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Boundary stabilization of a linear elastodynamic system with variable coefficients *

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Abstract

We consider the boundary stabilization of a linear elastodynamic system, with variables coefficients, by *natural* feedback. In [5, 7], Guesmia proved the boundary stabilization of the linear elastodynamic system, with variables coefficients, under restrictive conditions on the shape of the domain and on the data of the problem. Here, we propose using local coordinates in the boundary integrals to obtain stability under conditions that are only geometrical and less restrictive than those in [5, 7]. We also extend the boundary stabilization result obtained in [2].

Introduction

Let Ω be a bounded open set of $\mathbb{R}^3$ such that its boundary Γ satisfies

$$\Gamma \text{ is of class } \mathcal{C}^2,\ \Gamma = \Gamma_0 \cup \Gamma_1 \text{ with meas}(\Gamma_1) \neq 0,\ \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset. \tag{1}$$

Let a_{ijkl} $(i,j,k,l = 1,2,3)$ be a set of functions in $\mathcal{W}^{2,\infty}(\Omega \times \mathbb{R})$ such that

$$a_{ijkl} = a_{klij} = a_{jikl}\,, \quad \text{in } \Omega \times \mathbb{R}\,,$$

satisfying for some $\alpha > 0$ the condition

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij}\,, \quad a_{ijkl}'\,\varepsilon_{ij}\,\varepsilon_{kl} \leq 0\,, \quad \text{in } \Omega \times \mathbb{R} \tag{2}$$

for every symmetric tensor ε_{ij}, where $'$ denotes $\partial/\partial t$ (we use the summation convention for repeated indices). Given $\mathbf{x}$, a point on Γ, we denote by $\nu(\mathbf{x})$ the normal unit vector pointing outward from Ω.

For a regular vector field $\mathbf{v} = (v_1, v_2, v_3)$, we define

$$v_{i,j} = \partial_j v_i\,, \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{i,j} + v_{j,i})\,, \quad \sigma_{ij}(\mathbf{v}) = a_{ijkl}\varepsilon_{kl}(\mathbf{v})\,.$$

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Let A and B be two positive constants. We consider the following problem introduced by Lagnese [12]:

$$\begin{gathered}\mathbf{u}'' - \operatorname{div}\big(\sigma(\mathbf{u})\big) = 0\,, \quad \text{in } \Omega\times\mathbb{R}_+\,,\\ \mathbf{u} = 0\,, \quad \text{on } \Gamma_0\times\mathbb{R}_+\,,\\ \sigma(\mathbf{u})\nu + A\mathbf{u} + B\mathbf{u}' = 0\,, \quad \text{on } \Gamma_1\times\mathbb{R}_+\,,\\ \mathbf{u}(0) = \mathbf{u}^0\,, \quad \text{in } \Omega\,,\\ \mathbf{u}'(0) = \mathbf{u}^1\,, \quad \text{in } \Omega\,.\end{gathered} \tag{3}$$

Let $\mathbb{L}^2(\Omega)$ (resp. $\mathbb{H}^1(\Omega)$) be the space of vector fields $\mathbf{v}$ such that every component of $\mathbf{v}$ belongs to $\mathrm{L}^2(\Omega)$ (resp. $\mathrm{H}^1(\Omega)$). We introduce the space $\mathbb{H}^1_{\Gamma_0}(\Omega) = \{\mathbf{v}\in\mathbb{H}^1(\Omega)\,/\,\mathbf{v}=0\,,\text{ on }\Gamma_0\}$ and we assume

$$(\mathbf{u}^0,\mathbf{u}^1)\in\mathbb{H}^1_{\Gamma_0}(\Omega)\times\mathbb{L}^2(\Omega)\,. \tag{4}$$

Under this assumption, using semi-group theory, one can show that problem (3) is well-posed.

