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ON THE MULTIPLICITY OF SOLUTIONS FOR A FULLY NONLINEAR EMDEN–FOWLER EQUATION

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Abstract. We are concerned with the existence of two solutions for a fully nonlinear Emden–Fowler type equation. One solution is obtained via local minimization while the second solution follows by a mountain pass argument. A non-existence result in strictly star-shaped domains is also proven.

1. Introduction

Let $\mathcal{M}$ be a C^∞ compact connected manifold of dimension two and g a metric on $\mathcal{M}$. As known, the problem of finding a conformal metric g' such that the scalar curvature of $(\mathcal{M}, g')$ is equal to a given function $\mathcal{K}(x)$, gives rise to the following problem (scalar curvature problem, Nirenberg 1974)

$$-\Delta_g u + R_g u = \mathcal{K}(x)e^{2u}\,, \quad g' = e^{2u} g\,, \tag{1.1}$$

where R_g denotes the curvature of $\mathcal{M}$ and

$$\Delta_g = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x_j} \right)$$

is the Laplace–Beltrami operator. Problems like (1.1) are also involved in the study of stellar structure and in nonlinear diffusion and heat transfer in chemical kinetic.

Starting from these geometrical and physical motivations, let us consider a smooth bounded domain Ω in $\mathbb{R}^n$ with $n \geqslant 2$ and a possibly changing sign function $\mathcal{K}$ in $L^q(\Omega)$ for some $q > 1$ with $\mathcal{K} > 0$ a.e. in an open ball B of Ω. When $n = 2$, the semilinear elliptic equation with exponential growth

$$\begin{aligned} -\Delta u &= \mathcal{K}(x)\, e^u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has been studied in 1974 by Kazdan and Warner in [9] and in 1992 by Brézis and Merle in [5]. In the case of the $p-$Laplacian problem ($p > 1$ and $\lambda > 0$)

$$\begin{aligned} -\Delta_p u &= \lambda \mathcal{K}(x) e^u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

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a complete picture has been given by Aguilar Crespo and Peral Alonso in 1996 in [1]. In particular, it was shown existence of solutions of (1.2) for λ small (one solution for $p < n$ and two solutions for $p \geqslant n$) and nonexistence for values of λ sufficiently large. The case $p < n$ was treated by a fixed point argument, while the case $p \geqslant n$ was investigated by classical critical point methods. (see also [3]). In this note, we are concerned with the following more general problem at exponential growth

$$\begin{gathered} -\operatorname{div}\left(\nabla_{\xi} \mathcal{L}(x, u, \nabla u)\right)+D_{s} \mathcal{L}(x, u, \nabla u)=\lambda \mathcal{K}(x) e^{u} \quad \text { in } \Omega \\ u=0 \quad \text { on } \partial \Omega . \end{gathered} \tag{1.3}$$

For nonlinearities of power type problems like (1.3) have been studied in [2] and recently in [10, 12] by techniques of non-smooth analysis. Analogously, in our case the functional $f_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with (1.3)

$$f_{\lambda}(u)=\int_{\Omega} \mathcal{L}(x, u, \nabla u)\, dx-\lambda \int_{\Omega} \mathcal{K}(x) e^{u}\, dx\,, \tag{1.4}$$

is well defined for $p \geqslant n$, smooth if $p > n$ by the Morrey embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega),$$

but fails to be regular when $p = n$ unless $\mathcal{L}$ does not depend on u or it is subjected to some very restrictive growth conditions. Indeed, with natural growth conditions (see (1.5) and (1.6) below), in general, being

$$\forall s<+\infty: \ W_{0}^{1, n}(\Omega) \hookrightarrow L^{s}(\Omega) \qquad \text{but} \quad W_{0}^{1, n}(\Omega) \not\hookrightarrow L^{\infty}(\Omega)\,,$$

if $u \in W_0^{1,n}(\Omega)$ it may happen that

$$D_s\mathcal{L}(x, u, \nabla u) \notin W^{-1,n'}(\Omega)\,,$$

so that f_λ is not even locally Lipschitzian. Therefore we focus on the case $p = n$ and prove the existence of at least two nontrivial solutions in $W_0^{1,n}(\Omega)$ of (1.3) for λ positive and small. To solve (1.3) we look for critical points of (1.4) in the sense of non-smooth critical point theory (see [6, 12] and references therein). The case $p > n$ may be treated in a similar fashion via classical critical point theory.

