^{-1}(t^{\beta}\|\mathcal{FU}(-t)v_x\|_\infty + \||\partial_x|^{\frac12-\beta}v\|_{1,0})\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x u\|_{1,0} \end{aligned}$$

for all $t>0$, where $\beta\in(0,1/2]$.

Proof. Application of the Fourier transformation yields

$$\mathcal{F}(u_x\overline{v_x}) = \frac{1}{\sqrt{2\pi}}\int \hat{u}(t,\xi+\eta)\overline{\hat{v}(t,\eta)}(\xi+\eta)\eta d\eta,$$

then changing $i\eta\hat{u}(t,\eta) = e^{-\frac{it}{2}\eta^2}\phi(t,\eta)$ and $i\eta\hat{v}(t,\eta) = e^{-\frac{it}{2}\eta^2}\psi(t,\eta)$ we obtain

$$\mathcal{F}\mathcal{U}(-t)(u_x\overline{v_x}) = \frac{1}{\sqrt{2\pi}}\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta, \tag{2.7}$$

whence integrating by parts with respect to η we get

$$\begin{aligned}
&\||\partial_x|^{\frac12-\beta}(u_x\overline{v_x})\| = C\||\xi|^{\frac12-\beta}\mathcal{F}\mathcal{U}(-t)(u_x\overline{v_x})\| \\
=\ & C\||\xi|^{\frac12-\beta}\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta\| \\
\le\ & C\|\langle t\xi\rangle^{-1}|\xi|^{\frac12-\beta}\|(\|\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta\|_\infty \\
& +\|\int e^{-it\xi\eta}\phi_\xi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta\|_\infty \\
& +\|\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi_\eta(t,\eta)}d\eta\|_\infty) \\
\le\ & Ct^{\beta-1}\|\phi\|\|\psi\| + Ct^{\beta-1}\|\phi\|_\infty\||\xi|^{\frac12-\beta}\partial_\xi\psi\| + Ct^{\beta-1}\|\psi\|_\infty\||\xi|^{\frac12-\beta}\partial_\xi\phi\| \\
\le\ & Ct^{\beta-1}\|u_x\|\|v_x\| + Ct^{\beta-1}\|\mathcal{F}\mathcal{U}(-t)u_x\|_\infty\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x v\| \\
& +Ct^{\beta-1}\|\mathcal{F}\mathcal{U}(-t)v_x\|_\infty\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x u\|
\end{aligned}$$

and

$$\begin{aligned}
&\||\partial_x|^{\frac32-\beta}(u_x\overline{v_x})\| = C\||\xi|^{\frac32-\beta}\mathcal{F}\mathcal{U}(-t)(u_x\overline{v_x})\| \\
=\ & C\||\xi|^{\frac32-\beta}\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta\| \\
\le\ & Ct^{-1}(\||\xi|^{\frac12-\beta}\int e^{-it\xi\eta}\phi_\xi(t,\xi+\eta)\overline{\psi(t,\eta)}d\eta\| \\
& +C\||\xi|^{\frac12-\beta}\int e^{-it\xi\eta}\phi(t,\xi+\eta)\overline{\psi_\eta(t,\eta)}d\eta\|) \\
\le\ & Ct^{-1}\|\langle\xi\rangle^{\frac12-\beta}\phi\|\|\partial_\xi\psi\|_1 + Ct^{-1}\|\phi\|\||\xi|^{\frac12-\beta}\partial_\xi\psi\|_1 \\
& +Ct^{-1}\|\langle\xi\rangle^{\frac12-\beta}\psi\|\|\partial_\xi\phi\|_1 + Ct^{-1}\|\psi\|\||\xi|^{\frac12-\beta}\partial_\xi\psi\|_1 \\
\le\ & Ct^{-1}\||\partial_x|^{\frac12-\beta}u\|_{1,0}\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x v\|_{1,0} \\
& +Ct^{-1}\||\partial_x|^{\frac12-\beta}v\|_{1,0}\||\partial_x|^{\frac12-\beta}\mathcal{J}\partial_x u\|_{1,0}.
\end{aligned}$$

Lemma 2.4 is proved. □

3. Proof of Theorem 1.1

By virtue of the method in [4], [5] (see also the proof of a-priori estimates below in Lemma 3.2) we easily obtain the local existence of solutions in the functional space

$$\mathbf{X}_T = \{\phi \in \mathbf{C}((-T,T);L^2(\mathbb{R})) : \sup_{t\in(-T,T)}\|\phi(t)\|_{\mathbf{X}} < \infty\},$$

where the norm in $\mathbf{X}$ is

$$\begin{aligned}\|u\|_{\mathbf{X}} &= \langle t\rangle^{-\gamma}\|u\|_{3,0} + \langle t\rangle^{-\gamma}\|\mathcal{I}u\|_{1,0} + \langle t\rangle^{-3\gamma}\|\mathcal{I}^2 u\| \\ &\quad + t^{\alpha}\langle t\rangle^{1-2\gamma}\|\partial_t \mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty},\end{aligned}$$

with $\mathcal{I} = x\partial_x + 2t\partial_t$.