The energy functional associated with this problem is

$$E(\mathbf{u},t) = \frac{1}{2}\int_\Omega\big(|\mathbf{u}'|^2 + \sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\big)\,d\mathbf{x} + \frac{1}{2}\int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma\,.$$

A boundary stabilization result for this system has been proved by Guesmia [4] under restrictive conditions on the shape of Ω and on other data of the problem. We propose here a direct approach by using local coordinates in the expression of boundary integrals. Our conditions are only geometrical and are less restrictive than those in works of Guesmia. Our proof is constructive. Furthermore, the reader will observe that similar conditions have been introduced by Lagnese [11] for some anisotropic linear elastodynamic systems and by Lasieka and Triggiani [13] for the wave equation.

For a vector field $\mathbf{h} = (h_1,h_2,h_3)\in\big(\mathcal{C}^1(\overline{\Omega})\big)^3$ we denote by $\gamma_{\mathbf{h}}$ a small (non-negative) number satisfying

$$|h_m\,\partial_m(a_{ijkl})\,\varepsilon_{kl}\,\varepsilon_{ij}| \le \gamma_{\mathbf{h}}\,a_{ijkl}\,\varepsilon_{kl}\,\varepsilon_{ij}\,,$$

for all symmetric tensor ε_{ij}. We assume that there exists a vector field $\mathbf{h} = (h_1,h_2,h_3)$ such that

$$\mathbf{h}\in\big(\mathcal{C}^1(\overline{\Omega})\big)^3\,,\quad \mathbf{h}.\nu\le 0\,,\quad \text{on }\Gamma_0\,,\quad \mathbf{h}.\nu>0\,,\quad\text{on }\Gamma_1\,, \tag{5}$$

and furthermore, there exist $\alpha_{\mathbf{h}}>0$ and $\beta_{\mathbf{h}}\in\mathbb{R}$ such that

$$\begin{gathered}\forall\xi\in\big(\mathcal{C}^1(\overline{\Omega})\big)^3,\quad \int_\Omega\sigma_{ij}(\xi)h_{k,j}\xi_{i,k}\,d\mathbf{x} \ge \alpha_{\mathbf{h}}\int_\Omega\sigma(\xi)\colon\varepsilon(\xi)\,d\mathbf{x} + \beta_{\mathbf{h}}\int_{\Gamma_1}|\xi|^2\,d\Gamma\,,\\ \max_{\overline{\Omega}}(\operatorname{div}(\mathbf{h})) - \min_{\overline{\Omega}}(\operatorname{div}(\mathbf{h})) < 2\alpha_{\mathbf{h}} - \gamma_{\mathbf{h}}\,,\quad \min_{\overline{\Omega}}(\operatorname{div}(\mathbf{h}))>0\,,\end{gathered} \tag{6}$$

where we can choose $\beta_{\mathbf{h}}=0$, if $\operatorname{meas}(\Gamma_0)\ne 0$.

Under the above assumptions, we obtain the following result.

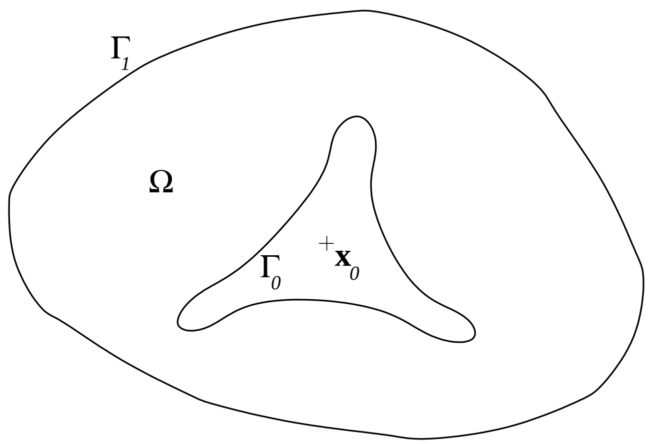

Figure 1: An example of an open set Ω.