We assume that $\mathcal{L} : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s, \xi) \in \mathbb{R}\times\mathbb{R}^n$ and of class C^1 in (s, ξ) a.e. in Ω. Moreover $\mathcal{L}(x, s, \cdot)$ is strictly convex, $n-$homogeneous with $\mathcal{L}(x, s, 0) = 0$ and the following conditions hold:

($\mathscr{H}_1$) there exist $a_1 \in L^1(\Omega)$, $r > 1$ and $b_0, b_1, b_2, \nu > 0$ such that:

$$\nu|\xi|^n \leqslant \mathcal{L}(x, s, \xi) \leqslant b_0|s|^r + b_0|\xi|^n, \tag{1.5}$$

$$|D_s\mathcal{L}(x, s, \xi)| \leqslant a_1(x) + b_1|\xi|^n, \quad |\nabla_\xi\mathcal{L}(x, s, \xi)| \leqslant b_2|\xi|^{n-1} \tag{1.6}$$

a.e. in Ω and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

($\mathscr{H}_2$) there exist $R > 0$ and $\gamma > 0$ such that:

$$|s| \geqslant R \implies D_s\mathcal{L}(x, s, \xi)s \geqslant 0, \tag{1.7}$$

$$\gamma\mathcal{L}(x, s, \xi) - D_s\mathcal{L}(x, s, \xi)s \geqslant \nu|\xi|^n \tag{1.8}$$

a.e. in Ω and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

The growth conditions of $(\mathscr{H}_1)$ and the assumptions in $(\mathscr{H}_2)$ are natural in the fully nonlinear setting and were considered in [12] and in a stronger form in [2, 10] also.

Under the preceding assumptions, the following is our main result.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that (1.3) admits at least two nontrivial solutions in $W_0^{1,n}(\Omega)$ for each $\lambda < \lambda_0$.*

This result extends [1, Theorem 4.1] to a more general class of nonlinear elliptic equations. In particular, quasilinear $n-$Laplacian problems of the type

$$-\operatorname{div}(a(u)|\nabla u|^{n-2}\nabla u) + a'(u)|\nabla u|^n = \lambda\mathcal{K}(x)e^u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

admit a pair of solutions for λ small and $a \in C^1(\mathbb{R})$ satisfying suitable assumptions.

Remark 1.2. In general, problems (1.3) are expected to have no bounded solution when $\mathcal{K} \geqslant 0$ and $\lambda > \lambda^*$ for a suitable $\lambda^* > 0$. See Theorem 5.8 of [1] where this is showed for problem (1.2) with $p = n$ and

$$\lambda^* = \max\left\{\lambda_1, \left(\frac{n-1}{e}\right)^{n-1}\lambda_1\right\},$$

being λ_1 the first eigenvalue of $-\Delta_n$ weighted by $\mathcal{K}$. See also Proposition 4.1.

Remark 1.3. In general, problems (1.3) have no solution if Ω is an unbounded domain of $\mathbb{R}^n$. See Theorem 3.3 of [7] where this is proved for problems (1.2).

Remark 1.4. Condition $\mathcal{K}^+ \not\equiv 0$ is crucial for the multiplicity result to hold. For example for (1.2), if $\mathcal{K} < 0$ one finds only one solution. See [1, Section 4].

In particular λ small, Ω bounded and $\mathcal{K}^+ \not\equiv 0$ seem to be natural assumptions in order to get the multiplicity result.

Remark 1.5. By the regularity result of Tolksdorf [13], each bounded weak solution of problem (1.3) belongs to $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.

2. The concrete Palais–Smale condition

Let us now recall two basic definitions of abstract critical point theory (see [6]).

Definition 2.1. Let (X, d) be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of $\sigma \geqslant 0$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$$

such that for all $(v, t) \in B_\delta(u) \times [0, \delta]$

$$d(\mathcal{H}(v, t), v) \leqslant t, \quad f(\mathcal{H}(v, t)) \leqslant f(v) - \sigma t.$$

We say that the extended real number $|df|(u)$ is the weak slope of f at u.

