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Nonautonomous attractors of skew-product flows with digitized driving systems *

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Abstract

The upper semicontinuity and continuity properties of pullback attractors for nonautonomous differential equations are investigated when the driving system of the generated skew-product flow is digitized.

1 Introduction

The objective of this paper is to study the semicontinuity and continuity properties of pullback attractors for nonautonomous differential equations

$$x' = f(t,x), \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \tag{1.1}$$

under perturbation of the driving system through a digitization procedure. We will formulate this problem in concrete terms using the language of skew-product flows [1, 16, 17], in particular, the Bebutov approach to the skew-product flow concept. Thus let $\mathcal{F}$ be some topological vector space of mappings $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ and, for each $t \in \mathbb{R}$, let $\theta_t : \mathcal{F} \to \mathcal{F}$ be the translation operator defined by $\theta_t(f)(s,x) := f(t+s,x)$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Now let $P \subset \mathcal{F}$ be a metrizable compact set which is invariant in the sense that, if $p \in P$, then $\theta_t(p) \in P$ for all $t \in \mathbb{R}$. Suppose further that the mapping $\mathbb{R}\times P \to P$ defined by $(t,p) \mapsto \theta_t(p)$ is continuous. This just says that $(P, \{\theta_t : t \in \mathbb{R}\})$ is a topological flow or autonomous dynamical system.

Now consider the family of differential equations

$$x' = p(t,x). \tag{1.2}$$

We assume, for each $x_0 \in \mathbb{R}^d$ and $p \in P$, that the solution $x(t) = \phi(t,x_0,p)$ of (1.2) satisfying $x(0) = x_0$ exists locally and is unique. We also assume that, if $t \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ and $p \in P$, then the correspondence $(t,x_0,p) \mapsto (\phi(t,x_0,p), \theta_t(p))$ is continuous on its domain of definition $V \subset \mathbb{R}\times\mathbb{R}^d\times P$. This correspondence

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$(t, x_0, p) \mapsto (\phi(t, x_0, p), \theta_t(p)) : V \to \mathbb{R}^d \times P$ is the skew product (local) flow on $\mathbb{R}^d \times P$ defined by the family of equations (1.2). The flow $(P, \{\theta_t : t \in \mathbb{R}\})$ is referred to as the **driving system** of equations (1.2). For each $p \in P$, the solutions $x(t)$ of (1.2) can be viewed as projections to $\mathbb{R}^d$ of trajectories of the above skew-product flow.

When the equations (1.2) satisfy an appropriate dissipativity condition, they induce a global pullback attractor $\mathbf{A} = \bigcup_{p\in P}(A_p \times \{p\}) \subset \mathbb{R}^d \times P$, where each fiber A_p is defined by a procedure which amounts to a nonlinear version of the classical Weyl limit-point construction. The set $\mathbf{A}$ is compact and is invariant with respect to the above skew-product system on $\mathbb{R}^d \times P$ defined by the solutions of the equations (1.2). For each $p \in P$, one can think of A_p as a "pointwise" pullback attractor; see Section 2 for details. The problem we pose is that of studying the semicontinuity, resp. continuity, properties of $\mathbf{A}$ and the individual fibers A_p, as the compact translation-invariant set P is varied within $\mathcal{F}$. For technical simplicity, throughout the paper, we assume that the equations (1.2) contract a fixed large ball in $\mathbb{R}^d$. In this way the concepts of local pullback attractor and global pullback attractor are made equivalent.

This problem has been considered by Kloeden and Kozyakin [12, 13], who, in particular, studied the upper semicontinuity in the Hausdorff sense of the A_p under perturbation in P when the driving system $(P, \{\theta_t : t \in \mathbb{R}\})$ has the shadowing property. Here we will not assume that shadowing holds. Rather, we will instead study perturbations in certain spaces $\mathcal{F}$ of a general compact, metrizable, Bebutov-invariant subset P. Among these perturbations are those determined by **digitizing** a given time-varying vector field $f(t, x)$. By this, we mean the following: the time-axis is decomposed into half-open intervals of lengths, say, between $\delta/2$ and δ for each $\delta > 0$. On each such interval, the time-varying vector field $f(t, x)$ is replaced by an autonomous vector field $\bar{f}(x)$ (which usually depends on the particular interval). For example, one may choose $\bar{f}(x)$ to be the time-average of $f(t, x)$ over the given (or previous) interval, or as some particular value $f(t_*, x)$, or in some other way as well. If now there is a compact, metrizable, Bebutov-invariant subset P of $\mathcal{F}$ such that $f \in P$, then we can identify (1.1) as one equation in the family (1.2). In this way one is led to identify appropriate perturbations $P_\delta \subset \mathcal{F}$ of P.

We will formulate and prove results to the effect that if

$$\mathbf{A}^\delta = \bigcup (A_{p_\delta} \times \{p_\delta\} : p_\delta \in P_\delta\}$$

is the pullback attractor for the skew-product dynamical system defined by $(P_\delta, \{\theta_t : t \in \mathbb{R}\})$, and if $p_\delta \in P_\delta$ corresponds to a digitized version of equation (1.1), then $A^\delta_{p_\delta}$ converges upper semicontinuously to A_f as $\delta \to 0$, with respect to the Hausdorff metric on compact subsets of $\mathbb{R}^d$. We will also provide sufficient conditions that $A^\delta_{p_\delta}$ converges continuously to A_f as $\delta \to 0$, with respect to the Hausdorff metric.

We will make use of two distinct sets of methods in our study study $\mathbf{A}$ and its fibers A_p. The first is drawn from the area of topological dynamics and dynamical systems. In applying these methods, we will make systematic use of the the skew-product local flow on $\mathbb{R}^d \times P$ induced by the solutions of equations (1.2). The second is taken from the fixed point theory of nonlinear mappings on Banach spaces. Fixed point theory can be applied here for the following reason (among others): the set A_p is equal, under our hypotheses, to the set of initial conditions $x_0 \in \mathbb{R}^d$ for which the corresponding solution $x(t) = \phi(t, p, x_0)$ of (1.2) exists and is bounded for all $t \in \mathbb{R}$. These solutions can be viewed as the fixed points of an appropriate nonlinear mapping.