Theorem 1 *Assume* (1), (2) *and* (4). *If there exists a vector field* $\mathbf{h}$ *satisfying* (5) *and* (6), *then there exists some constant* $\omega > 0$ *such that the solution* $\mathbf{u}$ *of* (3) *satisfies*

$$\forall t \geq 0\,, \quad E(\mathbf{u}, t) \leq E(\mathbf{u}, 0) \exp(1 - \omega t)\,.$$

Remark 1 Since Ω is bounded and Γ satisfies (1), Γ_1 is compact. Using continuity of $\mathbf{h}$ and ν given in (1) and (5), we get

$$\exists k > 0 \,:\, \mathbf{h}.\nu \geq k\,, \quad \text{on } \Gamma_1\,.$$

Remark 2 This result can be applied when $\mathbf{h}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ ($\alpha_{\mathbf{h}} = 1$, $\beta_{\mathbf{h}} = 0$) and

$$\gamma_{\mathbf{h}} < 2\,, \quad \Gamma_0 = \{\mathbf{x} \in \Gamma \,/\, \mathbf{h}(\mathbf{x}).\nu(\mathbf{x}) \leq 0\,\}\,, \quad \Gamma_1 = \{\mathbf{x} \in \Gamma \,/\, \mathbf{h}(\mathbf{x}).\nu(\mathbf{x}) > 0\,\}\,.$$

Especially, a possible case is $\Omega = U_1 \setminus U_2$, where U_1 is a convex open set, U_2 is a closed set, star-shaped with respect to one of its points, $\mathbf{x}_0$, such that $\{\mathbf{x}_0\} \subset U_2 \subset U_1$. (see figure 1)

This case has been studied in [4] for a particular shape of Ω (Γ_1 is supposed to be close to a sphere). Furthermore, this remark can be extended as follows.

Theorem 2 *Assume* (1), (2), (4) *and suppose that*

$$\begin{gathered} \gamma_{\mathbf{x}-\mathbf{x}_0} < 2\,, \\ (\mathbf{x} - \mathbf{x}_0).\nu(\mathbf{x}) \leq 0\,, \quad \text{if } \mathbf{x} \in \Gamma_0\,, \\ (\mathbf{x} - \mathbf{x}_0).\nu(\mathbf{x}) \geq 0\,, \quad \text{if } \mathbf{x} \in \Gamma_1\,. \end{gathered}$$

Then there exists a constant $\omega > 0$ *such that the solution* $\mathbf{u}$ *of* (3) *satisfies*

$$\forall t \geq 0\,, \quad E(\mathbf{u}, t) \leq E(\mathbf{u}, 0) \exp(1 - \omega t)\,.$$

This paper is mainly devoted to the proof of Theorem 1. After introducing some notations and definitions (section 1), we deal with the well-posedness in section 2 and conclude with the stabilization in section 3 where the proofs of Theorems 1 and 2 are given.

1 Notation

In this paper, we use the convention of repeated indices. As usual, we write $\mathrm{tr}(\tau) = \tau_{11} + \tau_{22} + \ldots = \tau_{ii}$, $\mathbf{v}.\mathbf{w} = v_i w_i$, $\sigma(\mathbf{v})\colon \varepsilon(\mathbf{v}) = \sigma_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})$.

Geometrical notation

We define Ω, Γ, ν as above. Since Γ is of class $\mathcal{C}^2$, for every point $\mathbf{x}$ of Γ, we can build a local $\mathcal{C}^2$-diffeomorphism ϕ from an open subset $\hat{\Gamma}$ of $\mathbb{R}^2$ onto some open neighbourhood of $\mathbf{x}$ in Γ. Then, the following vectors

$$\mathbf{a}_\alpha(\mathbf{x}) = \frac{\partial \phi}{\partial \xi_\alpha}\left(\phi^{-1}(\mathbf{x})\right), \quad \alpha \in \{1,2\},$$

are independent and generate the tangent plane to Γ at $\mathbf{x}$, $T_{\mathbf{x}}(\Gamma)$. Furthermore, we denote by $T(\Gamma)$ the tangent bundle (see [14] and [18]).

Then, we define

- The metric tensor g related to ϕ: $g_{\alpha\beta} = \mathbf{a}_\alpha.\mathbf{a}_\beta$, for all (α,β) in $\{1,2\}^2$,
- The inverse tensor of g: $\left(g^{\alpha\beta}\right)_{1\le\alpha,\beta\le 2}$.