Theorem 3.1. *Let the initial data $u_0 \in H^{3,0}\cap H^{2,2}$. Then for some time $T>0$ there exists a unique solution $u \in \mathbf{X}_T$ of the Cauchy problem (1.1). If we assume in addition that the norm of the initial data $\|u_0\|_{3,0} + \|u_0\|_{2,2} = \varepsilon^2$ is sufficiently small, then there exists a unique solution $u \in \mathbf{X}_T$ of (1.1) for some time $T > 1$, such that the following estimate $\sup_{t\in[0,T]}\|u\|_{\mathbf{X}} < \varepsilon$ is valid.*

In the next lemma we obtain the estimates of global solutions in the norm $\mathbf{X}$.

Lemma 3.2. *Let $\alpha \in [1/2, 1)$. We assume that the initial data $u_0 \in H^{3,0}\cap H^{2,2}$ and the norm $\|u_0\|_{3,0} + \|u_0\|_{2,2} = \varepsilon^2$ is sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that $u \in \mathbf{C}(\mathbb{R}; H^{3,0})$ and the following estimate is valid*

$$\sup_{t>0}\|u\|_{\mathbf{X}} < \varepsilon. \tag{3.1}$$

Proof. Applying the result of Theorem 3.1 and using a standard continuation argument we can find a maximal time $T > 1$ such that the inequality

$$\|u\|_{\mathbf{X}} \le \varepsilon \tag{3.2}$$

is true for all $t \in [0, T]$. If we prove (3.1) on the whole time interval $[0, T]$, then by the contradiction argument we obtain the desired result of the lemma. In view of the local existence Theorem 3.1 it is sufficient to consider the estimates of the solution on the time interval $t \ge 1$ only.

As a consequence of (3.2) we have

$$\begin{aligned}\|\mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty} &\le C\varepsilon + \int_0^t \|\partial_\tau \mathcal{F}\mathcal{U}(-\tau)u_x(\tau)\|_{0,1,\infty}d\tau \\ &\le C\varepsilon + C\varepsilon\int_0^t \langle\tau\rangle^{\gamma-1}\tau^{-\alpha}d\tau \le C\varepsilon.\end{aligned}$$

Note that $\mathcal{J}\partial_x = \mathcal{I} + 2it\mathcal{L}$, where $\mathcal{J} = x + it\partial_x$. Hence

$$\|\mathcal{J}\partial_x u\|_{1,0} \le \|\mathcal{I}u\|_{1,0} + Ct\|\mathcal{L}u\|_{1,0} \le \|\mathcal{I}u\|_{1,0} + Ct^{1/2}\|u_x\|_\infty\|u\|_{2,0}$$

and

$$\|\mathcal{J}\partial_x\mathcal{I}u\| \le \|\mathcal{I}^2u\| + Ct\|\mathcal{L}\mathcal{I}u\| \le \|\mathcal{I}^2u\| + Ct^{1/2}\|u_x\|_\infty(\|u_x\| + \|\mathcal{I}u_x\|).$$

Then by Lemma 2.1 with $\beta = \frac{1}{2}$, using estimate (3.4) we find

$$\begin{aligned}\|u_x\|_{1,0,\infty} &\le Ct^{-1/2}\|\mathcal{F}\mathcal{U}(-t)u_x\|_{0,1,\infty} + Ct^{\frac{\gamma}{2}-\frac{3}{4}}(\|u\|_{2,0} + \|\mathcal{J}\partial_x u\|_{1,0}) \\ &\le C\varepsilon t^{-1/2} + C\varepsilon t^{3\gamma-\frac{1}{4}}\|u_x\|_\infty,\end{aligned}$$

whence

$$\|u_x\|_{1,0,\infty} \le C\varepsilon t^{-1/2}. \tag{3.3}$$

Therefore by virtue of (3.2) we have also the estimates

$$t^{-\gamma}\|\mathcal{J}\partial_x u\|_{1,0} + t^{-3\gamma}\|\mathcal{J}\partial_x\mathcal{I}u\| \le C\varepsilon. \tag{3.4}$$

Let us estimate norms $\|u\|_{3,0}$, $\|\mathcal{I}u\|_{1,0}$ and $\|\mathcal{I}^2u\|$. Differentiating three times equation (1.1) we get for $h_0 = (1+\partial_x^3)u$

$$\mathcal{L}h_0 = t^{-\alpha}(\overline{u}_x\partial_x h_0 + u_x\partial_x\overline{h_0}) + R_0$$

where

$$\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2, R_0 = t^{-\alpha}(-|u_x|^2 + 3u_{xx}\overline{u}_{xxx} + 3u_{xxx}\overline{u}_{xx}).$$

Via (3.2), (3.3) we have the estimate

$$\|R_0\| \le Ct^{-\alpha}\|u_x\|_{1,0,\infty}\|h_0\| \le C\varepsilon^2 t^{\gamma-1}.$$

