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EXISTENCE OF GLOBAL SOLUTIONS TO REACTION-DIFFUSION SYSTEMS VIA A LYAPUNOV FUNCTIONAL

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Abstract. The purpose of this paper is to construct polynomial functionals (according to solutions of the coupled reaction-diffusion equations) which give L^p-bounds for solutions. When the reaction terms are sufficiently regular, using the well known regularizing effect, we deduce the existence of global solutions. These functionals are obtained independently of work done by Malham and Xin [11].

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = f(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - b\Delta v = g(u,v) \quad \text{in } \mathbb{R}^+ \times \Omega \tag{1.2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \quad \text{in } \Omega, \tag{1.4}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$ and a and b are positive constants. The initial data are assumed to be nonnegative and the functions $f(r,s)$ and $g(r,s)$ are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f(0,s) \geq 0$ and $g(r,0) \geq 0$ for all $r, s \geq 0$ which imply, via the maximum principle (see Smoller [20]), the positivity of the solution on its interval of existence. We assume that there exists an integer $p \geq 1$ such that

$$K^{2i-1} f(r,s) + g(r,s) \leq C_1(r,s)\,(r+s+1)\,, \quad i = 1,\dots,p\,, \tag{1.5}$$

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for all $r, s \geq 0$ and a real $m \geq 1$ such that

$$|f(r,s)|\,,\ |g(r,s)| \leq C_2(r,s)(r+s+1)^m \text{ on } (0,+\infty)\times(0,+\infty)\,, \tag{1.6}$$

where K is any positive constant satisfying

$$K \geq \frac{(a+b)}{2\sqrt{ab}},$$

and C_1, C_2 are positive and uniformly bounded functions defined on $\mathbb{R}^+ \times \mathbb{R}^+$.

In the case where $f(u,v) = -uv^\beta$ (which implies the uniform boundedness of u) and $g(u,v) = uv^\beta$, Alikakos [1] established global existence and L^∞-bounds of positive solutions when $1 < \beta < \frac{(n+2)}{n}$. Masuda [13] showed that solutions to this system exist globally for every $\beta > 1$. Hollis, Martin, and Pierre [6] established global existence of positive solutions for system (1.1)-(1.2) with the boundary conditions

$$\lambda_1 u + (1-\lambda_1)\frac{\partial u}{\partial \eta} = \beta_1, \quad \lambda_2 v + (1-\lambda_2)\frac{\partial v}{\partial \eta} = \beta_2 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.7}$$

where

$$0 < \lambda_1,\ \lambda_2 < 1 \quad \text{or} \quad \lambda_1 = \lambda_2 = 1, \quad \beta_1 \geq 0 \quad \text{and} \quad \beta_2 \geq 0, \tag{1.8}$$

or

$$\lambda_1 = \lambda_2 = \beta_1 = \beta_2 = 0 \tag{1.9}$$

under the conditions of: uniform boundedness of u on $[0, T_{\max}] \times \Omega$,

$$f(r,s) + g(r,s) \leq C_3(r,s)\,(r+s+1)\,, \quad \text{for all } r,s \geq 0 \quad i = 1,\dots,p\,, \tag{1.10}$$

and (1.6) for the reaction g, Haraux and Youkana [3] simplified the demonstration of Masuda by using techniques based on Lyapunov functionals. They could handle non-linearities $f(u,v) = -\ g(u,v) = -uF(v)$ satisfying the condition

$$\lim_{s\to+\infty}\left[\frac{\log\left(1+F(s)\right)}{s}\right] = 0\,, \tag{1.11}$$

which means that $F(s)$ is of sub-exponential growth. Morgan [14] generalized the results of Hollis, Martin, and Pierre [6] to the case of systems of $\widetilde{m}$ components ($\widetilde{m} \geq 2$), the case of two equations, he imposed on f and g the conditions (1.6), (1.10) and

$$f(r,s) \leq C_4(r,s)\,(r+s+1)\,, \text{ for all } r \geq 0 \text{ and } s \geq 0, \tag{1.12}$$

which is a particular case of (1.6). Martin and Pierre [12] and Hollis [5] extended the results, under the same conditions, to the boundary conditions (1.7) but in (1.8) they took

$$0 \leq \lambda_1, \quad \lambda_2 \leq 1, \quad \beta_1 \geq 0 \quad \text{and} \quad \beta_2 \geq 0. \tag{1.13}$$