We denote by $\pi(\mathbf{x})$ the orthogonal projection on $T_{\mathbf{x}}(\Gamma)$ and, for a given vector field $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^3$, we will write

$$\begin{gathered}\forall \mathbf{x} \in \Gamma, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_T(\mathbf{x}) + v_\nu(\mathbf{x})\nu(\mathbf{x}),\\ \text{with} \quad \mathbf{v}_T(\mathbf{x}) = \pi(\mathbf{x})\mathbf{v}(\mathbf{x}), \quad v_\nu(\mathbf{x}) = \mathbf{v}(\mathbf{x}).\nu(\mathbf{x}).\end{gathered}$$

We denote by ∂_T (resp. ∂_ν) the tangential (resp. normal) derivative. If v is some regular function, the transposed vector of $\partial_T v$ is the tangential gradient of v and is denoted by $\nabla_T v$. We have

$$\nabla v = \nabla_T v + \partial_\nu v\, \nu, \quad \text{on } \Gamma. \tag{7}$$

Strain and stress

If the vector field $\mathbf{v}$ is regular enough, as in [14] and [18], we can write

$$\begin{aligned}\mathrm{d}\mathbf{v} =& \pi(\partial_T \mathbf{v}_T)\pi + v_\nu(\partial_T \nu) + (\partial_\nu \mathbf{v}_T)\overline{\nu}\\ &+ \nu\left((\partial_T v_\nu) - \overline{\mathbf{v}_T}(\partial_T \nu) + (\partial_\nu v_\nu)\overline{\nu}\right), \quad \text{on } \Gamma,\end{aligned} \tag{8}$$

where $\overline{\mathbf{v}}$ (resp. $\overline{\tau}$) is the transposed vector (resp. matrix) of $\mathbf{v}$ (resp. τ). The strain tensor $\varepsilon(\mathbf{v})$ can be written on Γ as follows

$$\varepsilon(\mathbf{v}) = \varepsilon_T(\mathbf{v}) + \nu\overline{\varepsilon_S(\mathbf{v})} + \varepsilon_S(\mathbf{v})\overline{\nu} + \varepsilon_\nu(\mathbf{v})\nu\overline{\nu}, \quad \text{on } \Gamma,$$

with

$$\begin{aligned} 2\varepsilon_T(\mathbf{v}) =& \pi(\partial_T \mathbf{v}_T)\pi + \pi\overline{\partial_T \mathbf{v}_T}\pi + 2v_\nu \partial_T \nu\,, \\ 2\varepsilon_S(\mathbf{v}) =& \partial_\nu \mathbf{v}_T + \nabla_T v_\nu - (\partial_T \nu)\mathbf{v}_T\,, \\ \varepsilon_\nu(\mathbf{v}) =& \partial_\nu v_\nu\,. \end{aligned}$$

Similarly, we can write

$$\sigma(\mathbf{v}) = \sigma_T(\mathbf{v}) + \nu\overline{\sigma_S(\mathbf{v})} + \sigma_S(\mathbf{v})\overline{\nu} + \sigma_\nu(\mathbf{v})\nu\overline{\nu}\,, \quad \text{on } \Gamma\,,$$

where $\sigma_T(\mathbf{v})$ is a linear symmetric operator field on the tangent plane, $\sigma_S(\mathbf{v})$ is a tangent vector field and $\sigma_\nu(\mathbf{v})$ is a scalar field.

Remark 3 Let $\mathbf{v}$ be in $\mathbb{H}^1(\Omega)$. From the previous formulæ, we deduce

$$\begin{aligned} \varepsilon(\mathbf{v})\colon \varepsilon(\mathbf{v}) &= \varepsilon_T(\mathbf{v})\colon \varepsilon_T(\mathbf{v}) + 2|\varepsilon_S(\mathbf{v})|^2 + |\varepsilon_\nu(\mathbf{v})|^2\,, \quad \text{on } \Gamma\,, \\ \sigma(\mathbf{v})\colon \varepsilon(\mathbf{v}) &= \sigma_T(\mathbf{v})\colon \varepsilon_T(\mathbf{v}) + 2\overline{\sigma_S(\mathbf{v})}\varepsilon_S(\mathbf{v}) + \sigma_\nu(\mathbf{v})\varepsilon_\nu(\mathbf{v})\,, \quad \text{on } \Gamma\,. \end{aligned}$$

We will consider the following vector spaces.