Applying the operator $\mathcal{I}$ to both sides of equation (1.1) and using the commutator relations $\mathcal{L}\mathcal{I} = (\mathcal{I}+2)\mathcal{L}$ and $[\mathcal{I}, t^{-\alpha}] = -2\alpha t^{-\alpha}$, we find

$$\mathcal{L}h_k = t^{-\alpha}(\overline{u}_x\partial_x h_k + u_x\partial_x\overline{h_k}) + R_k, \tag{3.5}$$

where $k = 1, 2$, $h_1 = (1+\partial_x)\mathcal{I}u$, $h_2 = \mathcal{I}^2u$,

$$R_1 = t^{-\alpha}(\overline{u}_{xx}\mathcal{I}u_x + u_{xx}\overline{\mathcal{I}u}_x + 2(1-\alpha)(1+\partial_x)|u_x|^2),$$

and

$$R_2 = 2t^{-\alpha}(|\mathcal{I}u_x|^2 + (2-\alpha)\mathcal{I}|u_x|^2 + 2(1-\alpha)^2|u_x|^2).$$

By (3.2) and (3.4) we have

$$\|\mathcal{I}u_x\overline{\mathcal{I}u_x}\| \le Ct^{-\frac{1}{2}}\|\mathcal{I}u_x\|^{\frac{3}{2}}\|\mathcal{J}\mathcal{I}u_x\|^{1/2} \le C\varepsilon^2 t^{3\gamma-\frac{1}{2}},$$

then by virtue of (3.2), (3.3) we estimate the remainder terms

$$\|R_1\| \le Ct^{-1/2}\|u_x\|_{1,0,\infty}(\|u\|_{1,0} + \|\mathcal{I}u\|_{1,0}) \le C\varepsilon^2 t^{\gamma-1}$$

and

$$\|R_2\| \le Ct^{-1/2}\|u_x\|_\infty(\|u\|_{1,0} + \|\mathcal{I}u\|_{1,0}) + Ct^{-1/2}\|\mathcal{I}u_x\overline{\mathcal{I}u_x}\| \le C\varepsilon^2 t^{3\gamma-1}.$$

To cancel the higher-order derivative $t^{-\alpha}\overline{u}_x\partial_x h_k$, we multiply (3.5) by $E \equiv e^{-t^{-\alpha}\bar{u}}$. The other higher-order derivative $t^{-\alpha}u_x\partial_x\overline{h_k}$ will be eliminated via integration by parts. Since $E(\mathcal{L} - t^{-\alpha}\overline{u}_x\partial_x) = (\mathcal{L} - g)E$, where $g = -t^{-\alpha}\overline{u}_{xx} + \frac{1}{2}t^{-2\alpha}(\overline{u}_x)^2 - t^{-2\alpha}|u_x|^2$, from equation (3.5) we obtain

$$\mathcal{L}Eh_k = t^{-\alpha}u_x E\partial_x\overline{h_k} + ER_k + gEh_k. \tag{3.6}$$

Note that $\|E\|_{1,0,\infty} \le C$ and $\|g\|_\infty \le C\varepsilon t^{-1}$ by virtue of (3.2), (3.3). Applying the energy method to (3.6) we obtain

$$\frac{d}{dt}\|Eh_k\|^2 \le Ct^{-\alpha}|\int u_x E\partial_x(\overline{h_k})^2 dx| + C(\|ER_k\| + \|gEh_k\|)\|Eh_k\|,$$

whence integration by parts yields

$$\frac{d}{dt}\|Eh_k\| \le C\varepsilon t^{-1}\|Eh_k\| + C\|R_k\|, \tag{3.7}$$

where $k = 0, 1, 2$. Integrating (3.7) with respect to time $t \in [1, T]$ we obtain the estimate

$$\langle t\rangle^{-\gamma}\|u\|_{3,0} + \langle t\rangle^{-\gamma}\|\mathcal{I}u\|_{1,0} + \langle t\rangle^{-3\gamma}\|\mathcal{I}^2u\| < \frac{\varepsilon}{2}. \tag{3.8}$$

for all $t \in [0, T]$. We now estimate $\|\partial_t\mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty}$. We apply the Fourier transformation to equation (1.1), then changing the dependent variable $\mathcal{F}u_x = e^{-\frac{it}{2}\xi^2}w$, in view of (2.7) we obtain

$$iw_t(t,\xi) = -\frac{i\xi t^{-\alpha}}{\sqrt{2\pi}}\int e^{-it\xi\eta}w(t,\xi+\eta)\overline{w(t,\eta)}d\eta, \tag{3.9}$$

where $w(t,\xi) = \mathcal{F}\mathcal{U}(-t)u_x$. When $t \in (0,1)$ we get

$$\|\xi \int e^{-it\xi\eta} w(t,\xi+\eta)\overline{w(t,\eta)}d\eta\|_{0,1,\infty} \le C\|w\|_{0,2}^2 \le C\|u\|_{3,0}^2 \le C\varepsilon^2$$

and if $t \ge 1$, we integrate by parts with respect to η,

$$\begin{aligned}
&\|\xi \int e^{-it\xi\eta} w(t,\xi+\eta)\overline{w(t,\eta)}d\eta\|_{0,1,\infty} \\
\le\ & C\langle t\rangle^{-1}\| \int e^{-it\xi\eta}\partial_\eta(w(t,\xi+\eta)\overline{w(t,\eta)})d\eta\|_{0,1,\infty} \\
\le\ & C\langle t\rangle^{-1}\|\partial_\eta w\|_{0,1}\|w\|_{0,1} \\
\le\ & C\langle t\rangle^{-1}\|\mathcal{J}\partial_x u\|_{1,0}\|u\|_{2,0} \le C\varepsilon^2\langle t\rangle^{2\gamma-1};
\end{aligned}$$

therefore,

$$t^\alpha \langle t\rangle^{1-2\gamma}\|\partial_t \mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty} < \frac{\varepsilon}{2}. \tag{3.10}$$

