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Existence of positive periodic solutions for non-autonomous functional differential equations *

Sui Sun Cheng & Guang Zhang

Abstract

We establish the existence of positive periodic solutions for a first-order differential equation with periodic delay. For this purpose, we use the fixed point theorem proved by Krasnoselskii.

1 Introduction

In this article, we investigate the existence of positive periodic solutions for the first-order functional differential equation

$$y'(t) = -a(t)y(t) + \lambda h(t)f(y(t-\tau(t))), \tag{1.1}$$

where $a = a(t)$, $h = h(t)$ and $\tau = \tau(t)$ are continuous T-periodic functions. We assume that $T, \lambda > 0$, that $a = a(t)$, $f = f(t)$ and $h = h(t)$ are nonnegative, and that $a(t_0) > 0$ for some $t_0 \in [0, T]$.

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equation can be interpreted as the standard Malthus population model $y' = -a(t)y$ subject to a perturbation with periodical delay. One important question is whether these equations can support positive periodic solutions. Such question has been studied extensively by a number of authors; see for example [4, 3, 1, 2, 5] and the references therein. In this paper, we will obtain existence criteria for T-periodic solutions of (1.1) by means of a well known fixed point theorem due to Krasnoselskii.

Theorem 1.1 *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that*

1. *$\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$ for $P \cap \partial\Omega_2$, or that*
2. *$\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$ for $P \cap \partial\Omega_2$.*

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Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For the sake of convenience, the conditions needed for our criteria are listed as follows:

H1) $f \in C([0,\infty),[0,\infty))$ and there are $x_n \to 0$ such that $f(x_n) > 0$ for $n = 1, 2, \ldots$.

H2) $h(t) > 0$ for $t \in R$.

H3) $\sup_{r>0} \min_{r\sigma \le x \le r} f(x) > 0$, with σ to be defined later.

H4) $f \in C([0,\infty),[0,\infty))$ and $f(x) > 0$ for $x > 0$.

L1) $\lim_{x\to 0} f(x)/x = \infty$

L2) $\lim_{x\to \infty} f(x)/x = \infty$

L3) $\lim_{x\to 0} f(x)/x = 0$

L4) $\lim_{x\to \infty} f(x)/x = 0$

L5) $\lim_{x\to 0} f(x)/x = l$ with $0 < l < \infty$

L6) $\lim_{x\to \infty} f(x)/x = L$ with $0 < L < \infty$.

2 Main Result

We proceed formally from (1.1) to obtain

$$[y(t)\exp(\int_{-\infty}^{t} a(s)ds)]' = \lambda \exp(\int_{-\infty}^{t} a(s)ds)h(t)f(y(t-\tau(t))).$$

After integration from t to $t+T$, we obtain

$$y(t) = \lambda \int_{t}^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds, \tag{2.1}$$

where

$$G(t,s) = \frac{\exp(\int_t^s a(u)du)}{\exp(\int_0^T a(u)du) - 1}.$$

Note that the denominator in $G(t,s)$ is not zero since we have assumed that $a(t_0) > 0$ for some $t_0 \in [0,T]$. It is not difficult to check that any function $y(t)$ that satisfies (2.1) is also a T-periodic solution of (1.1). Note that

$$N \equiv G(t,t) \le G(t,s) \le G(t,t+T) = G(0,T) \equiv M,\ t \le s \le t+T,$$

and

$$1 \ge \frac{G(t,s)}{G(t,t+T)} \ge \frac{G(t,t)}{G(t,t+T)} = \frac{N}{M} > 0.$$

Now let X be the set of all real T-periodic continuous functions, endowed with the usual linear structure and the norm

$$\|y\| = \sup_{t\in[0,T]} |y(t)|.$$

Then X is a Banach space with cone

$$\Omega = \{y(t) : y(t) \geq \sigma\|y(t)\|,\ t \in R\},$$

where $\sigma = N/M$. Note that $a(t_0) > 0$ for some $t_0 \in [0,T]$. Clearly, $\sigma \in (0,1)$. Define a mapping $T : X \to X$ by

$$(Ty)(t) = \lambda \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds.$$

Then it is easily seen that T is completely continuous on bounded subset of Ω, and for $y \in \Omega$,

$$(Ty)(t) \leq \lambda M \int_0^T h(s)f(y(s-\tau(s)))ds$$

so that

$$(Ty)(t) \geq \lambda N \int_0^T h(s)f(y(s-\tau(s)))ds \geq \sigma\|Ty\|.$$

That is, $T\Omega$ is contained in Ω.