Recently Kouachi and Youkana [10] generalized the results of Haraux and Youkana [3]; they added $-c\Delta u$ to the right-hand side of the second equation of system (1.1)-(1.2) and require the condition

$$\lim_{s\to+\infty}\left[\frac{log\left(1+f(r,s)\right)}{s}\right] < \alpha^*, \text{ for any } r \geq 0,$$

with

$$\alpha^* = \frac{2ac}{n(a\text{-}c)^2\left\|u_0\right\|_\infty}\min\left\{\frac{\lambda}{\mu}, \frac{(a-c)}{b}\right\},$$

condition reflecting the weak exponential growth of the reaction term f. This result is posterior to the one of Kirane [9]. Kanel and Kirane [7] and [8] obtained results in the same sense and in the case where $a < b$ and $f + g \equiv 0$ but f has at most an exponential growth. Let us mention that other results have been obtained after the paper of Martin and Pierre, see Schmitt [19], Pierre and Schmitt [16] and [17].

One notices that, to prove existence global solutions to (1.1)-(1.2), it was required that one of the reaction terms satisfy (1.12). In this paper we show the global existence without using this condition.

2. Notation and preliminary observations

It is well known that to prove global existence of solutions to (1.1)-(1.4) (see Henry [4], pp. 35-62), it suffices to derive a uniform estimate of $\|f(u,v)\|_p$ and $\|g(u,v)\|_p$ on $[0, T_{\max}[$ in the space $L^p(\Omega)$ for some $p > n/2$. Our aim is to construct polynomial Lyapunov functionals allowing us to obtain L^p-bounds on u and v that lead to global existence.

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\,dx, \quad \|u\|_\infty = \max_{x\in\Omega}|u(x)|.$$

Since the nonlinear right hand side of (1.1)-(1.2) is continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$, then for any initial data in $C(\overline{\Omega})$, it is easy to check directly its Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\begin{pmatrix} -a\Delta & 0 \\ 0 & -b\Delta \end{pmatrix}$$

Under these assumptions, the following local existence result is well known (see Friedman [2] and Pazy [15]).

Proposition 2.1. *The system (1.1)-(1.4) admits a unique, classical solution* (u,v) *on* $(0, T_{\max}[\times\Omega$. *If* $T_{\max} < \infty$ *then*

$$\lim_{t\nearrow T_{\max}} \{\|u(t,.)\|_\infty + \|v(t,.)\|_\infty\} = \infty,$$

where $T_{\max}(\|u_0\|_\infty, \|v_0\|_\infty)$ *denotes the eventual blowing-up time.*

3. Statement and proof of the main theorem

The main result of the paper is the following.

Theorem 3.1. *Let* $(u(t,.), v(t,.))$ *be a solution of (1.1)-(1.4) and let*

$$L(t) = \int_\Omega H_p(u(t,x), v(t,x))\,dx, \tag{3.1}$$

where

$$H_p(u,v) = \sum_{i=0}^{p} C_p^i K^{i^2} u^i v^{p-i}$$

with p *a positive integer and* K *is a positive constant such that*

$$K \geq \frac{(a+b)}{2\sqrt{ab}}.$$

Then the functional L *is uniformly bounded on the interval* $[0, T^*]$, $T^* \leq T_{\max}$.

Proof. Differentiating L with respect to t yields

$$
\begin{aligned}
&L'(t)\\
&=\int_\Omega \Big[\sum_{i=1}^{p}\left(iC_p^i K^{i^2} u^{i-1}v^{p-i}\right)\frac{\partial u}{\partial t}+\sum_{i=0}^{p-1}\left((p-i)C_p^i K^{i^2} u^{i}v^{p-i-1}\right)\frac{\partial v}{\partial t}\Big]dx\\
&=\int_\Omega \sum_{i=1}^{p}\left(iC_p^i K^{i^2} u^{i-1}v^{p-i}\right)(a\Delta u+f(u,v))\,dx\\
&\quad+\int_\Omega \sum_{i=1}^{p}\left((p-i+1)C_p^{i-1} K^{(i-1)^2} u^{i-1}v^{p-i}\right)(b\Delta v+g(u,v))\,dx\\
&=\int_\Omega \Big[\sum_{i=1}^{p} aiC_p^i K^{i^2} u^{i-1}v^{p-i}\Delta u+b(p-i+1)C_p^{i-1} K^{(i-1)^2} u^{i-1}v^{p-i}\Delta v\Big]dx\\
&\quad+\int_\Omega \Big[\sum_{i=1}^{p} iC_p^i K^{i^2} u^{i-1}v^{p-i}f(u,v)+(p-i+1)C_p^{i-1} K^{(i-1)^2} u^{i-1}v^{p-i}g(u,v)\Big]dx\\
&=I+J\,.
\end{aligned}
$$