- $\mathcal{L}_s(T_{\mathbf{x}}(\Gamma))$ is the space of linear symmetric operators of $T_{\mathbf{x}}(\Gamma)$,
- $\mathcal{L}_s(T(\Gamma))$ is the space of symmetric operators of $T(\Gamma)$.

Remark 4 $\partial_T \nu(\mathbf{x})$ belongs to $\mathcal{L}_s(T_{\mathbf{x}}(\Gamma))$; its eigenvalues are principal curvatures of Γ at $\mathbf{x}$.

Some function spaces

Consider a tangent field $\mathbf{v}_T : \Gamma \to T(\Gamma)$ with: $\mathbf{v}_T = v^1\mathbf{a}_1 + v^2\mathbf{a}_2$. We will say that $\mathbf{v}_T$ belongs to $\mathrm{L}^2(\Gamma, T(\Gamma))$ if v^1 and v^2 belong to $\mathrm{L}^2(\Gamma)$. In $\mathrm{L}^2(\Gamma, T(\Gamma))$, we define the following norm

$$\|\mathbf{v}_T\|_{\mathrm{L}^2(\Gamma,T(\Gamma))} = \Big(\int_\Gamma |\mathbf{v}_T|^2\, d\Gamma\Big)^{1/2}, \tag{9}$$

which is equivalent to the norm: $\mathbf{v}_T \mapsto \Big(\|v^1\|^2_{\mathrm{L}^2(\Gamma)} + \|v^2\|^2_{\mathrm{L}^2(\Gamma)}\Big)^{1/2}$. Similarly, $\mathbf{v}_T$ belongs to $\mathrm{H}^1(\Gamma, T(\Gamma))$ if v^1 and v^2 belong to $\mathrm{H}^1(\Gamma)$ and we define a norm in $\mathrm{H}^1(\Gamma, T(\Gamma))$ by

$$\|\mathbf{v}_T\|_{\mathrm{H}^1(\Gamma,T(\Gamma))} = \big(\|v^1\|^2_{\mathrm{H}^1(\Gamma)} + \|v^2\|^2_{\mathrm{H}^1(\Gamma)}\big)^{1/2}. \tag{10}$$

A field $\tau_T : \Gamma \to \mathcal{L}_s(T(\Gamma))$ belongs to $\mathrm{L}^2(\Gamma, \mathcal{L}_s(T(\Gamma)))$ if $(\tau_T\colon \tau_T)^{1/2} : \Gamma \to \mathbb{R}$ belongs to $\mathrm{L}^2(\Gamma)$. We take

$$\|\tau_T\|_{\mathrm{L}^2(\Gamma,\mathcal{L}_s(T(\Gamma)))} = \|(\tau_T\colon \tau_T)^{1/2}\|_{\mathrm{L}^2(\Gamma)}\,. \tag{11}$$

Remark 5 If $\mathbf{v}_T \in \mathrm{H}^1(\Gamma, T(\Gamma))$, then $\varepsilon_T(\mathbf{v}_T) \in \mathrm{L}^2(\Gamma, \mathcal{L}_s(T(\Gamma)))$.

Another useful space is $\mathbb{V} = \{\mathbf{v} \in \mathbb{H}^1(\Omega) \,/\, \mathbf{v}_T \in \mathrm{H}^1(\Gamma, T(\Gamma))\}$ with the norm

$$\|\mathbf{v}\|_{\mathbb{V}} = \left(\|\mathbf{v}\|^2_{\mathbb{H}^1(\Omega)} + \|\mathbf{v}_T\|^2_{\mathrm{H}^1(\Gamma,T(\Gamma))}\right)^{1/2}. \tag{12}$$

Proposition 1 *The expression*

$$\|\mathbf{v}_T\|_1^2 = \int_\Gamma \left(|\mathbf{v}_T|^2 + \varepsilon_T(\mathbf{v}_T) : \varepsilon_T(\mathbf{v}_T)\right) \, d\Gamma \,.$$

constitutes a norm equivalent to (10).