Definition 2.2. Let (X, d) be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We say that u is a critical point of f if $|df|(u) = 0$.

Definition 2.3. We say that $f : X \to \mathbb{R}$ satisfies the Palais–Smale condition at level c (in short $(PS)_c$) if each sequence (u_h) in X with $f(u_h) \to c$ and $|df|(u_h) \to 0$ admits a convergent subsequence in X.

We now return to the concrete case and set $f = f_\lambda$ and $X = W_0^{1,n}(\Omega)$ endowed with the standard norm $\|u\|_{1,n}^n = \int_\Omega |\nabla u|^n \, dx$.

Definition 2.4. A sequence $(u_h) \subset W_0^{1,n}(\Omega)$ is said to be a concrete Palais–Smale sequence at level $c \in \mathbb{R}$ ($(CPS)_c$–sequence, in short) for f_λ, if $f_\lambda(u_h) \to c$,

$$-\operatorname{div}(\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)) + D_s\mathcal{L}(x, u_h, \nabla u_h) \in W^{-1,n'}(\Omega)$$

eventually as $h \to +\infty$ and

$$-\operatorname{div}(\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)) + D_s\mathcal{L}(x, u_h, \nabla u_h) - \lambda\mathcal{K}(x)e^{u_h} \to 0$$

strongly in $W^{-1,n'}(\Omega)$. We say that f_λ satisfies the concrete Palais–Smale condition at level c ($(CPS)_c$ in short), if every $(CPS)_c$–sequence for f_λ admits a strongly convergent subsequence.

Lemma 2.5. *Let $u \in W_0^{1,n}(\Omega)$ be such that $|df_\lambda|(u) < +\infty$. Then*

$$-\operatorname{div}(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s\mathcal{L}(x, u, \nabla u) - \lambda\mathcal{K}(x)e^{u} \in W^{-1,n'}(\Omega)$$

and

$$\|-\operatorname{div}(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s\mathcal{L}(x, u, \nabla u) - \lambda\mathcal{K}(x)e^{u}\|_{-1,n'} \leqslant |df_\lambda|(u).$$

In particular, if $|df_\lambda|(u) = 0$ then u solves (1.3) in the distributional space $\mathscr{D}'(\Omega)$.

For the proof of the above lemma, see [12, Theorem 2.3]. It is readily seen that if f_λ satisfies $(CPS)_c$, then it satisfies $(PS)_c$.

Let us now recall a very useful consequence of Brezis-Browder's Theorem [4].

Proposition 2.6. *Let $u, v \in W_0^{1,n}(\Omega)$ be such that $D_s\mathcal{L}(x, u, \nabla u)v \geqslant 0$ and assume that $w \in W^{-1,n'}(\Omega)$ is defined by*

$$\forall \varphi \in C_c^\infty(\Omega) : \langle w, \varphi\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)\varphi\, dx.$$

Then $D_s\mathcal{L}(x, u, \nabla u)v \in L^1(\Omega)$ and

$$\langle w, v\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla v\, dx + \int_\Omega D_s\mathcal{L}(x, u, \nabla u)v\, dx.$$

For the proof of this proposition, see [12, Proposition 3.1].

The next result will provide compactness of concrete Palais–Smale sequences.

Lemma 2.7. *Let (u_h) be a bounded sequence in $W_0^{1,n}(\Omega)$ and set*

$$\forall \varphi \in C_c^\infty(\Omega) : \langle w_h, \varphi\rangle = \int_\Omega \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla\varphi\, dx + \int_\Omega D_s\mathcal{L}(x, u_h, \nabla u_h)\varphi\, dx.$$

If (w_h) is strongly convergent to some w in $W^{-1,n'}(\Omega)$, then (u_h) admits a strongly convergent subsequence in $W_0^{1,n}(\Omega)$.

For the proof of this lemma, see [12, Theorem 3.4].

Let us prove that f_λ satisfies the concrete Palais–Smale condition.