In order to illustrate the points made above, we consider the example

$$x' = f(t,x) = -x + h(t,x), \quad x \in \mathbb{R}, t \in \mathbb{R}, \tag{1.3}$$

where h is uniformly continuous and uniformly Lipschitz in x on each subset of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^d$ is compact. Assume that there are positive numbers a and σ, such that for each $t \in \mathbb{R}$ one has

$$a + h(t, -a) \geq \sigma, \quad -a + h(t,a) \leq -\sigma.$$

Let $f_\delta(t,x)$ be obtained by digitizing f: thus, for example, we might set

$$f_\delta(t,x) = -x + \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} h(s,x)ds, \quad t \in [n\delta, (n+1)\delta)$$

for each $n \in \mathbb{Z}$. Using methods of dynamical systems, we will first show that $A^\delta_{f_\delta}$ exists and tends upper semicontinuously to A_f as $\delta \to 0$, and moreover that this convergence is uniform in t when f is replaced by $\theta_t(f)$ for any $t \in \mathbb{R}$.

On the other hand, a bounded solution of (1.3) is expressible as a fixed point of the mapping T defined for each $t \in \mathbb{R}$ by

$$T[x](t) = -\int_t^\infty e^{(t-s)} f(s, x(s))\, ds.$$

This mapping is defined and continuous on the Banach space C_b of bounded, continuous real-valued functions on $\mathbb{R}$. If $|h(t,x)| \leq a$ whenever $|x| \leq a$ and $t \in \mathbb{R}$, and if $\left|\frac{\partial h}{\partial x}\right| \leq \alpha < 1$ for all $t \in \mathbb{R}$ and $|x| \leq a$, then T is a contraction on the ball $\{x(\cdot) \in C_b : \|x\|_\infty \leq a\}$. One has that $A_f = \{x_0 \in \mathbb{R} :$ the solution $x(\cdot)$ of (1.3) with $x(0) = x_0$ is bounded on $(-\infty, \infty)\}$, and that the set of solutions $x(\cdot)$ of (1.3) which are bounded on $(-\infty, \infty)$ is the fixed point set of T. We will show that the digitization gives rise to continuous convergence in the Hausdorff sense $A^\delta_{f_\delta}$ to A_f as $\delta \to 0$, and moreover the convergence is uniform in t when f is replaced by $\theta_t(f)$ for any $t \in \mathbb{R}$.

2 Preliminaries

In this section, we formulate the concept of driving system in a way which is convenient for present purposes. We define the notion of pullback attractor when the driving system is compact and when a uniform dissipativity condition is valid. Finally we introduce various basic definitions which will be needed later on.

Let $\mathcal{F}$ denote the vector space of all functions (time-varying vector fields) $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following properties:

i) For each compact set $K \subset \mathbb{R}^d$, f is uniformly continuous on $\mathbb{R} \times K$;

ii) For each compact set $K \subset \mathbb{R}^d$, there exists a constant L_K (also depending on f) so that

$$\|f(t,x) - f(t,y)\| \le L_K \|x - y\| \quad \text{for all } x, y \in K,\ t \in \mathbb{R}.$$

The second condition states that f is uniformly Lipschitz in x on each set of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^d$ is compact.

It will be clear that all of our results can be formulated and proved when the vector fields f in question satisfy less stringent conditions. However, Conditions i) and ii) are not particularly restrictive and they permit a simple exposition of the facts we wish to present.

We put the topology of uniform convergence on compact sets on $\mathcal{F}$. Thus a sequence $\{f_n\}$ of elements of $\mathcal{F}$ converges to $f \in \mathcal{F}$ if and only if $f_n(t,x) \to f(t,x)$ uniformly on each set $D \times K \subset \mathbb{R} \times \mathbb{R}^d$ when D and K are compact. This topology is metrizable, but not complete. There is a natural flow (Bebutov flow) $\{\theta_t : t \in \mathbb{R}\}$ defined on $\mathcal{F}$ by translation of the t-variable, specifically $\theta_t(f)(s,x)$ $:= f(t+s,x)$ for all $f \in \mathcal{F}$, $x \in \mathbb{R}^d$ and s, t in $\mathbb{R}$.

A simple but basic observation can now be made: if $f \in \mathcal{F}$ (i.e., if f satisfies the conditions i) and ii)), then the orbit closure $P := \mathrm{cls}\{\theta_t(\mathrm{f}) : \mathrm{t} \in \mathbb{R}\} \subset \mathcal{F}$ is compact. This follows from the uniform continuity conditions on f. Moreover, condition ii) holds for each $p \in P$ with the same set of Lipschitz constants $\{L_K\}$. See [16] for further details.

Consider the family of differential equations

$$x' = p(t,x), \tag{2.1}$$

where p ranges over P. If $x_0 \in \mathbb{R}^d$ and $p \in P$, then equation (2.1) admits a unique maximally defined solution $x(t) = \phi(t, x_0, p)$ satisfying $x(0) = x_0$. We define a local flow $\{\pi_t : t \in \mathbb{R}\}$ on $\mathbb{R}^d \times P$ by setting $\pi_t(x_0, p) = (\phi(t, x_0, p), \theta_t(p))$ for all triples (t, x_0, p) such that the right-hand side is well-defined. This local flow is said to be of skew-product type because the component $\theta_t(p)$ does not depend on x_0. As noted in the Introduction, the flow $(P, \{\theta_t : t \in \mathbb{R}\})$ is referred to as the **driving system** for the family (2.1).

Of course, a skew-product local flow on $\mathbb{R}^d \times P$ can be constructed for any compact, translation-invariant subset of $\mathcal{F}$; it is not required that P be generated as above by the translates of a fixed vector field f (when P is so generated, it is referred to as the **hull** of f).