Proof. We only need to prove that there exists a constant $C > 0$ such that

$$\|\mathbf{v}_T\|_{\mathrm{H}^1(\Gamma,T(\Gamma))} \le C\|\mathbf{v}_T\|_1 \,, \quad \forall \mathbf{v}_T \in \mathrm{H}^1(\Gamma, T(\Gamma)) \,.$$

Assume that such a constant does not exist. Then there exits a sequence $\left(\mathbf{v}_T^{(k)}\right)$ such that

$$\begin{aligned} \|\mathbf{v}_T^{(k)}\|_{\mathrm{H}^1(\Gamma,T(\Gamma))} &= 1 \,, \quad \forall k \in \mathbb{N} \,, \\ \left(\varepsilon_T(\mathbf{v}_T^{(k)})\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma, \mathcal{L}_s(T(\Gamma))) \,, \\ \left(\mathbf{v}_T^{(k)}\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma, T(\Gamma)) \,. \end{aligned}$$

Setting $\mathbf{v}_T^{(k)} = v^{k1}\mathbf{a}_1 + v^{k2}\mathbf{a}_2$ and $v^{k\alpha}_{,\beta} = \frac{\partial v^{k\alpha}}{\partial \xi_\beta}$, we get

$$\left(v^{k\alpha}_{,\beta} + g^{\lambda\alpha} g_{\beta\mu} v^{k\mu}_{,\lambda}\right) \to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma) \,.$$

With $\alpha = \beta$, we easily get $\left(v^{k1}_{,1} + v^{k2}_{,2}\right) \to 0$, in $\mathrm{L}^2(\Gamma)$. We have

$$g_{\alpha\eta}\left(v^{k\alpha}_{,\beta} + g^{\lambda\alpha} g_{\beta\mu} v^{k\mu}_{,\lambda}\right) = g_{\alpha\eta} v^{k\alpha}_{,\beta} + g_{\beta\mu} v^{k\mu}_{,\eta} \,.$$

This expression vanishes in $\mathrm{L}^2(\Gamma)$ as $k \to \infty$. We use $(\beta, \eta) = (1, 1)$, $(\beta, \eta) = (2, 2)$ and $(\beta, \eta) = (1, 2)$ and get

$$\begin{aligned} \left(g_{11} v^{k1}_{,1} + g_{21} v^{k2}_{,1}\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma) \,, \\ \left(g_{22} v^{k2}_{,2} + g_{21} v^{k1}_{,2}\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma) \,, \\ \left(g_{11} v^{k1}_{,2} + g_{22} v^{k2}_{,1}\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma) \,, \\ \left(v^{k1}_{,1} + v^{k2}_{,2}\right) &\to 0 \,, \quad \text{in } \mathrm{L}^2(\Gamma) \,. \end{aligned}$$

Set $\mathbf{w}_T^{(k)} = g\mathbf{v}_T^{(k)} = w^{k1}\mathbf{a}_1 + w^{k2}\mathbf{a}_2$. From $\left(\mathbf{v}_T^{(k)}\right) \to 0$, in $\mathrm{L}^2(\Gamma, T(\Gamma))$ and previous computations, one can easily deduce that sequences $\left(w^{k1}\right)$, $\left(w^{k1}\right)$, $\left(w^{k1}_{,1} + w^{k2}_{,2}\right)$, $\left(w^{k2}_{,1} + w^{k1}_{,2}\right)$ vanish in $\mathrm{L}^2(\Gamma)$. Thanks to Korn's inequality in $\hat{\Gamma}$, we get that sequences $\left(w^{k1}\right)$, $\left(w^{k2}\right)$ vanish in $\mathrm{H}^1(\Gamma)$. Hence, $\left(v^{k1}\right)$, $\left(v^{k2}\right)$ vanish in $\mathrm{H}^1(\Gamma)$. This is impossible. □

2 Well-posedness

By using semi-group theory [17], we can show that problem (3) is well-posed.