Theorem 2.8. *f_λ satisfies $(CPS)_c$ for each $c \in \mathbb{R}$.*

Proof. Let (u_h) be a concrete Palais–Smale sequence for f_λ at level $c \in \mathbb{R}$. We shall divide the proof into two steps:

I) Let us first show that (u_h) is bounded in $W_0^{1,n}(\Omega)$. Note that in view of (1.7), by Proposition 2.6 one can take u_h as test functions in $f'_\lambda(u_h)$. Therefore, since

$f'_\lambda(u_h)(u_h) = o(1)$ as $h \to +\infty$, by (1.8) one obtains

$$\begin{aligned}
f_\lambda(u_h) =& f_\lambda(u_h) - \frac{1}{n} f'_\lambda(u_h)(u_h) + o(1) \\
=& -\frac{1}{n}\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx + \lambda\int_\Omega \mathcal{K}(x)e^{u_h}\left\{\frac{u_h}{n}-1\right\}dx + o(1) \\
\geqslant& -\frac{\gamma}{n}\int_\Omega \mathcal{L}(x,u_h,\nabla u_h)\,dx + \lambda\int_\Omega \mathcal{K}(x)e^{u_h}\left\{\frac{u_h}{n}-1\right\}dx \\
&+\frac{\nu}{n}\int_\Omega |\nabla u_h|^n\,dx + o(1) \\
=& -\frac{\gamma}{n}\left\{f_\lambda(u_h) + \lambda\int_\Omega \mathcal{K}(x)e^{u_h}\,dx\right\} + \lambda\int_\Omega \mathcal{K}(x)e^{u_h}\left\{\frac{u_h}{n}-1\right\}dx \\
&+\frac{\nu}{n}\int_\Omega |\nabla u_h|^n\,dx + o(1)
\end{aligned}$$

as $h \to +\infty$, which yields

$$(n+\gamma)f_\lambda(u_h) \geqslant \lambda\int_\Omega \mathcal{K}(x)e^{u_h}(u_h-\gamma-n)\,dx + \nu\int_\Omega |\nabla u_h|^n\,dx + o(1)$$

as $h \to +\infty$. Since

$$\lim_{\xi\to-\infty} e^{\xi}\{\xi-\gamma-n\} = 0^-, \qquad \lim_{\xi\to+\infty} e^{\xi}\{\xi-\gamma-n\} = +\infty\,,$$

if we set

$$C = -\min_{\xi\in\mathbb{R}} e^{\xi}\{\xi-\gamma-n\}\,,$$

it results $C > 0$ and

$$(n+\gamma)f_\lambda(u_h) \geqslant -\lambda C\|\mathcal{K}\|_q \mathscr{L}^n(\Omega)^{1/q'} + \nu\int_\Omega |\nabla u_h|^n\,dx + o(1)$$

as $h \to +\infty$, where $\mathscr{L}^n$ denotes the $n-$dimensional Lebesgue measure. Being $f_\lambda(u_h) \to c$, we conclude that

$$\nu\int_\Omega |\nabla u_h|^n\,dx \leqslant (n+\gamma)c + \lambda C\|\mathcal{K}\|_q\mathscr{L}^n(\Omega)^{1/q'} + o(1)$$

as $h \to +\infty$, which implies the boundedness of (u_h) in $W_0^{1,n}(\Omega)$.

II) By step I, up to a subsequence, one has $u_h \rightharpoonup u$ in $W_0^{1,n}(\Omega)$ for some u and

$$u_h \to u \qquad \text{in } L^s(\Omega), \quad 1 < s < \infty. \tag{2.1}$$

If we now fix $\eta \in W_0^{1,n}(\Omega)$ with $\|\nabla\eta\|_n = 1$, Hölder and Sobolev inequalities yield

$$\begin{aligned}
&\Big|\int_\Omega \mathcal{K}(x)(e^{u_h}-e^u)\eta\,dx\Big| \\
&\leqslant \int_\Omega |\mathcal{K}(x)|e^u|e^{u_h-u}-1||\eta|\,dx \\
&\leqslant \int_\Omega |\mathcal{K}(x)|e^u|u_h-u|e^{|u_h-u|}|\eta|\,dx \\
&\leqslant \|\mathcal{K}\|_q\Big(\int_\Omega e^{\beta_1 u}\,dx\Big)^{1/\beta_1}\Big(\int_\Omega e^{\beta_2|u_h-u|}\,dx\Big)^{1/\beta_2}\|\eta\|_{\beta_3}\|u_h-u\|_{\beta_4} \\
&\leqslant c\|\mathcal{K}\|_q\Big(\int_\Omega e^{\beta_1 u}\,dx\Big)^{1/\beta_1}\Big(\int_\Omega e^{\beta_2|u_h-u|}\,dx\Big)^{1/\beta_2}\|u_h-u\|_{\beta_4}
\end{aligned} \tag{2.2}$$