By a simple use of Green's formula we have

$$
I=-\int_\Omega\left(\mathcal{A}\left|\nabla u\right|^2+\mathcal{B}\nabla u\nabla v+\mathcal{C}\left|\nabla v\right|^2\right)dx,
$$

where

$$
\mathcal{A}=\sum_{i=2}^{p} ai(i-1)C_p^i K^{i^2} u^{i-2}v^{p-i}
$$

$$
\mathcal{B}=\sum_{i=1}^{p-1} ai(p-i)C_p^i K^{i^2} u^{i-1}v^{p-i-1}+\sum_{i=2}^{p} b(i-1)(p-i+1)C_p^{i-1} K^{(i-1)^2} u^{i-2}v^{p-i}
$$

$$
\mathcal{C}=\sum_{i=1}^{p-1} b(p-i)(p-i+1)C_p^{i-1} K^{(i-1)^2} u^{i-1}v^{p-i-1}.
$$

Using the fact that

$$
iC_p^i=pC_{p-1}^{i-1},\ \text{for all } i=1,\dots,p\,, \tag{3.2}
$$

we get $\mathcal{A}=\sum_{i=2}^{p} ap(p-1)C_{p-2}^{i-2}K^{i^2}u^{i-2}v^{p-i}$,

$$
\begin{aligned}
\mathcal{B}=&\sum_{i=1}^{p-1} ap(p-1)C_{p-2}^{i-1} K^{i^2} u^{i-1}v^{p-i-1}+\sum_{i=2}^{p} bp(p-1)C_{p-2}^{i-2} K^{(i-1)^2} u^{i-2}v^{p-i}\\
=&\mathcal{B}_1+\mathcal{B}_2\,,
\end{aligned}
$$

and

$$
\mathcal{C}=\sum_{i=1}^{p-1} bp(p-1)C_{p-2}^{i-1} K^{(i-1)^2} u^{i-1}v^{p-i-1}.
$$

Putting $j = i - 2$, we have

$$\mathcal{A} = \sum_{j=0}^{p-2} ap(p-1)C_{p-2}^{j} K^{(j+2)^2} u^j v^{p-j-2}$$

$$\mathcal{B}_2 = \sum_{j=0}^{p-2} bp(p-1)C_{p-2}^{j} K^{(j+1)^2} u^j v^{p-j-2}$$

and putting $j = i - 1$, we get

$$\mathcal{B}_1 = \sum_{j=0}^{p-2} ap(p-1)C_{p-2}^{j} K^{(j+1)^2} u^j v^{p-j-2}$$

$$\mathcal{C} = \sum_{j=0}^{p-2} bp(p-1)C_{p-2}^{j} K^{j^2} u^j v^{p-j-2}.$$

Then

$$\begin{aligned} I = & - p(p-1) \sum_{j=0}^{p-2} C_{p-2}^{j} \int_\Omega u^j v^{p-j-2} \\ & \times \Big(aK^{(j+2)^2} \left|\nabla u\right|^2 + (a+b)\, K^{(j+1)^2} \nabla u \nabla v + bK^{j^2} \left|\nabla v\right|^2 \Big)\, dx. \end{aligned}$$

The quadratic forms (with respect to ∇u and ∇v) are positive since

$$\Big((a+b)\, K^{(j+1)^2} \Big)^2 - 4abK^{j^2} K^{(j+2)^2} \leq 0, \quad j = 0, 1, \dots, p-2.$$

Then $I \leq 0$. Using the relation (3.2), in the second integral, yields

$$J = \int_\Omega \Big[p \sum_{i=1}^{p} \Big(K^{i^2} f(u,v) + K^{(i-1)^2} g(u,v) \Big)\, C_{p-1}^{i-1} u^{i-1} v^{p-i} \Big] dx.$$