Next we turn to the concept of pullback attractor. Let us begin with a fixed vector field $f \in \mathcal{F}$ and the corresponding differential equation

$$x' = f(t,x). \tag{2.2}$$

To simplify matters, we assume that there exists $R > 0$ such that

$$\langle f(t,x), x\rangle < 0 \tag{2.3}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ satisfying $\|x\| \geq R$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. Using the uniform continuity properties of f, one sees that condition (2.3) actually implies that, for some $\eta > 0$,

$$\langle f(t,x), x\rangle \leq -\eta$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ satisfying $\|x\| = R$. Passing to the hull P of f, we see that, for each $p \in P$, the closed ball $B_R = \{x \in \mathbb{R}^d : \|x\| \leq R\}$ is positively invariant with respect to the solutions of the differential equation (2.1). In particular, if $p \in P$ and $x_0 \in B_R$, then the solution $x(t)$ of (2.1) satisfying $x(0) = x_0$ exists and lies in B_R for all $t \geq 0$.

Let us write $\phi(t, x_0, p)$ for the solution $x(t)$ of (2.1) satisfying $x(0) = x_0$. If $D \subset \mathbb{R}^d$, we write $\phi(t, D, p) = \{\phi(t, x_0, p) : x_0 \in D\}$. Define

$$A_p = \bigcap_{t \geq 0} \phi\left(t, B_R, \theta_{-t}(p)\right), \tag{2.4}$$

then put $\mathbf{A} = \bigcup_{p\in P} \left(A_p \times \{p\}\right) \subset \mathbb{R}^d \times P$. We refer to A_p as the pullback attractor for the single equation (2.1), and to $\mathbf{A}$ as the pullback attractor for the family (2.1), or more simply as the global pullback attractor. Clearly, the sets A_p are all nonempty and compact, and ϕ-invariant in the sense that $\phi\left(t, A_p, p\right) = A_{\theta_t(p)}$.

We will consider the sets A_p and $\mathbf{A}$ in more detail in the following sections. In the remainder of the present section we briefly recall two basic definitions, namely those of exponential dichotomy and of the Hausdorff distance.

Let P be a compact, translation-invariant subset of $\mathcal{F}$. Let us suppose that P consists entirely of linear vector fields: $p(t, x) = P(t)x$, where we permit an abuse of notation. The function $P(\cdot)$ takes values in the set $\mathcal{M}_d$ of $d \times d$ real matrices and is uniformly continuous on $\mathbb{R}$. Let $\Psi_p(t)$ be the fundamental matrix of the linear ordinary differential equation

$$x' = P(t)x; \tag{2.5}$$

thus $\Psi_p(t)$ is the $d \times d$-matrix solution of (2.5) such that $\Psi_p(0) = I$, the $d \times d$ identity matrix. Let $\mathcal{Q}$ be the set of linear projections $Q : \mathbb{R}^d \to \mathbb{R}^d$; this set has finitely many connected components determined by the possible dimensions of the image of Q.

Definition Say that the family (2.5) has an **exponential dichotomy** (ED) over P if there are positive constants γ, L and a continuous projection-valued function Q defined by $p \mapsto Q_p \in \mathcal{Q}$ such that for all $p \in P$,

$$\left\|\Psi_p(t) Q_p \Psi_p(s)^{-1}\right\| \leq L \exp^{-\gamma(t-s)} \quad (t \geq s),$$
$$\left\|\Psi_p(t)(I - Q_p) \Psi_p(s)^{-1}\right\| \leq L \exp^{\gamma(t-s)} \quad (t \leq s)\,.$$

Now let $(\mathcal{X}, d)$ be a metric space and let A and B be nonempty compact subsets of $\mathcal{X}$. Define $\operatorname{dist}(a, B) := \min_{b \in B} d(a, b)$ and then define the Hausdorff semi-distance

$$H^*(A, B) := \max_{a \in A} \operatorname{dist}(a, B).$$

Thus, if $\epsilon > 0$ and $H^*(A, B) < \epsilon$, then for each $a \in A$ there exists $b \in B$ such that $d(a, b) < \epsilon$. Finally, define the Hausdorff distance $H(A, B)$ between A and B as follows:

$$H(A, B) := \max \{H^*(A, B), H^*(B, A)\}.$$

Note that if B happens to be a singleton, $B = \{b\}$, then $H^*(A, B) = H(A, B)$.

3 Semicontinuity results

We begin by fixing a compact, translation-invariant subset P of the topological vector space $\mathcal{F}$ described in the preceding section. We assume that there exist numbers $R > 0$, $\eta > 0$ so that, if $\|x\| = R$ and $p \in P$, then

$$\langle p(t, x), x \rangle \leq -\eta. \tag{3.1}$$

Let $B_R = \{x \in \mathbb{R}^d : \|x\| \leq R\}$ be the closed ball in $\mathbb{R}^d$ centered at the origin with radius R.

We make some remarks about the pullback attractor A_p and the global pullback attractor $\mathbf{A}$, first when P is held fixed and then when it is varied in some systematic way. First of all, it follows from (3.1) that, if $t > s$, then $\phi(t, B_R, \theta_{-t}(p)) \subset \phi(s, B_R, \theta_{-s}(p))$. Thus the intersection in (2.4) is over a decreasing collection of sets. Using the continuity property of the reduced Čech homology functor $\check{\mathrm{H}}$ in the category of compact spaces together with the fact that each set $\phi(t, B_R, \theta_{-t}(p))$ is homeomorphic to a ball, we have $\check{\mathrm{H}}(A_p) = 0$. In particular, we have:

Proposition 3.1 *For each $p \in P$, the space A_p is connected; in fact A_p is ∞-proximally connected in the sense of [4].*

We record a second fact which also follows quickly from (3.1) and from the definition of A_p.