with $1/q+1/\beta_1+1/\beta_2+1/\beta_3+1/\beta_4=1$ and $c>0$. Since by Trudinger inequality there exist $c_{1,n}, c_{2,n}>0$ so that:

$$\forall\beta>0,\,\forall w\in W_0^{1,n}(\Omega):\ \int_\Omega e^{\beta|w|}\,dx\leqslant c_{1,n}\mathscr{L}^n(\Omega)e^{c_{2,n}\beta^n\|\nabla w\|_n^n},$$

the exponential terms in (2.2) are bounded by step I and the last term goes to zero in view of (2.1). Thus,

$$\sup_{\|\eta\|_{1,n}=1}\Big|\int_\Omega \mathcal{K}(x)(e^{u_h}-e^u)\eta\,dx\Big|=o(1),$$

as $h\to+\infty$, which shows that

$$\mathcal{K}(x)e^{u_h}\to\mathcal{K}(x)e^u\quad\text{ in } W^{-1,n'}(\Omega).$$

Therefore, since we have that for all $\varphi\in C_c^\infty(\Omega)$:

$$\begin{aligned}&\int_\Omega \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\varphi\,dx+\int_\Omega D_s\mathcal{L}(x,u_h,\nabla u_h)\varphi\,dx\\&=\int_\Omega\mathcal{K}(x)e^u\varphi\,dx+\langle w_h,\varphi\rangle+o(1)\,,\end{aligned}$$

with $w_h\to 0$ in $W^{-1,n'}(\Omega)$ as $h\to+\infty$, by Lemma 2.7 up to a further subsequence (u_h) strongly converges to u in $W_0^{1,n}(\Omega)$. □

3. Mountain pass critical point and local minimum

Proposition 3.1. *There exist $\lambda_0>0$ and $R_2>R_1>0$ such that*

$$\forall u\in W_0^{1,n}(\Omega):\ \|u\|_{1,n}=R_1\implies f_\lambda(u)>f_\lambda(0)\tag{3.1}$$

$$\exists w\in W_0^{1,n}(\Omega):\ \|w\|_{1,n}=R_2\ \text{ and }\ f_\lambda(w)<f_\lambda(0)\tag{3.2}$$

for each $\lambda\in]0,\lambda_0[$.

Proof. Note that by (1.5) and arguing as in [1], for each $\lambda>0$ we have

$$f_\lambda(u)=\int_\Omega\mathcal{L}(x,u,\nabla u)\,dx-\lambda\int_\Omega\mathcal{K}(x)e^u\,dx\geqslant\varphi_{a,b}(\|\nabla u\|_n)\tag{3.3}$$

where $\varphi_{a,b}:[0,+\infty[\to\mathbb{R}$ is such that

$$\varphi_{a,b}(\|\nabla u\|_n)=\nu\|\nabla u\|_n^n-\lambda a\|\mathcal{K}\|_q e^{b\|\nabla u\|_n^n}\,,$$

for each $u\in W_0^{1,n}(\Omega)$, with

$$a=c_1^{\frac{q-1}{q}}\mathscr{L}^n(\Omega)^{\frac{q-1}{q}},\quad b=c_2\Big(\frac{q-1}{q}\Big)^{n-1}\mathscr{L}^n(\Omega)^{\frac{n-1}{n}},\tag{3.4}$$

being $c_1,c_2>0$ such that

$$\int_\Omega\exp\Big\{\frac{c_1|u|}{\|\nabla u\|_n}\Big\}^{\frac{n}{n-1}}dx\leqslant c_2\mathscr{L}^n(\Omega)\,,$$