By (3.8) and (3.10) we see that estimate (3.1) is true for all $t \in [0,T]$. The contradiction obtained proves (3.1) for all $t > 0$. □

To complete the proof of Theorem 1.1 we evaluate the large time asymptotic estimate of the solution u. Note that by Lemma 2.1 derivative u_x has a quasi linear asymptotic formula

$$u_x = M\mathcal{D}w + O(t^{\gamma-\frac{3}{4}}\|\mathcal{J}\partial_x u\|) = M\mathcal{D}w + O(t^{2\gamma-\frac{3}{4}}),$$

where $M = e^{\frac{ix^2}{2t}}$, $\mathcal{D}\phi = \frac{1}{\sqrt{it}}\phi(\frac{x}{t})$. For the solution $u(t,x)$ we have

$$u(t,x) = \mathcal{F}^{-1}e^{-\frac{it}{2}\xi^2}v = \frac{M}{\sqrt{2\pi}}\int e^{-\frac{it}{2}(\xi-\frac{x}{t})^2}v(t,\xi)d\xi,$$

where $v = \mathcal{F}\mathcal{U}(-t)u$. In the same way as in the proof of (3.10) we have the estimate

$$\|\partial_t \mathcal{F}\mathcal{U}(-t)\mathcal{I}u_x(t)\|_{0,1,\infty} \le Ct^{5\gamma-\alpha-1}.$$

To apply Lemma 2.3 we need to prove the representation

$$\partial_\xi v(t,\xi) = (\alpha-1)B|\xi|^{\alpha-1}\xi^{-1} + O(t^{-\gamma}|\xi|^{\alpha-2})$$

for all $t^{\theta-1} \le |\xi| \le t^{-\mu}$, $\partial_\xi v(t,\xi) = O(|\xi|^{\alpha-2})$ and $v(t,\xi) = t^{1-\alpha}\Psi(t\xi) + O(t^{1-\alpha-\delta})$ for all $|\xi| \le t^{\theta-1}$, with $\delta \ge \theta+\gamma$. From (3.4) we get $\|\xi\partial_\xi v\| \le \|\partial_x \mathcal{J}u\| \le Ct^\gamma$. We have

$$\partial_\xi v(t,\xi) = \xi^{-1}(2t\partial_t - v - \widehat{\mathcal{I}}v),$$

where $\widehat{\mathcal{I}} = \mathcal{F}\mathcal{I}\mathcal{F}^{-1} = -\partial_\xi\xi + 2t\partial_t$.

Similarly to (3.5) we get

$$\mathcal{L}\mathcal{I}u = t^{-\alpha}(\overline{u}_x\mathcal{I}u_x + u_x\overline{\mathcal{I}u_x}) + 2(1-\alpha)t^{-\alpha}|u_x|^2,$$

hence

$$\begin{aligned}
\widehat{\mathcal{I}}v(t,\xi) =\ & \widehat{\mathcal{I}}v(0,\xi) + \frac{i}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau\int e^{-i\tau\xi\eta}\widehat{\mathcal{I}}w(\tau,\xi+\eta)\overline{w(\tau,\eta)}d\eta \\
&+\frac{i}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau\int e^{-i\tau\xi\eta}w(\tau,\xi+\eta)\overline{\widehat{\mathcal{I}}w(\tau,\eta)}d\eta \\
&+\frac{2i(1-\alpha)}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau\int e^{-i\tau\xi\eta}w(\tau,\xi+\eta)\overline{w(\tau,\eta)}d\eta.
\end{aligned}$$

Since $t\partial_t v = O(t^{1-\alpha}\langle t\xi\rangle^{-1}) = O(t^{-\gamma}|\xi|^{\alpha-1})$ for $t^{\theta-1} \le |\xi|$ and $t\partial_t v = O(|\xi|^{\alpha-1})$ for $|\xi| \le t^{\theta-1}$, applying Lemma 2.2 we get