Proposition 3.2 *For each $p \in P$, one has $A_p = \{x_0 \in \mathbb{R}$: the solution $x(\cdot)$ of (1.2) with $x(0) = x_0$ is defined on the entire real axis and satisfies $\|x(t)\| \leq R$ for all t in $\mathbb{R}\}$.*

Proof. Let $x_0 \in A_p$. Then for each $t < 0$, there exists $\bar{x} \in B_R$ such that $\phi(t, \bar{x}, \theta_{-t}(p)) = x_0$, and one has $x(t) = \bar{x}$. It follows that $x(t)$ is defined and satisfies $\|x(t)\| \leq R$ for all $t \in \mathbb{R}$. It is equally easy to see that, if the solution $x(t)$ of (1.2) satisfying $x(0) = x_0$ exists and is bounded on $\mathbb{R}$, then $x_0 \in A_p$. □

It follows from Proposition 3.2 that $\mathbf{A} \subset \mathbb{R}^d \times P$ is compact, and from this one sees that $\check{H}(\mathbf{A}) = \check{H}(P)$ because of the continuity of the Čech homology functor on compact spaces. One also sees that, if $\mathcal{K}(\mathbb{R}^d)$ is the space of all nonempty compact subsets of $\mathbb{R}^d$, then the mapping $p \mapsto A_p : P \to \mathcal{K}(\mathbb{R}^d)$ is upper semicontinuous in the sense that (using the notation of Section 2):

$$H^*(A_{p_n}, A_p) \to 0 \quad \text{whenever} \quad p_n \to p \text{ in } P.$$

Next let $f \in \mathcal{F}$ be a vector field satisfying condition (2.3). We want to consider the upper semicontinuity properties of the pullback attractor A_f when f is digitized. It will be convenient and informative to study the upper semicontinuity properties of the pullback attractor A_f using the language of skew-product flows.

First we introduce some terminology. By a **digitization** we mean a procedure which, to each $f \in \mathcal{F}$ and each real number $\delta > 0$, assigns the following data with the indicated properties:

I) There is a collection $\mathcal{I}^\delta = \{I_j^\delta \; : \; j \in \mathbb{Z}\}$ of nonempty half-open intervals in $\mathbb{R}$ such that $\cup_{j=-\infty}^{\infty} I_j^\delta = \mathbb{R}$, and such that each interval I_j^δ has length $\leq \delta$ and (say) $\geq \delta/2$.

II) To each $f \in \mathcal{F}$ there is associated a collection $\{f_\delta^j : \delta > 0, j \in \mathbb{Z}\}$ of autonomous vector fields. There is a positive function $\omega = \omega(\epsilon)$, defined for positive values of ϵ and tending to zero as $\epsilon \to 0+$, such that for each interval $I_j^\delta \in \mathcal{I}^\delta$ and each $x \in \mathbb{R}^d$ the following property holds: if $\epsilon_x = \sup\left\{\|f(r,x) - f(s,x)\| : r, s \in I_j^\delta\right\}$, then

$$\left\|f_\delta^j(x) - f(t,x)\right\| \leq \omega(\epsilon_x), \quad t \in I_j^\delta.$$

III) There is a positive function $\omega_1 = \omega_1(M)$, which is defined for positive values of M and which depends only on M, such that, if x, y in $\mathbb{R}^d$ satisfy $\|f(t,x) - f(t,y)\| \leq M$ for all t in some interval I_j^δ, then

$$\left\|f_\delta^j(x) - f_\delta^j(y)\right\| \leq \omega_1(M)\|x - y\|$$

for all $\delta > 0$.

IV) There is a positive function $\omega_2 = \omega_2(\eta)$, defined for positive values of η and tending to zero as $\eta \to 0+$, such that, if $J \subset \mathbb{R}$ is an interval and if $x \in \mathbb{R}^d$ is a point, and if f, $\tilde{f} \in \mathcal{F}$ satisfy $\|f(t,x) - \tilde{f}(t,x)\| \leq \eta$ for all $t \in J$, then

$$\left\|f_\delta^j(x) - \tilde{f}_\delta^j(x)\right\| \leq \omega_2(\eta)$$

for all $\delta > 0$ and all j such that $I_j^\delta \subset J$.

Although these properties are cumbersome to state, they are reasonable requirements on a digitization scheme. Now let $f \in \mathcal{F}$ be a vector field satisfying (2.3). Put $f_\delta(t,x) = f_\delta^j(x)$ for $t \in I_j^\delta$, $j \in \mathbb{Z}$. Abusing language slightly, we call $\{f_\delta : \delta > 0\}$ a digitization of f. The vector fields f_δ

discussed in the Introduction are obtained by procedures for which I)–IV) are satisfied, so these f_δ are digitizations in our sense. In fact , the subintervals $\mathcal{I}^\delta$ in I) for each fixed $\delta > 0$ of such digitizations often also satisfies the following recurrence condition, in which case it will be called a **recurrent digitization**.

V) Fix $\delta > 0$. To each $\eta > 0$ there corresponds a number T (which may depend on δ as well as on η) so that each interval $[a, a+T] \subset \mathbb{R}$ contains a number s such that $\operatorname{dist}(\mathcal{I}^\delta, \mathcal{I}^\delta + s) < \eta$. Here $\mathcal{I}^\delta + s$ is the s-translate of $\mathcal{I}^\delta$ and dist is the Hausdorff distance on $\mathbb{R}$.

Now consider the differential equation

$$x' = f_\delta(t, x) \tag{3.2}$$

for each $\delta > 0$. Though f_δ is only piecewise continuous in t, it nevertheless admits a unique local solution $x(t, x_0)$ for each initial condition $x(0, x_0) = x_0 \in \mathbb{R}^d$; moreover $x(t, x_0)$ is jointly continuous on its domain of definition. Using property II) and condition (2.3) on f, we see that f_δ also satisfies condition (2.3) for small $\delta > 0$. It follows that the pullback attractor $A_{f_\delta} \subset \mathbb{R}^d$ of the equation (3.2), which is defined by the formula (2.4), is contained in B_R and is compact for small $\delta > 0$.

For each $\delta > 0$ and $t \in \mathbb{R}$, let $(\theta_t(f))_\delta$ be the digitization of the t-translate of f (we remark parenthetically that $\theta_t(f_\delta) \neq (\theta_t(f))_\delta$ in general). We want to prove that $H^*\left(A_{(\theta_t(f))_\delta}, A_{\theta_t(f)}\right)$ converges to zero as $\delta \to 0$, uniformly in $t \in \mathbb{R}$. That is, we want to prove that $A_{(\theta_t(f))_\delta}$ tends to $A_{\theta_t(f)}$ upper semicontinuously, uniformly in $t \in \mathbb{R}$. Actually we will prove more. Let $P \subset \mathcal{F}$ be the hull of f and let p_δ be the digitization of p for each $p \in P$; then $H^*\left(A_{p_\delta}, A_p\right)$ tends to zero as $\delta \to 0$, uniformly in $p \in P$.