(cf. [8, Theorem 7.15]). In particular, (3.1) follows arguing as in [1]. To prove (3.2), fix $\phi\in C_c^\infty(B)$ with $\phi\geqslant 0$, where $B\subset\Omega$ is the set where $\mathcal{K}$ is positive. For each

$\tau > 0$ it results

$$\begin{aligned} f_\lambda(\tau\phi) &= \int_\Omega \mathcal{L}(x, \tau\phi, \tau\nabla\phi)\,dx - \lambda\int_\Omega \mathcal{K}(x)e^{\tau\phi}\,dx \\ &\leqslant b_0\tau^r\int_\Omega |\phi|^r\,dx + b_0\int_\Omega \tau^n|\nabla\phi|^n\,dx - \lambda\tau^{2\max\{r,n\}}\int_\Omega \mathcal{K}^+(x)\phi^{2\max\{r,n\}}\,dx \\ &\quad + \lambda\int_\Omega \mathcal{K}^-(x)\,dx. \end{aligned}$$

Then, since $\mathcal{K}^+ \not\equiv 0$, we deduce that $f_\lambda(\tau\phi) \to -\infty$ as $\tau \to +\infty$, thus yielding the second assertion. □

We now come to the proof of the main result of the paper.

Proof of Theorem 1.1. If we set

$$\Theta = \Big\{\gamma \in C([0,1], W_0^{1,n}(\Omega)) : \ \gamma(0) = 0, \ \gamma(1) = w\Big\}$$

for some $w \in W_0^{1,n}(\Omega)$ with $f_\lambda(w) < f_\lambda(0)$ and

$$c_\lambda = \inf_{\gamma\in\Theta}\max_{t\in[0,1]} f_\lambda(\gamma(t)),$$

by combining the non-smooth mountain pass Lemma [6] with Proposition 3.1 and Theorem 2.8, we get a weak solution $u_1 \in W_0^{1,n}(\Omega)$ of (1.3) with $f_\lambda(u_1) = c_\lambda > f_\lambda(0)$. To get a second solution, argue on the truncated functional f_λ^τ given by

$$f_\lambda^\tau(u) = \int_\Omega \mathcal{L}(x, u, \nabla u)\,dx - \lambda\int_\Omega \mathcal{K}(x)\tau(\|u\|_{1,n})e^u\,dx\,,$$

where $\tau \in C^\infty(\mathbb{R})$ is nonincreasing and

$$\tau(x) = \begin{cases} 1 & \text{if } x \leqslant R_1 \\ 0 & \text{if } x \geqslant R_2 \end{cases}$$

being R_1 and R_2 as in Proposition 3.1. Note that since for all $u \in W_0^{1,n}(\Omega)$:

$$\|u\|_{1,n} \geqslant R_2 \implies f_\lambda^\tau(u) \geqslant \nu\int_\Omega |\nabla u|^n\,dx\,,$$

there results

$$\lim_{\|u\|_{1,n}\to+\infty} f_\lambda^\tau(u) = +\infty.$$

Observe that by Lemma 2.8 and the definition of τ, f_λ^τ satisfies the concrete Palais–Smale condition at each level c' such that

$$c' < -\lambda\int_\Omega \mathcal{K}(x)\,dx.$$

If we fix $\phi \in C_c^\infty(B)$ (recall that $\mathcal{K} > 0$ a.e. in B) with $\phi \geqslant 0$, $\|\phi\|_{1,n} = 1$ and $\varrho < R_1$, then there exists $c_0 > 0$ such that

$$\begin{aligned} f_\lambda^\tau(\varrho\phi) =& f_\lambda(\varrho\phi) = \int_\Omega \mathcal{L}(x, \varrho\phi, \varrho\nabla\phi)\, dx - \lambda \int_\Omega \mathcal{K}(x) e^{\varrho\phi}\, dx \\ \leqslant& c_0 \varrho^{\max\{r,n\}} - \lambda \int_\Omega \mathcal{K}^+(x)(1+\varrho\phi)\, dx + \lambda \int_\Omega \mathcal{K}^-(x)\, dx \\ =& \varrho \left\{ c_0 \varrho^{\max\{r,n\}-1} - \lambda \int_\Omega \mathcal{K}^+(x)\phi\, dx \right\} - \lambda \int_\Omega \mathcal{K}^+(x)\, dx \\ &+ \lambda \int_\Omega \mathcal{K}^-(x)\, dx < -\lambda \int_\Omega \mathcal{K}(x)\, dx \end{aligned}$$

provided that $\varrho > 0$ is sufficiently small. Then

$$c = \inf_{B_{W_0^{1,n}}(0,R_1)} f_\lambda^\tau < -\lambda \int_\Omega \mathcal{K}(x)\, dx.$$