Lemma 2.1 *With the above notation, $T\Omega \subset \Omega$.*

Lemma 2.2 *Assume that there exist two positive numbers a and b such that $a \neq b$,*

$$\max_{0\leq x\leq a} f(x) \leq \frac{a}{\lambda A}, \tag{2.2}$$

and

$$\min_{\sigma b\leq x\leq b} f(x) \geq \frac{b}{\lambda B} \tag{2.3}$$

where

$$A = \max_{0\leq t\leq T} \int_0^T G(t,s)h(s)ds \tag{2.4}$$

and

$$B = \min_{0\leq t\leq T} \int_0^T G(t,s)h(s)ds. \tag{2.5}$$

Then there exists $\overline{y} \in \Omega$ which is a fixed point of T and satisfies $\min\{a,b\} \leq \|\overline{y}\| \leq \max\{a,b\}$.

Proof. Let $\Omega_\xi = \{w \in \Omega | \|w\| < \xi\}$. Assume that $a < b$. Then, for any $y \in \Omega$ which satisfies $\|y\| = a$, in view of (2.2), we have

$$(Ty)(t) \leq \{\lambda \int_t^{t+T} G(t,s)h(s)ds\} \cdot \frac{a}{\lambda A} \leq \lambda A \cdot \frac{a}{\lambda A} = a. \tag{2.6}$$

That is, $\|Ty\| \leq \|y\|$ for $y \in \partial\Omega_a$. For any $y \in \Omega$ which satisfies $\|y\| = b$, we have

$$(Ty)(t) \geq \{\lambda \int_t^{t+T} G(t,s)h(s)ds\} \cdot \frac{b}{\lambda B} \geq \lambda B \cdot \frac{b}{\lambda B}. \tag{2.7}$$

That is, we have $\|Ty\| \geq \|y\|$ for $y \in \partial\Omega_b$. In view of Theorem 1.1, there exists $\overline{y} \in \Omega$ which satisfies $a \leq \|\overline{y}\| \leq b$ such that $T\overline{y} = \overline{y}$. If $a > b$, (2.6) is replaced by $(Ty)(t) \geq b$ inview of (2.3), and (2.7) is replaced by $(Ty)(t) \leq a$ in view of (2.2). The same conclusion then follows. The proof is complete.

Theorem 2.3 *Suppose (H1), (H2), (L1) and (L2) hold. Then for any $\lambda \in (0, \lambda^*)$, equation (1.1) has at least two positive periodic solutions, where*

$$\lambda^* = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0\leq x\leq r} f(x)},$$

and A is defined by (2.4).

Proof. Let $q(r) = r/(A \max_{0\leq x\leq r} f(x))$. In view of (H1), we have that $q \in C((0,\infty),(0,\infty))$. In view of (L1) and (L2), we see further that $\lim_{r\to 0} q(r) = \lim_{r\to\infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, by the intermediate value theorem, there exist $a_1 \in (0, r_0)$ and $a_2 \in (r_0, \infty)$ such that $q(a_1) = q(a_2) = \lambda$. Thus, we have $f(x) \leq a_1/(\lambda A)$ for $x \in [0, a_1]$ and $f(x) \leq a_2/(\lambda A)$ for $x \in [0, a_2]$. On the other hand, in view of (L1) and (L2), we see that there exist $b_1 \in (0, a_1)$ and $b_2 \in (a_2, \infty)$ such that $f(x)/x \geq 1/(\lambda\sigma B)$ for $x \in (0, b_1] \cup [b_2\sigma, \infty)$. That is, $f(x) \geq b_1/(\lambda B)$ for $x \in [b_1\sigma, b_1]$ and $f(x) \geq b_2/(\lambda B)$ for $x \in [b_2\sigma, b_2]$. An application of Lemma 2.2 leads to two distinct solutions of (1.1).

We remark that the arguments in the above proof actually yield the following result: If (H1) and (H2) hold, and if either (L1) or (L2) holds, then for any $0 < \lambda < \lambda^*$, equation (1.1) has at least one positive periodic solution.

Theorem 2.4 *Suppose (H2), (H4), (L3) and (L4) hold. Then for any $\lambda > \lambda^{**}$, equation (1.1) has at least two positive periodic solutions, where*

$$\lambda^{**} = \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\sigma r\leq x\leq r} f(x)},$$

and B is defined by (2.5).