To prove this, it will be convenient to work in an enlarged topological vector space $\mathcal{G}$ which contains $\mathcal{F}$ together with the (in general, temporally discontinuous) vector fields p_δ. We define $\mathcal{G}$ to be the class of jointly Lebesgue measurable mappings $g \in \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following conditions:

a) For each compact set $K \subset \mathbb{R}^d$, one has

$$\sup_{x \in K} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s, x)\| \, ds < \infty;$$

b) For each compact set $K \subset \mathbb{R}^d$ there is a constant L_K (depending on g) so that, for almost all $t \in \mathbb{R}$:

$$\|g(t, x) - g(t, y)\| \leq L_K \|x - y\|, \quad x, y \in K.$$

Now, for each $r = 1, 2, 3, \ldots$ and each $N = 1, 2, 3, \ldots$ introduce a pseudo-metric $d_{r,N}$ on $\mathcal{G}$:

$$d_{r,N}(g_1, g_2) = \sup_{\|x\| \leq r} \int_{-N}^{N} \|g_1(s, x) - g_2(s, x)\| \, ds.$$

Then put

$$d_r(g_1,g_2)=\sum_{N=1}^{\infty}\frac{1}{2^N}\frac{d_{r,N}(g_1,g_2)}{1+d_{r,N}(g_1,g_2)},$$

and finally set

$$d(g_1,g_2)=\sum_{r=1}^{\infty}\frac{1}{2^r}\frac{d_r(g_1,g_2)}{1+d_r(g_1,g_2)}.$$

We identify two elements of $\mathcal{G}$ if their d-distance is zero, thereby obtaining a metric space which we also call $\mathcal{G}$.

Observe that, if $g\in\mathcal{G}$, then the Cauchy problem

$$x'=g(t,x),\quad x(0)=x_0 \tag{3.3}$$

admits a unique, maximally-defined local solution $x(t,x_0)$ for each $x_0\in\mathbb{R}^d$; moreover, $x(t,x_0)$ depends continuously on (t,x_0) on its domain of definition. This can be proved using the standard Picard iteration method to solve (3.3). We will write

$$x(t,x_0)=\phi(t,x_0,g)$$

to maintain consistency with notation used previously.

Observe further that, for each $t\in\mathbb{R}$, the translation $\theta_t:\mathcal{G}\to\mathcal{G}$, i.e., $\theta_t(g)(s,x)=g(t+s,x)$ is well-defined. Let $\mathcal{G}_1\subset\mathcal{G}$ be a translation invariant subset such that the supremum in a) and the constants L_K in b) of the definition of $\mathcal{G}$ are uniform in $g\in\mathcal{G}_1$, for each compact $K\subset\mathbb{R}^d$. Then $(t,g)\to\theta_t(g):\mathbb{R}\times\mathcal{G}_1\to\mathcal{G}_1$ is continuous.

Next let $\delta_0>0$. We will show that the set $\mathcal{U}=P\cup\{p_\delta:p\in P, 0<\delta\le\delta_0\}\subset\mathcal{G}$ is equi-uniformly continuous in the sense that, to each $\epsilon>0$, there corresponds $\eta>0$ such that, if $|t-s|<\eta$, then $d(\theta_t(p),\theta_s(p))<2\epsilon$ and $d(\theta_t(p_\delta),\theta_s(p_\delta))<2\epsilon$ for all p, $p_\delta\in\mathcal{U}$ and for all t, $s\in\mathbb{R}$.

To do this, fix $\epsilon>0$. Recall that $P\subset\mathcal{F}\subset\mathcal{G}$ is the hull of the uniformly continuous function f. Hence if $B\subset\mathbb{R}^d$ is a ball centered at the origin and if $N\ge 1$, then we can find $\eta_1>0$ such that, if $|t-s|<\eta_1$, then

$$\int_{-N}^{N}\|\theta_t(p)(v,x)-\theta_s(p)(v,x)\|\,dv<\epsilon/3$$

for all $x\in B$. Then, taking account of the definition of the distance d, we see that it is sufficient to prove that, for some sufficiently large ball $B\subset\mathbb{R}^d$ and some sufficiently large N, there exists $\eta_2\in(0,\eta_1]$ such that

$$\sup_{x\in B}\int_{-N}^{N}\|\theta_t(p_\delta)(v,x)-\theta_s(p_\delta)(v,x)\|\,dv<\epsilon \tag{3.4}$$

whenever $|t-s|<\eta_2$, $0<\delta\le\delta_0$. Let us write $d_B(\theta_t(p_\delta),\theta_s(p_\delta))$ for the quantity on the left hand side of (3.4).

To prove (3.4), we use the properties I)–IV) of a recurrent digitization. Choose $\epsilon_1>0$ so that $\omega(\epsilon_1)<\epsilon/3$, then choose δ_1 so that, if $0<\delta\le\delta_1$

then $\epsilon_x \le \epsilon_1/(2N)$ for all $x \in B$. Using property II), we see that, if $0 < \delta \le \delta_1$ and $p \in P$, then

$$\begin{aligned} d_B(\theta_t(p_\delta), \theta_s(p_\delta)) &\le d_B(\theta_t(p_\delta), \theta_t(p)) + d_B(\theta_t(p), \theta_s(p)) + d_B(\theta_s(p), \theta_s(p_\delta)) \\ &< 3 \cdot \epsilon/3 = \epsilon \end{aligned}$$

whenever $|t - s| < \eta$.

If $\delta_1 \ge \delta_0$, we set $\eta = \eta_1$ and stop. If $\delta_1 < \delta_0$ and if $\delta_1 < \delta \le \delta_0$, we first choose $\eta_2 \le \delta/100$, then note that on each interval $[u, u + N] \subset \mathbb{R}$ of length N, the difference $p_\delta(t, x) - p_\delta(s, x)$ is zero except on at most $[2N/\delta] + 1$ subintervals of length $2\eta_2$, where $[\cdot]$ denotes the integer part of a positive number. Using the uniform boundedness of the vector fields $p_\delta \in \mathcal{U}$ on $\mathbb{R} \times B$, we can determine $\eta_3 \le \eta_2$ so that, if $|t - s| < \eta_3$, then $d_B(\theta_t(p_\delta), \theta_s(p_\delta)) < \epsilon$. So if $\eta = \min\{\eta_1, \eta_3\}$ we obtain (3.4) for all p, $p_\delta \in \mathcal{U}$.