$$\begin{aligned}
v_\xi(t,\xi) &= -\xi^{-1}(\widehat{\mathcal{I}}v + v) + O(t^{-\gamma}|\xi|^{\alpha-2}) \\
&= -\frac{i}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}\widehat{\mathcal{I}}w(\tau,\xi+\eta)\overline{w(\tau,\eta)}d\eta \\
&\quad -\frac{i}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}w(\tau,\xi+\eta)\overline{\widehat{\mathcal{I}}w(\tau,\eta)}d\eta \\
&\quad +\frac{2i\alpha}{\sqrt{2\pi}}\int_0^t \tau^{-\alpha}d\tau \int e^{-i\tau\xi\eta}w(\tau,\xi+\eta)\overline{w(\tau,\eta)}d\eta + O(t^{-\gamma}|\xi|^{\alpha-2}) \\
&= G(t)|\xi|^{\alpha-1}\xi^{-1} + O(t^{-\gamma}|\xi|^{\alpha-2})
\end{aligned}$$

for all $t^{\theta-1} \le |\xi| \le t^{-\mu}$, and $\partial_\xi v(t,\xi) = O(|\xi|^{\alpha-2})$ for all $|\xi| \le t^{\theta-1}$, where

$$\begin{aligned}
G(t) &= \frac{2i}{\sqrt{2\pi}}\Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})\Re\int \widehat{\mathcal{I}}w(t,\eta)\overline{w(t,\eta)}|\eta|^{\alpha-1}d\eta \\
&\quad +\frac{2i\alpha}{\sqrt{2\pi}}\Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})\int |w(t,\eta)|^2|\eta|^{\alpha-1}d\eta,
\end{aligned}$$

since $|w(t,\eta)|^2\mathrm{sign}\eta$ and $\Re(\widehat{\mathcal{I}}w(t,\eta)\overline{w(t,\eta)})\mathrm{sign}\eta$ are odd functions. We have by (3.10)

$$\|w(t,\eta) - w(\tau,\eta)\|_{0,1,\infty} \le \|\int_\tau^t \partial_s w(s,\eta)ds\|_{0,1,\infty} \le C\int_\tau^t s^{2\gamma-\alpha-1}ds \le C\tau^{2\gamma-\alpha}$$

for all $1 \le \tau \le t$. Therefore there exists a limit $W = \lim_{t\to\infty} w(t)$ in $H_\infty^{0,1}(\mathbb{R})$ such that

$$\|w(t,\eta) - W\|_{0,1,\infty} \le C\tau^{2\gamma-\alpha}.$$

Similarly to (3.10) we get by (3.4)

$$\|\partial_t\widehat{\mathcal{I}}w\|_{0,1,\infty} \le C\varepsilon t^{5\gamma-\alpha-1} \tag{3.11}$$

for all $t \ge 1$. Hence there exists a limit $K = \lim_{t\to\infty}\widehat{\mathcal{I}}w(t)$ in $H_\infty^{0,1}(\mathbb{R})$ such that

$$\|\widehat{\mathcal{I}}w(t,\eta) - K\|_{0,1,\infty} \le C\tau^{5\gamma-\alpha}.$$

Thus

$$\partial_\xi v(t,\xi) = B_1|\xi|^{\alpha-1}\xi^{-1} + O(t^{-\gamma}|\xi|^{\alpha-2}),$$

where

$$\begin{aligned}
B_1 &= \frac{2i}{\sqrt{2\pi}}\Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})\Re\int K(\eta)\overline{W(\eta)}|\eta|^{\alpha-1}d\eta \\
&\quad +\frac{2i\alpha}{\sqrt{2\pi}}\Gamma(1-\alpha)\sin(\frac{\pi\alpha}{2})\int |W(\eta)|^2|\eta|^{\alpha-1}d\eta.
\end{aligned}$$

Also we have

$$\begin{aligned} v(t,\xi) &= \int_0^t \tau^{-\alpha} d\tau \int e^{-i\tau\xi\eta} w(\tau,\xi+\eta) w(\tau,\eta) d\eta \\ &= \int_0^t \tau^{-\alpha} d\tau \int e^{-i\tau\xi\eta} |W(\eta)|^2 d\eta + O(t^{-\gamma} \min(|\xi|^{\alpha-1}, t^{1-\alpha})) \\ &= t^{1-\alpha} \int_0^1 |z|^{-\alpha} dz \int e^{-iz\eta t\xi} |W(\eta)|^2 d\eta + O(t^{1-\alpha-\delta}) \\ &= t^{1-\alpha} \Psi(t\xi) + O(t^{1-\alpha-\delta}) \end{aligned}$$

for $|\xi| \le t^{\theta-1}$, where $\delta = \alpha - 2\gamma$, $\Psi(x) = \int_0^1 |z|^{-\alpha} dz \int e^{-iz\eta x} |W(\eta)|^2 d\eta$. Now application of Lemma 2.3 yields asymptotics (1.4) for the solution $u(t,x)$. Using Lemma 3.2 we get the result of Theorem 1.1 with $B = B_1/\sqrt{2\pi}$, $P = \frac{1}{\sqrt{2\pi}} \check{\Psi}$, $Q = \frac{1}{\sqrt{2\pi}} V$. Theorem 1.1 is proved.