Proof. Let $p(r) = r/(B \min_{\sigma r \le x \le r} f(x))$. Clearly, $q \in C((0,\infty),(0,\infty))$. From (L3) and (L4), we see that $\lim_{r\to 0} p(r) = \lim_{r\to\infty} p(r) = \infty$. Thus, there exists $r_0 > 0$ such that $p(r_0) = \min_{r>0} p(r) = \lambda^{**}$. For any $\lambda > \lambda^{**}$, there exist $b_1 \in (0, r_0)$ and $b_2 \in (r_0, \infty)$ such that $p(b_1) = p(b_2) = \lambda$. Thus, we have $f(x) \ge b_1/(\lambda B)$ for $x \in [\sigma b_1, b_1]$ and $f(x) \ge b_2/(\lambda B)$ for $x \in [\sigma b_2, b_2]$. On the other hand, in view of (L3), we see that $f(0) = 0$ and that there exists $a_1 \in (0, b_1)$ such that $f(x)/x \le 1/(\lambda A)$ for $x \in (0, a_1]$. Thus, we have $f(x) \le a_1/(\lambda A)$. In view of (L4), we see that there exists $a \in (b_2, \infty)$ such that $f(x)/x \le 1/(\lambda A)$ for $x \in [a, \infty)$. Let $\delta = \max_{0\le x\le a} f(x)$. Then we have $f(x) \le a_2/(\lambda A)$ for $x \in [0, a_2]$, where $a_2 > a$ and $a_2 \ge \lambda\delta A$. An application of Lemma 2.2 leads to two distinct solutions of (1.1).

Again, we remark that the proof of Theorem 2.4 shows the following: If (H1), (H2) and (H3) hold, and if (L3) or (L4) holds, then for any $\lambda > \lambda^{**}$, equation (1.1) has a positive periodic solution.

Theorem 2.5 *Assume that (H1), (H2), (L5) and (L6) hold. Then, for each λ satisfying*

$$\frac{1}{\sigma BL} < \lambda < \frac{1}{Al} \tag{2.8}$$

or

$$\frac{1}{\sigma Bl} < \lambda < \frac{1}{AL},$$

equation (1.1) has a positive periodic solution.

Proof. Suppose (2.8) holds. Let $\varepsilon > 0$ be such that

$$\frac{1}{\sigma B(L-\varepsilon)} \le \lambda \le \frac{1}{A(l+\varepsilon)}.$$

Note that $l > 0$, thus there exists $H_1 > 0$ such that $f(x) \le (l+\varepsilon)x$ for $0 < x \le H_1$. So, for $y \in \Omega$ with $\|y\| = H_1$, we have

$$\begin{aligned}(Ty)(t) &\le \lambda(l+\varepsilon)\int_t^{t+T} G(t,s)h(s)y(s-\tau(s))ds \\ &\le \lambda(l+\varepsilon)\|y\|\int_0^T G(t,s)h(s)ds \\ &\le \lambda A(l+\varepsilon) \le \|y\|.\end{aligned}$$

Next, since $L > 0$, there exists a $\overline{H}_2 > 0$ such that $f(x) \ge (L-\varepsilon)x$ for $x \ge \overline{H}_2$. Let $H_2 = \max\{2H_1, \sigma\overline{H}_2\}$, then for $y \in \Omega$ with $\|y\| = H_2$,

$$\begin{aligned}(Ty)(t) &\ge \lambda(L-\varepsilon)\int_t^{t+T} G(t,s)h(s)y(s-\tau(s))ds \\ &\ge \lambda(L-\varepsilon)\sigma\|y\|\int_0^T G(t,s)h(s)ds \\ &\ge \lambda(L-\varepsilon)\sigma B\|y\| \ge \|y\|.\end{aligned}$$

In view of Lemma 2.2, we see that equation (1.1) has a positive periodic solution. The other case is similarly proved.

Corollary 2.6 *Assume that (H1) and (H2) hold. Assume further that either (L1) and (L4) hold, or, (L2) and (L3) hold, then for any* $\lambda > 0$*, equation (1.1) has a positive periodic solution.*

Proof. Suppose first that (L1) and (L4) hold. If $\sup_{0\le x<\infty} f(x) = D < \infty$, then $\lambda^* \ge (1/A)\sup_{r>0}(r/D) = \infty$. If $f(x)$ is unbounded, then there exist a sequence $\{r_n\}$ such that $f(r_n) = \max_{0\le x\le r_n} f(x)$ and $\lim_{n\to\infty} r_n = \infty$. In view of (L4), we have $\lambda^* \ge (1/A)\sup(r_n/f(r_n)) = \infty$. Thus, we have proved $\lambda^* = \infty$. In this case, our assertion follows from the remark following Theorem 2.3. If (L2) and (L3) hold, then we have $\lim_{x\to\infty} f(x) = \infty$. Thus, (H3) holds. Let $\{r_n\}$ satisfy $\lim_{n\to\infty} r_n = \infty$ and $f(\sigma r_n) = \min_{\sigma r_n\le x\le r_n} f(x)$. In view of (L2), we have $\lambda^{**} \le (1/B)\inf(r_n/f(\sigma r_n)) = 0$. Thus, $\lambda^{**} = 0$. In this case, our assertion follows from the remark following Theorem 2.4.