Using conditions (1.5), we deduce

$$J \leq C_5 \int_\Omega \Big[\sum_{i=1}^{p} (u+v+1)\, C_{p-1}^{i-1} u^{i-1} v^{p-i} \Big] dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T^*]$, first we write

$$\sum_{i=1}^{p} (u+v+1)\, C_{p-1}^{i-1} u^{i-1} v^{p-i} = R_p(u,v) + S_{p-1}(u,v),$$

where $R_p(u,v)$ and $S_{p-1}(u,v)$ are two homogeneous polynomials of degrees p and $p-1$, respectively. First, since the polynomials $H_p(u,v)$ and $R_p(u,v)$ are all two of degree p, there exists a positive constant C_6 such that

$$\int_\Omega R_p(u,v) dx \leq C_6 \int_\Omega H_p(u,v) dx,$$

then applying Hölder's inequality to the integral $\int_\Omega S_{p-1}(u,v) dx$, one gets

$$\int_\Omega S_{p-1}(u,v) dx \leq (\operatorname{meas} \Omega)^{1/p} \Big(\int_\Omega \left(S_{p-1}(u,v)\right)^{p/(p-1)} dx \Big)^{(p-1)/p}.$$

Since for all $u \geq 0$ and $v > 0$,

$$\frac{(S_{p-1}(u,v))^{p/(p-1)}}{H_p(u,v)} = \frac{(S_{p-1}(x,1))^{p/(p-1)}}{H_p(x,1)},$$

where $x = \frac{u}{v}$, and

$$\lim_{x\to+\infty} \frac{(S_{p-1}(x,1))^{p/(p-1)}}{H_p(x,1)} < +\infty,$$

one asserts that there exists a positive constant C_7 such that

$$\frac{(S_{p-1}(u,v))^{p/(p-1)}}{H_p(u,v)} \leq C_7, \quad \text{for all } u \geq 0,\ v \geq 0.$$

Hence, the functional L satisfies the differential inequality

$$L'(t) \leq C_6 L(t) + C_8 L^{(p-1)/p}(t),$$

which for $Z = L^{1/p}$ can be written as

$$pZ' \leq C_6 Z + C_8.$$

A simple integration gives the uniform bound of the functional L on the interval $[0, T^*]$. this completes the present proof. □

Corollary 3.2. *Suppose that $f(r,s)$ and $g(r,s)$ are continuously differentiable on $(0,+\infty)\times$ $(0,+\infty)$ and satisfy (1.5) for some $p \geq 1$. Then all solutions of (1.1)-(1.4) with positive initial data in $L^\infty(\Omega)$ are in $L^\infty(0,T^*;L^p(\Omega))$.*

The proof of this corollary is an immediate consequence of Theorem 3.1 and the inequality

$$\int_\Omega (u(t,x)+v(t,x))^p\,dx \leq L(t) \quad \text{on } [0,T^*[\,.$$

Proposition 3.3. *Suppose that conditions (1.5) and (1.6) are satisfied and that $p > \frac{mn}{2}$, then all solutions of (1.1)-(1.4) with positive initial data in $L^\infty(\Omega)$ are global.*

Proof. From Corollary 3.2, there exists a positive constant C_8 such that

$$\int_\Omega (u(t,x)+v(t,x)+1)^p\,dx \leq C_8 \quad \text{on } [0,T_{\max}[\,.$$

From (1.6) we have

$$|f(u,v)|^{p/m}, |g(u,v)|^{p/m} \leq C_2(u,v)(u+v+1)^p \quad \text{on } [0,T_{\max}[\times\Omega.$$

Since u and v are in $L^\infty(0,T^*;L^p(\Omega))$ and $p/m > n/2$, then from the preliminary observations the solution is global.

4. Applications

In this section we apply Proposition 3.3 to some particular biochemical and chemical reaction models. Throughout this section we will assume that all reactions take place in a bounded domain Ω with smooth boundary $\partial\Omega$.