4. Proof of Theorem 1.2

By the method in [3] (see also the proof of a-priori estimates below in Lemma 4.2), we easily obtain the local existence of solutions in the analytic functional space

$$\mathbf{A}_T = \left\{ \phi \in \mathbf{C}([-T,T]; \mathbf{L}^2(\mathbb{R})) : \sup_{t\in[-T,T]} \|\phi(t)\|_{\mathbf{A}_t} < \infty \right\},$$

where the norm $\mathbf{A}_t$ is defined as

$$\begin{aligned} \|u\|_{\mathbf{A}_t} &= \langle t\rangle^{-\gamma} \| |\partial_x|^{\frac{1}{2}-\beta} u\|_{3,0} + t^{\alpha} \langle t\rangle^{1-\gamma} \|\partial_t \mathcal{F}\mathcal{U}(-t) u_x(t)\|_{0,1,\infty} \\ &\quad + \sum_{n=1}^{\infty} \frac{\langle t\rangle^{-n\gamma}}{n!} \| |\partial_x|^{\frac{1}{2}-\beta} \mathcal{I}^n u\|_{1,0}. \end{aligned}$$

Denote

$$\|u\|_{\mathbb{Z}} = \sum_{n=0}^{\infty} \frac{t^{-n\gamma}}{n!} \| |\partial_x|^{\frac{1}{2}-\beta} \mathcal{I}^n u\|_{1,0}.$$

Theorem 4.1. *Let $\alpha \in (0,1)$. We assume that the initial data $u_0 \in \mathbf{A}_0$. Then for some time $T > 0$ there exists a unique solution $u \in \mathbf{A}_T$ of the Cauchy problem (1.1). If we assume in addition that the norm of the initial data $\|u_0\|_{\mathbf{A}_0} = \varepsilon^2$ is sufficiently small, then there exists a unique solution $u \in \mathbf{A}_T$ of (1.1) for some time $T > 1$, such that the following estimate $\sup_{t\in[0,T]} \|u\|_{\mathbf{A}} < \varepsilon$ is valid.*

In the next lemma we obtain the estimates of global solutions in the norm $\mathbf{A}_t$.

Lemma 4.2. *Let the initial data $u_0 \in \mathbf{A}_0$ are such that the norm $\|u_0\|_{\mathbf{A}_0} = \varepsilon^2$ is sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that $u \in \mathbf{A}_\infty$. Moreover the following estimate is valid*

$$\|u\|_{\mathbf{A}_t} < \varepsilon \tag{4.1}$$

for all $t > 0$.

Proof. As in Lemma 3.2 we argue by contradiction and find a maximal time $T > 1$ such that the estimate

$$\|u\|_{\mathbf{A}_t} \le \varepsilon \tag{4.2}$$

is valid for all $t \in [0, T]$. Via Theorem 4.1 it is sufficient to consider $t \geq 1$. As above in Lemma 3.2 we can estimate the norms $\|u_x\|_{1,0,\infty}$, $\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x u\|_{1,0}$, $\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x \mathcal{I}u\|_{1,0}$ via the norm $\|u\|_{\mathbb{Z}}$. Indeed we have the estimate

$$\begin{aligned}
\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x u\|_{1,0} &\leq \||\partial_x|^{\frac{1}{2}-\beta}\mathcal{I}u\|_{1,0} + 2t\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{L}u\|_{1,0} \\
&\leq C\|u\|_{\mathbb{Z}} + C\|u\|_{\mathbb{Z}}^2 \leq C\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{J}\partial_x \mathcal{I}u\|_{1,0} &\leq \||\partial_x|^{\frac{1}{2}-\beta}\mathcal{I}^2 u\|_{1,0} + 2t\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{L}\mathcal{I}u\|_{1,0} \\
&\leq C\|u\|_{\mathbb{Z}} + C\|u\|_{\mathbb{Z}}^2 \leq C\varepsilon.
\end{aligned}$$

We first estimate $\|\partial_t \mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty}$. We apply the Fourier transformation to equation (1.1), then changing the dependent variable $\mathcal{F}u_x = e^{-\frac{it}{2}\xi^2}w$, in view of (2.7) we obtain

$$iw_t(t,\xi) = \frac{i\xi t^{-\alpha}}{\sqrt{2\pi}} \int e^{-it\xi\eta} w(t,\xi+\eta)\overline{w(t,\eta)}d\eta, \tag{4.3}$$

where $w(t,\xi) = \mathcal{F}\mathcal{U}(-t)u_x$. For $t \geq 1$, we integrate by parts with respect to η

$$\begin{aligned}
&\|\xi \int e^{-it\xi\eta} w(t,\xi+\eta)\overline{w(t,\eta)}d\eta\|_{0,1,\infty} \\
&\leq C\langle t\rangle^{-1}\| \int e^{-it\xi\eta}\partial_\eta(w(t,\xi+\eta)\overline{w(t,\eta)})d\eta\|_{0,1,\infty} \leq C\langle t\rangle^{-1}\|\partial_\eta w\|_{0,1}\|w\|_{0,1} \\
&\leq C\langle t\rangle^{-1}\|\mathcal{J}\partial_x u\|_{1,0}\|u\|_{2,0} \leq C\varepsilon^2\langle t\rangle^{2\gamma-1};
\end{aligned}$$

therefore,

$$t^{\alpha+1-2\gamma}\|\partial_t \mathcal{F}\mathcal{U}(-t)u_x(t)\|_{0,1,\infty} < \varepsilon \tag{4.4}$$

for all $t \geq 1$. In the same manner

$$t^{\alpha+1-5\gamma}\|\partial_t \mathcal{F}\mathcal{U}(-t)\mathcal{I}u_x(t)\|_{0,1,\infty} < \varepsilon. \tag{4.5}$$