Now let $\delta \in (0, \delta_0]$. For each $p \in P$, let $P_\delta(p) = \mathrm{cls}\{\theta_\mathrm{t}(\mathrm{p}_\delta) : \mathrm{t} \in \mathbb{R}\}$. Then $P_\delta(p)$ is compact (this uses the recurrence condition V) of a recurrent digitization) and translation invariant in $\mathcal{G}$. Moreover, using property III) of a digitization and a Gronwall-type argument, one shows that the map $(t, x_0, g) \mapsto (\phi(t, x_0, g), \theta_t(g))$ defines a (continuous) skew-product flow on $\mathbb{R}^d \times P_\delta(p)$.

Choose δ_0 so that each p_δ satisfies (2.3) for all $p \in P$ and $0 < \delta \le \delta_0$. Then the pullback attractor A_{p_δ} exists and equals $\{x_0 \in \mathbb{R}^d : \phi(t, x_0, p_\delta)$ is defined on all of $\mathbb{R}$ and satisfies $\|\phi(t, x_0, p_\delta)\| \le R\}$; see Proposition 3.2. In fact, A_{p_δ} is then the p_δ-fiber of a global pullback attractor $\mathbf{A}^\delta \subset \mathbb{R}^d \times P_\delta(p)$. Now $p_\delta \to p$ in $\mathcal{G}$ as $\delta \to 0$, so using property III) of a digitization and a Gronwall argument, together with the characterization of A_{p_δ} in terms of bounded solutions of $x' = p_\delta(t, x)$, one shows that $H^*(A_{p_\delta}, A_p) \to 0$ as $\delta \to 0$. However, more is true. Using property IV) of a digitization one has the following: if $p_n \top \in P$ and if $\delta_n \to 0$, then $d(p_{(n,\delta_n)}, p) \to 0$. Again using II) together with a Gronwall argument and the above characterization of $A_{p_{(n,\delta_n)}}$, one sees that $H^*(A_{(p_{n,\delta_n})}, A_p) \to 0$ as $n \to \infty$. This is a strong uniformity statement, and implies

Proposition 3.3 *$H^*(A_{p_\delta}, A_p) \to 0$ as $\delta \to 0$, uniformly in $p \in P$. In particular,*

$$H^*(A_{(\theta_t(f)_\delta}, A_{\theta_t(f)}) \to 0 \quad \text{as } \delta \to 0,$$

uniformly in $t \in \mathbb{R}$.

We use the recurrence condition V) to prove that the sets $P_\delta(p)$ are compact. This would seem to be a basic requirement to be satisfied when one sets about computing the pullback attractor, because otherwise the convergence in (2.4) of the intersection to A_{p_δ} cannot be expected to have any uniformity properties. We note, however, that Proposition 3.3 could be proved without asumptions ensuring that the sets $P_\delta(p)$ are compact; one only needs the uniform continuity in t on $\mathbb{R}$ (uniform on compact x subsets of $\mathbb{R}^d$) of the vector field $f(t, x)$ and the continuity of the Bebutov flow on $(\mathcal{G}, d)$.

4 Continuity results

In this section, we continue to investigate the perturbation properties of pullback attractors, this time with the goal of giving a sufficient condition for the Hausdorff continuity (and not just upper semicontinuity) of the sets A_p, resp. $\mathbf{A}$, as the base space P is varied in some functional space.

To be specific, let $\mathcal{F}$ be the topological space introduced in Section 2. Let P be a compact, translation-invariant subset of $\mathcal{F}$ (which need not be the hull of any one element $f \in \mathcal{F}$). Let us assume that each $p \in P$ can be written in the form

$$p(t, x) = L_p(t)x + h_p(t, x),$$

where $L_p(\cdot)$ is a uniformly continuous function with values in the set $\mathcal{M}_d$ of real $d \times d$ matrices. We assume that the mappings $(p, t, x) \mapsto L_p(t)x$ and $(p, t, x) \mapsto h_p(t, x)$ are uniformly continuous on compact subsets of $P \times \mathbb{R} \times \mathbb{R}^d$. A sufficient condition that this is the case is the following. Consider the metric space P; put $F(p, x) = p(0, x)$ for each $p \in P$ and $x \in \mathbb{R}^d$. Note that $F(\theta_t(p), x) = p(t, x)$ for all $t \in \mathbb{R}$, $p \in P$. Suppose that the Jacobian $\frac{\partial F}{\partial x}(p, 0)$ is continuous as a function of p. Then, setting

$$L_p(t)x = \frac{\partial F}{\partial x}(\theta_t(p), 0)x, \quad h_p(t, x) = p(t, x) - L_p(t)x,$$

we obtain the desired decomposition.

We now impose the following hypothesys.

(H1) The family of linear systems

$$x' = L_p(t)x, \quad p \in P,$$

admits an exponential dichotomy over P with constants $L > 0$, $\gamma > 0$ and continuous family of projections $\{Q_p : p \in P\}$.

Let $C_b = C_b(\mathbb{R}, \mathbb{R}^d)$ be the Banach space of bounded continuous functions $x : \mathbb{R} \to \mathbb{R}^d$ with the norm $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$. For each $p \in P$, define a nonlinear operator $T_p : C_b \to C_b$ as follows:

$$\begin{aligned} T_p[x](t) \quad = \quad & \int_t^\infty \Psi_p(t)Q_p\Psi_p(s)^{-1}h_p(s, x(s))\, ds \\ & + \int_{-\infty}^t \Psi_p(t)(Q_p - I)\Psi_p(s)^{-1}h_p(s, x(s)),\, ds \end{aligned}$$

where $\Psi_p(t)$ is the fundamental matrix with initial value $\Psi_p(0) = I$ (identity matrix) of the linear equation $x' = L_p(t)x$.