Let us begin with the general two-components reaction

$$lA + qB \underset{k}{\overset{h}{\rightleftarrows}} rA + sB$$

that leads to the reaction diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = hu^l v^q - ku^r v^s \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.1}$$

$$\frac{\partial v}{\partial t} - b\Delta v = -hu^l v^q + ku^r v^s \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.2}$$

with boundary conditions (1.3) and initial data (1.4), where h, k, l, q, r and s are positive constants. The initial data are assumed to be nonnegative.

We remark that (1.6) for this system, is satisfied for all positive constants h, k, l, q, r and s whenever

$$m = \max\{l+q, r+s\}.$$

Also, condition (1.5) is trivial in the case $l+q \le 1$ for all integer $p \ge 1$ by and applying of the Young inequality to the term $u^l v^q$ in the right hand side of the first equation of system (4.1)-(4.2). However, uniqueness is not ensured. On the other hand if $l+q > 1$ the condition (1.6) becomes difficult to verify. However we have the following statement.

Proposition 4.1. *Suppose that one of the following three conditions is satisfied*

$$l+q \le 1 \text{ or } r+s \le 1, \tag{4.3}$$

$$r+s > l+q > 1 \text{ and } l-r < sl-qr < s-q, \tag{4.4}$$

$$l+q > r+s > 1 \text{ and } s-q < sl-qr < l-r. \tag{4.5}$$

Then, solutions of (4.1)-(4.2) with boundary conditions (1.3) and positive initial data (1.4) exist for all $t > 0$.

Proof. The case $r+s \le 1$ is also a trivial application of Young's inequality to the term $u^r v^s$ in the right hand side of the second equation of system (4.1)-(4.2).

Assuming (4.4), put

$$\nu_1 = \frac{r+s-1}{l+q-1} \quad \text{and} \quad \nu_2 = \frac{\nu_1}{\nu_1 - 1} = \frac{r+s-1}{(r+s)-(l+q)}.$$

Then

$$\nu_1 > 1,\ \nu_2 > 1 \quad \text{and} \quad \frac{1}{\nu_1} + \frac{1}{\nu_2} = 1.$$

Then we can write $l = l_1 + l_2$ and $q = q_1 + q_2$, where

$$l_1 = \frac{r}{\nu_1} = \frac{l+q-1}{r+s-1} r, \quad l_2 = \frac{(sl-qr)-(l-r)}{r+s-1},$$

$$q_1 = \frac{s}{\nu_1} = \frac{l+q-1}{r+s-1} s \quad \text{and} \quad q_2 = \frac{(s-q)-(sl-qr)}{r+s-1}.$$

Applying Young's inequality, one gets

$$u^l v^q \le \frac{k}{h}\left(u^{l_1} v^{q_1}\right)^{\nu_1} + C_9 \left(u^{l_2} v^{q_2}\right)^{\nu_2}, \tag{4.6}$$

where C_9 is a positive constant, $\left(u^{l_1} v^{q_1}\right)^{\nu_1} = u^r v^s$ and $\nu_2(l_2 + q_2) = 1$.

Finally, applying Young's inequality one more time to the second term of the right hand side of inequality (4.6), one deduces that

$$K^{2i-1} f(r,s) + g(r,s) \le C_{10}(u+v+1),\ i = 1, 2, \dots \tag{4.7}$$

where C_{10} is a positive constant. From this last inequality, Proposition 3.3 is applicable and then it follows the conclusion of Proposition 4.1 under hypothesis (4.4). For the other case, we take

$$\nu_1 = \frac{l+q-1}{r+s-1} \quad \text{and} \quad \nu_2 = \frac{\nu_1}{\nu_1 - 1} = \frac{l+q-1}{(l+q)-(r+s)}.$$

Then we prove that there exists a positive constant C_{11} such that

$$u^r v^s \leq \frac{h}{k} \left(u^{r_1} v^{s_1}\right)^{\nu_1} + C_{11} \left(u^{r_2} v^{s_2}\right)^{\nu_2},$$

where $r_1 = \frac{l}{\nu_1}$, $r_2 = r - r_1$, $s_1 = \frac{q}{\nu_1}$ and $s_2 = s - s_1$. From this setting, we have

$$\left(u^{r_1} v^{s_1}\right)^{\nu_1} = u^l v^q \quad \text{and} \quad \nu_2(r_2 + s_2) = 1.$$