We estimate the norm $\|u\|_{\mathbb{Z}} = \sum_{n=0}^{\infty} \frac{t^{-n\gamma}}{n!}\||\partial_x|^{\frac{1}{2}-\beta}\mathcal{I}^n u\|_{1,0}$ for all $t \geq 1$. Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^{-\gamma n}}{n!}\|(\mathcal{I}+a)^n u\|_{1,0} \\
&\leq \sum_{n=0}^{\infty} \frac{t^{-\gamma n}}{n!}\sum_{j=0}^{n} C_n^j |a|^{n-j}\|\mathcal{I}^j u\|_{1,0} = \sum_{n=0}^{\infty}\sum_{j=0}^{n-j} \frac{|a|^{n-j}}{(n-j)!}\frac{t^{-\gamma n}}{j!}\|\mathcal{I}^j u\|_{1,0} \\
&= \sum_{j=0}^{\infty} \frac{t^{-\gamma j}}{j!}\|\mathcal{I}^j u\|_{1,0}\sum_{k=j}^{\infty}\frac{|at|^{k-j}}{(k-j)!} = e^{|a|t^{-\gamma}}\sum_{j=0}^{\infty}\frac{t^{-\gamma j}}{j!}\|\mathcal{I}^j u\|_{1,0} \leq C\|u\|_{\mathbb{Z}}
\end{aligned}$$

We have by Lemma 2.4 denoting $C_n^m = \frac{n!}{m!(n-m!)}$

$$\begin{aligned}
\|\mathcal{L}u\|_{\mathbb{Z}} &= \|t^{-\alpha}|u_x|^2\|_{\mathbb{Z}} = \sum_{n=0}^{\infty} \frac{t^{-n\gamma}}{n!} \||\partial_x|^{\frac{1}{2}-\beta}\mathcal{I}^n t^{-\alpha}|u_x|^2\|_{1,0} \\
&= t^{-\alpha}\sum_{n=0}^{\infty} \frac{t^{-n\gamma}}{n!} \||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}-2\alpha)^n|u_x|^2\|_{1,0} \\
&= t^{-\alpha}\sum_{n=0}^{\infty} \frac{t^{-n\gamma}}{n!} \sum_{j=0}^{n} C_n^j \||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}-2\alpha)^{n-j}u_x(\mathcal{I}-2\alpha)^j\overline{u}_x\|_{1,0} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{t^{-\alpha-3n\gamma}}{(n-j)!j!} \||\partial_x|^{\frac{1}{2}-\beta}((\mathcal{I}-1-2\alpha)^{n-j}u)_x((\mathcal{I}-1-2\alpha)^j\overline{u})_x\|_{1,0} \\
&\le C\sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{t^{-\alpha-3n\gamma+\beta-1}}{(n-j)!j!} \||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}-1-2\alpha)^{n-j}u\|_{1,0} \\
&\quad (\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}-1-2\alpha)^j u\|_{1,0} + \||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}-1-2\alpha)^j\mathcal{J}\partial_x u\|_{1,0}) \\
&\le Ct^{-\alpha+\beta-1}\|u\|_{\mathbb{Z}}(\|u\|_{\mathbb{Z}} + \|\mathcal{I}u\|_{\mathbb{Z}} + t\|\mathcal{L}u\|_{\mathbb{Z}}),
\end{aligned}$$

hence we get

$$\begin{aligned}
\|\mathcal{L}u\|_{\mathbb{Z}} &\le Ct^{-\alpha+\beta-1}\|u\|_{\mathbb{Z}}^2 + Ct^{-\alpha+\beta-1}\|u\|_{\mathbb{Z}}\|\mathcal{I}u\|_{\mathbb{Z}} \\
&\le Ct^{-\gamma-1}(\|u\|_{\mathbb{Z}} + \|\mathcal{I}u\|_{\mathbb{Z}}).
\end{aligned}$$

In particular we have

$$\|\mathcal{J}\partial_x u\|_{\mathbb{Z}} \le C\|u\|_{\mathbb{Z}} + C\|\mathcal{I}u\|_{\mathbb{Z}}.$$

Using the commutation relations $\mathcal{I}^n\partial_x = \partial_x(\mathcal{I}+1)^n$, $\mathcal{L}\mathcal{I}^n = (\mathcal{I}+2)^n\mathcal{L}$, $\mathcal{I}^n t^{-\alpha} = t^{-\alpha}(\mathcal{I}-2\alpha)^n$ applying operator $\mathcal{I}$ to equation (1.1) we get

$$\mathcal{L}\mathcal{I}^n u = t^{-\alpha}(\mathcal{I}+2(1-\alpha))^n|u_x|^2. \tag{4.6}$$