Let us note that there exists a $(CPS)_c$–sequence for f_λ^τ in $B_{W_0^{1,n}}(0, R_1)$. Indeed, since f_λ^τ is bounded from below, we find a minimizing sequence (u_h) for f_λ^τ in $B_{W_0^{1,n}}(0, R_1)$. Of course we have $f_\lambda^\tau(u_h) \to c$. Moreover, if it was $|df_\lambda^\tau|(u_h) \not\to 0$, we would find $\sigma > 0$ such that $|df_\lambda^\tau|(u_h) \geqslant \sigma$. Then by [6, Theorem 1.1.11] one would get a continuous deformation

$$\eta : B_{W_0^{1,n}}(0, R_1) \times [0, \delta] \to B_{W_0^{1,n}}(0, R_1)$$

for some $\delta > 0$ such that for all $t \in [0, \delta]$ and $h \in \mathbb{N}$

$$f_\lambda^\tau(\eta(u_h, t)) \leqslant f_\lambda^\tau(u_h) - \sigma t.$$

This yields the contradiction $f_\lambda^\tau(\eta(u_h, t)) < c$ for sufficiently large values of h. Thus (u_h) is a $(CPS)_c$–sequence. Since f_λ^τ satisfies $(CPS)_c$, there exists $u_2 \in W_0^{1,n}(\Omega)$ with $u_2 \not\equiv 0$ such that $u_h \to u_2$ in $W_0^{1,n}(\Omega)$. This yields

$$f_\lambda^\tau(u_2) = \min_{B_{W_0^{1,n}}(0,R_1)} f_\lambda^\tau < -\lambda \int_\Omega \mathcal{K}(x)\, dx.$$

Since $f_\lambda^\tau \equiv f_\lambda$ in a neighbourhood of u_2, of course u_2 solves problem (1.3) and $f_\lambda(u_2) < f_\lambda(0)$. Moreover, being $f_\lambda(u_1) > f_\lambda(0)$, one obtains $u_2 \not\equiv u_1$. □

4. A non-existence result

Assume now that $\mathcal{L}$ does not depend on x, $\mathcal{L}(s, \xi)$ is of class C^1 in $\mathbb{R} \times \mathbb{R}^n$ and, additionally, that the vector valued function

$$\nabla_\xi \mathcal{L}(s, \xi) = \Big(\frac{\partial \mathcal{L}}{\partial \xi_1}(s, \xi), \cdots, \frac{\partial \mathcal{L}}{\partial \xi_n}(s, \xi) \Big)$$

is of class C^1 in $\mathbb{R} \times \mathbb{R}^n$ (see [11]). We recall that a smooth bounded domain $\Omega \subset \mathbb{R}^n$ is said to be strictly star–shaped with respect to the origin if $x \cdot \nu > 0$ for a.e. $x \in \partial\Omega$, where ν denotes the outer normal to $\partial\Omega$.

Proposition 4.1. *Assume that Ω is strictly star–shaped with respect to the origin, $D_s\mathcal{L} \leqslant 0$ and $\mathcal{K}$ is constant and positive. Then there exists $\lambda^* > 0$ such that (1.3) admits no solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ for each $\lambda > \lambda^*$.*