Assume from now on that condition (2.3) holds for all $p \in P$. We further impose a condition of uniform contractivity.

(H2) For all $p \in P$ and for all x, y in B_R, one has

$$\|h_p(t, x) - h_p(t, x)\| \le k\|x - y\|,$$

where $k < \gamma/2L$.

To simplify the analysis, we now modify each $h_p(t,x)$ outside the ball B_R so that h_p satisfies Hypothesis (H2) for all $x \in \mathbb{R}^d$ and so that $h_p(t,x) = 0$ whenever $\|x\| \geq R+1$. This means that condition (2.3) does not hold if $\|x\| \geq R+1$, but it will clear that this will have no effect on our analysis of the pullback attractors of the equations (4.1).

Proposition 4.1 *For each $p \in P$, the equation*

$$x' = p(t,x), \tag{4.1}$$

admits a unique solution $x_p(t)$ which is bounded on all of $\mathbb{R}$.

Proof. The argument is standard (see, e.g., Fink [3]). The operator T_p is a contraction on C_b and hence admits a unique fixed point $x_p(\cdot)$, which is a bounded solution of (4.1). Since each fixed point of T_p in C_b is a bounded solution of (4.1) the proposition is proved. □

Using Propositions 3.2 and 4.1, we see that, for each $p \in P$, the pullback attractor $A_p = \{x_p(0)\}$, i.e., each A_p contains exactly one point. Then from continuity with respect to parameters of the fixed point of a contractive mapping, we see that $\mathbf{A} = \{(x_p(0), p) : p \in P\} \subset \mathbb{R}^d \times P$ is compact. One verifies that $\mathbf{A} \subset \mathbb{R}^d \times P$ is the global pullback attractor for the family (4.1).

Now, if $\{x_0\}$ is a singleton subset of a metric space $\mathcal{X}$, and if $B \subset \mathcal{X}$ is compact, then $H^*(B, \{x_0\})$ coincides with the Hausdorff distance $H(B, \{x_0\})$. This fact will allow us to prove that, if $p \in P$, then A_p is a point of Hausdorff continuity for the pullback attractors $A_{\tilde{p}}$ of equations $x' = \tilde{p}(t,x)$ obtained by appropriate perturbations $\tilde{p}$ of p. We will formulate a fairly general continuity result whose hypotheses are satisfied in particular by the digitizations of Section 3.

We view the compact metric space P as a subset of $\mathcal{G}$. Choose fixed values for the suprema in a) and for the constants L_K in b) of the definition of $\mathcal{G}$, and let $\mathcal{G}_1 \subset \mathcal{G}$ be the set of all $g \in \mathcal{G}$ for which a) and b) hold with these fixed values. As in Section 3, write $\theta_t(g)(s,x) = g(t+s,x)$, and let $\phi(t,x_0,g)$ denote the solution of the Cauchy problem (3.3) for each $g \in \mathcal{G}_1$. Using a Gronwall-type inequality, one verifies that the mapping $(t,x_0,g) \mapsto (\phi(t,x_0,g), \theta_t(g))$ is continuous on its domain of definition $V \subset \mathbb{R} \times \mathbb{R}^d \times \mathcal{G}_1$, and defines a continuous local skew-product flow on $\mathbb{R}^d \times \mathcal{G}_1$.

Next let $\tilde{P} \in \mathcal{G}_1$ be a **compact** translation-invariant set. The Hausdorff semi-metric $H^*(\tilde{P}, P)$ is defined relative to the metric d in $\mathcal{G}_1$. Let $\eta_* > 0$ be a constant so that

$$\langle p(t,x), x \rangle \leq -\eta_* \tag{4.2}$$

for all $p \in P$, $t \in \mathbb{R}$, and $x \in \mathbb{R}^d$ with $\|x\| = R$. One can show that there exists $\delta > 0$ so that, if $H^*(\tilde{P}, P) < \delta$, and if $\tilde{p} \in \tilde{P}$, then for each $x_0 \in \mathbb{R}^d$ with $\|x_0\| = R$, the solution $x(t)$ of the Cauchy problem

$$x' = \tilde{p}(t,x), \quad x(0) = x_0,$$

satisfies $\|x(t)\| < R$ for all $t > 0$. The proof uses the continuity of the local skew-product flow on $\mathbb{R}^d \times \mathcal{G}_1$. This means that the family $\{(4.1) : \; p \in P\}$ admits a global pullback attractor $\tilde{\mathbf{A}}$, which lies in $B_R \times \tilde{P}$.

Since A_p is a singleton set, we have $H^*(A_p, A_{p_\delta}) \le H^*(A_{p_\delta}, A_p)$, whether the sets A_{p_δ} are singleton sets or not. Thus in Proposition 3.3 we actually have continuous convergence in this case.

Proposition 4.2 *For each each $\delta > 0$, let $\{p_\delta : p \in P\}$ be a subset of $\mathcal{G}$ such that $d(p_\delta, p) \to 0$ as $\delta \to 0$, uniformly with respect to $p \in P$. Then $H\left(A_{p_\delta}, A_p\right) \to 0$ as $\delta \to 0$, uniformly in $p \in P$.*

Note that the equations $x' = p_\delta(t, x)$ need not give rise to contractions in C_b, so this result cannot be proved using the continuity properties of fixed points of contraction mappings.

5 Example

We give an example to illustrate the strength of Proposition 4.2. We will work with **quasi-periodic** vector fields. Let $\mathbb{T}^k$ be the k-torus, $k \ge 2$, and let $\gamma = (\gamma_1, \dots, \gamma_k)$ be a rationally independent vector in $\mathbb{R}^k$. Let $(\phi_1, \dots, \phi_k)$ be angular coordinates $\mod 2\pi$ on $\mathbb{T}^k$. Introduce the corresponding Kronecker flow $\{\theta_t : t \in \mathbb{R}\}$ on $\mathbb{T}^k$ by setting $\theta_t(\phi_1, \dots, \phi_k) = (\phi_1 + \gamma_1 t, \dots, \phi_k + \gamma_k t)$. For brevity we set $\phi = (\phi_1, \dots, \phi_k)$ and $\theta_t(\phi) = \phi + \gamma t$.