Via (4.3) we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} t^{-\alpha-\gamma n}(n!)^{-1}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+2-2\alpha)^n|u_x|^2\|_{1,0} \\
&\le C\sum_{n=0}^{\infty} t^{-\alpha-\gamma n}(n!)^{-1}\sum_{m=0}^{n} C_n^m(2-2\alpha)^{n-m} \\
&\quad \times\sum_{j=0}^{m} C_m^j\||\partial_x|^{\frac{1}{2}-\beta}((\mathcal{I}+1)^j u)_x\overline{((\mathcal{I}+1)^{m-j}u)_x}\|_{1,0} \\
&\le Ct^{\beta-\alpha-1}\|u\|_{\mathbb{Z}}^2 + Ct^{\beta-\alpha-1}\|u\|_{\mathbb{Z}}\|\mathcal{J}\partial_x u\|_{\mathbb{Z}} \\
&\le C\varepsilon t^{-\gamma-1}(\|u\|_{\mathbb{Z}} + \|\mathcal{I}u\|_{\mathbb{Z}}).
\end{aligned}$$

Using Lemma 2.4 we obtain

$$\begin{aligned}
&\||\partial_x|^{\frac{1}{2}-\beta}((\mathcal{I}+1)^j u)_x\overline{((\mathcal{I}+1)^{m-j}u)_x}\|_{1,0} \\
&\le Ct^{\beta-1}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^j u\|_{1,0}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^{m-j}u\|_{1,0} \\
&\quad +Ct^{\beta-1}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^j u\|_{1,0}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^{m-j}\mathcal{J}\partial_x u\|_{1,0} \\
&\quad +Ct^{\beta-1}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^j\mathcal{J}\partial_x u\|_{1,0}\||\partial_x|^{\frac{1}{2}-\beta}(\mathcal{I}+1)^{m-j}u\|_{1,0}.
\end{aligned}$$

Hence by the energy method, in view of (4.2) we find

$$\frac{d}{dt}\|u\|_{\mathbb{Z}} \le -\gamma t^{-1-\gamma}\|\mathcal{I}u\|_{\mathbb{Z}} + C\varepsilon t^{-\gamma-1}\|u\|_{\mathbb{Z}} + C\varepsilon t^{-\gamma-1}\|\mathcal{I}u\|_{\mathbb{Z}} \le C\varepsilon t^{\beta-\alpha-1}\|u\|_{\mathbb{Z}}. \tag{4.7}$$

Integration of (4.7) with respect to time $t \ge 1$ yields $\|u\|_{\mathbb{Z}} < \frac{\varepsilon}{2}$ for all $t \ge 1$. The norm $\||\partial_x|^{\frac{1}{2}-\beta}u\|_{3,0}$ is estimated in the same manner as in the proof of Lemma 3.2. Therefore, Lemma 4.2 is proved. □

Now we complete the proof of Theorem 1.2 by applying Lemmas 2.2 and 2.3 as in the previous section.

Acknowledgments. The second author was partially supported by a grant from CONACYT.

References

[1] H. Bateman and A. Erdélyi, *Tables of integral transforms,* McGraw-Hill Book Company, Inc., N.Y. 1954.

[2] S. Cohn, *Resonance and long time existence for the quadratic nonlinear Schrödinger equation,* Commun. Pure Appl. Math. **45** (1992), pp. 973-1001.

[3] N. Hayashi, *Global existence of small analytic solutions to nonlinear Schrödinger equations,* Duke Math. J. **60** (1990), pp. 717-727.

[4] N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space,* J. Nonlinear Anal. T.M.A. **32** (1993), pp. 823-833.

[5] N.Hayashi and T.Ozawa, *Remarks on nonlinear Schödinger equations in one space dimension,* Differential and Integral Equations, **7** (1994), pp.453-461

[6] N. Hayashi and P.I. Naumkin, *A quadratic nonlinear Schrödinger equation in one space dimension,* Preprint, (2000).

[7] N. Hayashi and P.I. Naumkin, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations without a self-conjugate property*, Funkcialaj Ekvacioj **42** (1999), pp. 311-324.

[8] N. Hayashi and P.I. Naumkin, *On the reduction of the modified Benjamin - Ono equation to the cubic derivative nonlinear Schrödinger equation,* Preprint 2000.

[9] N. Hayashi, P.I. Naumkin and Y. Yamazaki, *Large time behavior of small solutions to subcritical derivative nonlinear Schrödinger equations*, Proc. Amer. Math. Soc., to appear.

[10] T. Ozawa, *Remarks on quadratic nonlinear Schrödinger equations.* Funkcialaj Ekvacioj **38** (1995), pp. 217-232.

[11] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations,* Commun. Pure Appl. Math. **38** (1985), pp. 685-696.

Nakao Hayashi
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: nhayashi@math.wani.osaka-u.ac.jp

Pavel I. Naumkin
Instituto de Física y Matemáticas, Universidad Michoacana, AP 2-82, Morelia, CP 58040, Michoacán, Mexico
E-mail address: pavelni@zeus.ccu.umich.mx