Proof. Assume by contradiction that (1.3) has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. By applying the Pucci-Serrin identity [11, formula 5] to

$$\mathcal{F}(u, \nabla u) = \mathcal{L}(u, \nabla u) - \lambda \mathcal{K} e^u,$$

with $h(x) = x$ and $a = 0$, since $\mathcal{L}(s, \cdot)$ is $n-$homogeneous, we obtain

$$\begin{aligned} &(n-1) \int_{\partial\Omega} \mathcal{L}(0, \nabla u)(x \cdot \nu)\, d\mathscr{H}^{n-1} + \lambda \int_{\partial\Omega} \mathcal{K}\, x \cdot \nu \, d\mathscr{H}^{n-1} \\ &= \lambda n \int_\Omega \mathcal{K} e^u \, dx\,, \end{aligned} \tag{4.1}$$

where $\mathscr{H}^{n-1}$ denotes the Hausdorff measure. Integrating (1.3), since $x \cdot \nu > 0$ on the boundary $\partial\Omega$, by (1.5) and (1.6) we obtain

$$\begin{aligned} \int_\Omega \lambda \mathcal{K} e^u \, dx &= \int_\Omega - \operatorname{div}\left(\nabla_\xi \mathcal{L}(u, \nabla u)\right) dx + \int_\Omega D_s \mathcal{L}(u, \nabla u)\, dx \\ &\leqslant - \int_{\partial\Omega} \nabla_\xi \mathcal{L}(0, \nabla u) \cdot \nu \, d\mathscr{H}^{n-1} \\ &\leqslant \int_{\partial\Omega} |\nabla_\xi \mathcal{L}(0, \nabla u)| \, d\mathscr{H}^{n-1} \\ &\leqslant \Big(\int_{\partial\Omega} |\nabla_\xi \mathcal{L}(0, \nabla u)|^{n'} (x \cdot \nu)\, d\mathscr{H}^{n-1} \Big)^{\frac{n-1}{n}} \Big(\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{n}} \\ &\leqslant c_{1,n} \Big(\int_{\partial\Omega} |\nabla u|^n (x \cdot \nu)\, d\mathscr{H}^{n-1} \Big)^{\frac{n-1}{n}} \Big(\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{n}} \\ &\leqslant c_{2,n} \Big(\int_{\partial\Omega} \mathcal{L}(0, \nabla u)(x \cdot \nu)\, d\mathscr{H}^{n-1} \Big)^{\frac{n-1}{n}} \Big(\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{n}} \end{aligned}$$

which implies

$$\Big(\lambda \int_\Omega \mathcal{K} e^u \, dx \Big)^{\frac{n}{n-1}} \leqslant A \int_{\partial\Omega} \mathcal{L}(0, \nabla u)(x \cdot \nu)\, d\mathscr{H}^{n-1}\,,$$

where we set

$$A = c_{2,n}^{n/n-1} \Big(\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1} \Big)^{\frac{1}{n-1}}.$$

In particular, by (4.1) we obtain

$$\frac{n-1}{A} \Big(\lambda \int_\Omega \mathcal{K} e^u \, dx \Big)^{\frac{n}{n-1}} + \lambda B - n\lambda \int_\Omega \mathcal{K} e^u \, dx \leqslant 0\,,$$

where $B = \int_{\partial\Omega} \mathcal{K}\, x \cdot \nu \, d\mathscr{H}^{n-1}$. Since the map $\varphi : [0, +\infty[\to \mathbb{R}$ given by

$$\varphi(x) = \frac{n-1}{A} x^{\frac{n}{n-1}} + \lambda B - nx$$

achieves its absolute minimum at $x_0 = A^{n-1}$ with $\varphi(x_0) = \lambda B - A^{n-1}$, we get

$$\lambda \leqslant c_{2,n}^n \frac{\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1}}{\int_{\partial\Omega} \mathcal{K}\, x \cdot \nu \, d\mathscr{H}^{n-1}}.$$

In particular, by setting

$$\lambda^* = c_{2,n}^n \frac{\int_{\partial\Omega} (x \cdot \nu)^{-(n-1)} \, d\mathscr{H}^{n-1}}{\int_{\partial\Omega} \mathcal{K}\, x \cdot \nu \, d\mathscr{H}^{n-1}},$$

the assertion follows. □

Remark 4.2. For the lagrangian

$$\mathcal{L}(x, s, \xi) = \frac{1}{n}|\xi|^n,$$

it was shown in [7] by an approximation procedure that the non-existence of smooth solutions (C^2) implies the non-existence of bounded solutions.

Remark 4.3. By Proposition 4.1 assumption (1.7) seems to be natural in order to get existence of solutions of (1.3).

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