Let $F : \mathbb{T}^k \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that the Jacobian $\frac{\partial F}{\partial x}(\phi, 0)$ is defined and is a continuous function of $\phi \in \mathbb{T}^k$. Let $L_\phi(t) = \frac{\partial F}{\partial x}(\theta_t(\phi), 0)$ and $h_\phi(t, x) = F(\theta_t(\phi), x) - L_\phi(t)x$. Suppose that the family of linear equations

$$x' = L_\phi(t)x, \quad \phi \in \mathbb{T}^k,$$

admits an exponential dichotomy over $\mathbb{T}^k$ with constants $L > 0$, $\gamma > 0$ and a continuous family of projections $\{Q_\phi : \phi \in \mathbb{T}^k\}$. Suppose further that there exists $R > 0$ so that $\langle F(\phi, x), x\rangle < 0$ for all $\phi \in \mathbb{T}^k$ and $x \in \mathbb{R}^d$ with $\|x\| \ge R$. Suppose finally that there is a constant $k < \gamma/(2L)$ such that

$$\|h_\phi(t, x) - h_\phi(t, y)\| \le k\|x - y\|$$

for all $\phi \in \mathbb{T}^k$ and x, y in $\mathbb{R}^d$ with $\|x - y\| \le R$.

Let $G_n = G_n(\phi, x)$ be any sequence of continuous functions on $\mathbb{T}^k \times \mathbb{R}^d$ with values in $\mathbb{R}^d$ such that

1) $G_n \to 0$ uniformly on $\mathbb{T}^k \times B_R$ as $n \to \infty$;

2) There is a real number M such that

$$\|G_n(\phi, x) - G_n(\phi, y)\| \le M\|x - y\|$$

for all $\phi \in \mathbb{T}^k$ and x, $y \in B_R$ and $n \ge 1$.

We do not assume that $\frac{\partial G_n}{\partial x}(\phi, 0)$ exists, nor that M is small. Hence, if we consider the vector functions

$$F_n(\phi, x) = F(\phi, x) + G_n(\phi, x),$$

it may not be the case that the family of equations

$$x' = F_n(\theta_t(\phi), x), \quad \phi \in \mathbb{T}^k, \tag{5.1}$$

generates a family of fixed-point mappings $\{T_\phi\}$, and even if it does, there is no guarantee that any T_ϕ is a contraction on C_b.

The F_n are of course perturbations of F. Let us now introduce a further perturbative element. Namely, let $\gamma^{(n)} = (\gamma_1^{(n)}, \dots, \gamma_k^{(n)})$ be a sequence of frequency vectors such that $\gamma^{(n)} \to \gamma$ in $\mathbb{R}^k$. Of course, it is not assumed that the $\gamma^{(n)}$ are rationally independent. Write $\theta_t^{(n)}(\phi_1, \dots, \phi_k) = (\phi_1 + \gamma_1^{(n)} t, \dots, \phi_k + \gamma_k^{(n)} t)$ for the corresponding Kronecker flow with $n = 1, 2, 3, \dots$.

Now let

$$P = \left\{p \in \mathcal{F} : \ p(t,x) = F(\theta_t(\phi), x) \text{ for some } \phi \in \mathbb{T}^k\right\},$$

$$P^{(n)} = \left\{p \in \mathcal{F} : \ p(t,x) = F_n(\theta_t^{(n)}(\phi), x) \text{ for some } \phi \in \mathbb{T}^k\right\}, \quad n \geq 1.$$

These are all compact translation-invariant subsets of $\mathcal{F}$. One can verify that $H(P^{(n)}, P) \to 0$ as $n \to \infty$; this is true even though the frequency vector has been perturbed. Since condition (2.3) is satisfied by the family (5.1) for all sufficiently large n, there are pullback attractors $\mathbf{A}^{(n)} \subset \mathbb{R}^d \times P^{(n)}$ for each such n defined by the respective families of equations (5.1). Using the arguments preceding Proposition 4.2, one has that $H(\mathbf{A}^{(n)}, \mathbf{A}) \to 0$ as $n \to \infty$, where $\mathbf{A}$ is the pullback attractor defined by the equations

$$x' = F(\theta_t(\phi), x), \quad \phi \in \mathbb{T}^k.$$

Let us now use this example to illustrate how information can be obtained concerning convergence of pullback attractors under digitization. Let the letters F, F_n, G_n, $\gamma^{(n)}$ have the significance attributed to them above. Suppose we are given a digitization scheme satisfying the conditions I)–IV) of Section 3.

Let $P = \{p \in \mathcal{G} : p(t,x) = F(\theta_t(\phi), x) \text{ for some } \phi \in \mathbb{T}^k\}$; thus P is the same as before except that now it is viewed as a subset of $\mathcal{G}$. Let $\{\delta_n\}$ be a sequence of positive numbers which converges to zero. For each $\phi \in \mathbb{T}^k$, let $p_\phi^{(n)}$ be the δ_n-digitization of the time-varying vector field $(t,x) \to F_n(\phi + \gamma^{(n)} t, x)$.

Now, for large n, each $p_\phi^{(n)}$ (ϕ in $\mathbb{T}^k$) satisfies condition (2.3), so the pullback attractor $A_\phi^{(n)}$ defined by the equation $x' = p_\phi^{(n)}(t,x)$ is contained in B_R. Let $\mathbf{A} \subset \mathbb{R}^d \times P$ be the global pullback attractor defined by the family of equations (1.2). Each $p \in P$ corresponds to (at least) one point $\phi \in \mathbb{T}^k$. Let us write $p = p_\phi$ if p corresponds to ϕ, then write A_ϕ instead of A_p for the fiber of $\mathbf{A}$ at p. Each fiber A_ϕ is a singleton subset of $\mathbb{R}^d$.

Finally, arguing as in Section 3, we can show that $H(A_\phi^{(n)}, A_\phi) \to 0$ as $n \to \infty$. In fact, the convergence here is uniform with respect to $\phi \in \mathbb{T}^k$.

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