Finally we deduce an analogous inequality to (4.7),

$$K^{2i-1} f(r,s) + g(r,s) \leq C_{12}(u+v+1), \quad i = 1, 2, \dots,$$

where C_{12} is a positive constant. This completes completes the present proof. $\square$

We next consider the three-component reaction

$$lU + qV \underset{k}{\overset{h}{\rightleftarrows}} rW,$$

that leads to the reaction diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = -hu^l v^q + kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.8}$$

$$\frac{\partial v}{\partial t} - b\Delta v = -hu^l v^q + kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.9}$$

$$\frac{\partial w}{\partial t} - c\Delta w = hu^l v^q - kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.10}$$

with homogeneous Neumann boundary conditions and positive initial data in $L^\infty(\Omega)$, where h, k, l, q and r are positive constants such that $r \leq 1$ or $l + q \leq 1$. The special case $l = q = r = 1$ has been studied by Rothe [18] under the same boundary condition (1.2) where he showed that $T_{\max} = \infty$ if $n \leq 5$. Morgan[14] generalized the results of Rothe for every integer $n \geq 1$ and when all the components satisfy the same boundary conditions (Neumann or Dirichlet). Hollis [5] completed the works of Morgan and established global existence in the case when some components of the system (4.8)-(4.10) satisfy Neumann boundary conditions while others satisfy nonhomogeneous Dirichlet boundary conditions.

Conditions (1.5) and (1.6) are trivial for the two coupled systems (4.8)-(4.10) and (4.9)-(4.10) when $r \leq 1$. For the case $l + q \leq 1$, they are also satisfied for the two coupled systems (4.8)-(4.10) and (4.10)-(4.9), while applying the Young inequality to $u^l v^q$. Then Corollary 3.2 implies that all components of the solution are in $L^\infty(0, T^*; L^p(\Omega))$ for all $p \geq 1$. Since the reactions terms are of polynomial growth, then $T_{\max} = \infty$.

Proposition 4.2. *Solutions of (4.8)-(4.10) with nonnegative uniformly bounded initial data are positive and exist globally for every positive constants l, q and r such that $r \leq 1$ or $l + q \leq 1$.*

As another example, we have the system

$$\frac{\partial u}{\partial t} - a\Delta u = v^l - u^q \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.11}$$

$$\frac{\partial v}{\partial t} - b\Delta v = Au^q - Bv^l \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{4.12}$$

where $l > 1$, $q > 1$, A and B are positive constants such that $B > A$. It is clear that condition (1.12) is not satisfied, nevertheless the method of invariant regions (see Smoller[20]) gives the global existence of positive solutions under the above conditions on l, q, A, B. However, our technique is applicable and we have the following statement.

Proposition 4.3. *Suppose that $A \leq K$ and let p the smallest integer such that $K^{2p-1} \leq B$. Then if*

$$p > \frac{n}{2} \max\{l, q\}$$

all solutions of (4.11)-(4.12) with the boundary conditions (1.3) and positive initial data in $L^\infty(\Omega)$ are global and positive.

Proof. The positivity of solutions is a trivial consequence of the method of invariant regions (see Smoller[20]). Condition (1.6) is satisfied if we take $m = \max\{l, q\}$. Since

$$K^{2i-1} f(u,v) + g(u,v) = \left(A - K^{2i-1}\right) u^q + \left(K^{2i-1} - B\right) v^l,\ i = 1, \dots, p\,,$$

then condition (1.5) is also satisfied with $C_1 \equiv 0$. Proposition 3.3 is then applied to obtain global existence.

5. Remarks and comments

Remark 5.1. Since the reaction terms are continuously differentiable on $\mathbb{R}^2$, then all solutions of problem (4.11)-(4.12) with the boundary conditions (1.3) and initial data in $L^\infty(\Omega) \times L^\infty(\Omega)$ are unique.

Remark 5.2. If conditions (1.10) and (1.12) are satisfied, then (1.5) is satisfied for all integer $p \geq 1$, but the reciprocal is false.

Remark 5.3. If f and $f + g$ are non-positive on $\mathbb{R}^+ \times \mathbb{R}^+$, then the functional defined by (3.1) is decreasing on $[0, T^*[$ and so all positive solutions of (1.1)-(1.4) with initial data in $L^\infty(\Omega)$ are in $L^\infty(0, T^*; L^p(\Omega))$ for all $p \geq 1$.

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