Corollary 2.7 *Assume that (H1) and (H2) hold. Assume further that either (L1) and (L6) hold, or, (L2) and (L5) hold. Then for any* $0 < \lambda < 1/(Al)$ *or* $0 < \lambda < 1/(AL)$ *equation (1.1) has a positive periodic solution.*

Corollary 2.8 *Assume that (H1) and (H2) hold. Assume further that either (L3) and (L6) hold, or, (L4) and (L5) hold. Then for any* $1/(\sigma LB) < \lambda < \infty$ *or* $1/(\sigma lB) < \lambda < \infty$ *equation (1.1) has a positive periodic solution.*

Similarly, we can also discuss the equation

$$x'(t) = a(t)x(t) - \lambda h(t)f(x(t-\tau(t))). \tag{2.9}$$

where $a = a(t)$, $h = h(t)$ and $f = f(t)$ satisfy the same assumptions stated for equation (1.1). By (2.9), we have

$$x(t) = \int_t^{t+T} H(t,s)h(s)f(x(s-\tau(s))ds,$$

where

$$H(t,s) = \frac{\exp(-\int_t^s a(u)du)}{1-\exp(-\int_0^T a(u)du)} = \frac{\exp(\int_s^{t+T} a(u)du)}{\exp(\int_0^T a(u)du - 1)}$$

which satisfies

$$M = G(0,T) = H(t,t) \ge H(t,s) \ge H(t,t+T) = H(0,T) = G(t,t) = N$$

and

$$1 \ge \frac{H(t,s)}{H(t,t)} \ge \frac{H(t,t+T)}{H(t,t)} = \frac{N}{M} = \sigma.$$

Let

$$A' = \max_{0\le t\le T} \int_0^T H(t,s)h(s)ds$$

and

$$B' = \min_{0\le t\le T} \int_0^T H(t,s)h(s)ds.$$

Then we have the following results.

Theorem 2.9 *Assume that (H1) and (H2) hold. Suppose further that either (L1) or (L2) holds. Then for any $\lambda \in (0, \overline{\lambda})$, equation (2.9) has a positive periodic solution, where*

$$\overline{\lambda} = \frac{1}{A'} \sup_{r>0} \frac{r}{\max_{0\le x\le r} f(x)}.$$

Theorem 2.10 *Suppose (H1), (H2), (L1) and (L2) hold. Then for any $\lambda \in (0, \overline{\lambda})$, equation (2.9) has at least two positive periodic solutions.*

Theorem 2.11 *Assume that (H1), (H2) and (H3). Suppose further that either (L3) or (L4) holds. Then for any $\lambda > \underline{\lambda}$, equation (2.9) has a positive periodic solution, where*

$$\underline{\lambda} = \frac{1}{B'} \inf_{r>0} \frac{r}{\min_{\sigma r\le x\le r} f(x)}.$$

Theorem 2.12 *Suppose (H2), (H4), (L3) and (L4) hold. Then for any $\lambda > \underline{\lambda}$, equation (2.9) has at least two positive periodic solutions.*

Theorem 2.13 *Assume that (H1), (H2), (L5) and (L6) hold. Then, for each λ satisfying*

$$\frac{1}{\sigma B' L} < \lambda < \frac{1}{A' l}$$

or

$$\frac{1}{\sigma B' l} < \lambda < \frac{1}{A' L},$$

equation (2.9) has a positive periodic solution.

Corollary 2.14 *Assume that (H1) and (H2) hold. Suppose further that either (L1) and (L4) hold, or, (L2) and (L3) hold. Then for any $\lambda > 0$, equation (2.9) has a positive periodic solution.*

Corollary 2.15 *Assume that (H1) and (H2) hold. Suppose further that either (L1) and (L6) hold, or, (L2) and (L5) hold. Then for any $0 < \lambda < 1/(A'L)$ or $0 < \lambda < 1/(A'l)$ equation (2.9) has a positive periodic solution.*

Corollary 2.16 *Assume that (H1) and (H2) hold. Suppose further that either (L3) and (L6) hold, or, (L4) and (L5) hold. Then for any $1/(\sigma LB') < \lambda < \infty$ or $1/(\sigma lB') < \lambda < \infty$ equation (2.9) has a positive periodic solution:*

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Sui Sun Cheng
Department of Mathematics, National Tsing Hua University
Hsinchu, Taiwan 30043
e-mail: sscheng@math2.math.nthu.edu.tw

Guang Zhang
Department of Mathematics, Yanbei Normal Institute
Datong, Shanxi 037000, P.R.China
e-mail: dtgzhang@yahoo.com.cn