Proposition 2 *Assume* (2). *If* $(\mathbf{u}^0, \mathbf{u}^1)$ *belongs to* $\mathbb{H}^1_{\Gamma_0}(\Omega) \times \mathbb{L}^2(\Omega)$*, then problem* (3) *has one and only one (weak) solution* $\mathbf{u}$ *which satisfies*

$$\mathbf{u} \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{H}^1_{\Gamma_0}(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathbb{L}^2(\Omega))\,.$$

If $(\mathbf{u}^0, \mathbf{u}^1)$ *belongs to* $(\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_{\Gamma_0}(\Omega)) \times \mathbb{H}^1_{\Gamma_0}(\Omega)$ *and if*

$$\sigma(\mathbf{u}^0).\nu + A\mathbf{u}^0 + B\mathbf{u}^1 = 0\,, \quad \text{on } \Gamma_1\,,$$

then the (strong) solution of (3) *satisfies*

$$(\mathbf{u}, \mathbf{u}', \mathbf{u}'') \in \mathcal{C}^0(\mathbb{R}_+, (\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_{\Gamma_0}(\Omega)) \times \mathbb{H}^1_{\Gamma_0}(\Omega) \times \mathbb{L}^2(\Omega))\,.$$

3 Stabilization

Following Komornik [9], we will prove here that the energy functional is exponentially decreasing with respect to time. From Lemma 5, to be proved later, it is sufficient to consider the case $\mathrm{meas}(\Gamma_0) \neq 0$. We recall the following fundamental result [9].

Lemma 1 *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there exists* $T > 0$ *such that*

$$\int_t^\infty E(s)\, ds \le T E(t)\,, \quad \forall t \ge 0\,.$$

Then we have

$$E(t) \le E(0) \exp\Big(1 - \frac{t}{T}\Big)\,, \quad \forall\, t \ge T\,.$$

First, we prove that the energy functional is non-increasing.

Proposition 3 *Under assumptions* (1), (2), (4), *the weak solution* $\mathbf{u}$ *of* (3) *is such that* $\mathbf{u}'\sqrt{B}$ *belongs to* $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{L}^2(\Gamma_1))$ *and* $a_{ij\acute{}kl}\, \varepsilon_{ij}\, \varepsilon_{kl}$ *belongs to the space* $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^1(\Omega))$*. The energy functional is non-increasing and satisfies*

$$E(\mathbf{u}, T) - E(\mathbf{u}, S) = -\int_S^T \int_{\Gamma_1} B|\mathbf{u}'|^2\, d\Gamma\, dt + \frac{1}{2}\int_S^T \int_\Omega a_{ij\acute{}kl}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{u})\, d\mathbf{x}\, dt\,,$$

for $0 \le S < T < +\infty$.

Proof. Assume first that $\mathbf{u}$ is a strong solution of (3) (with appropriate initial data). We can write

$$\begin{aligned} E'(\mathbf{u},t) &= \int_\Omega \Big(\mathbf{u}'.\mathbf{u}'' + \sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}') + \frac{1}{2} a_{ij\acute{}kl}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{u})\Big)\, d\mathbf{x} + \int_{\Gamma_1} A\mathbf{u}.\mathbf{u}'\, d\Gamma \\ &= \int_\Omega (\mathbf{u}'.\operatorname{div}(\sigma(\mathbf{u})) + \sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}'))\, d\mathbf{x} + \frac{1}{2}\int_\Omega a_{ij\acute{}kl}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{u})\, d\mathbf{x} \\ &\quad + \int_{\Gamma_1} A\mathbf{u}.\mathbf{u}'\, d\Gamma \\ &= \int_{\Gamma_1} (\sigma(\mathbf{u})\nu)\,.\mathbf{u}' + \frac{1}{2}\int_\Omega a_{ij\acute{}kl}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{u})\, d\mathbf{x}\, d\Gamma + \int_{\Gamma_1} A\mathbf{u}.\mathbf{u}'\, d\Gamma \\ &= -\int_{\Gamma_1} B|\mathbf{u}'|^2\, d\Gamma + \frac{1}{2}\int_\Omega a_{ij\acute{}kl}\, \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{u})\, d\mathbf{x}\,. \end{aligned}$$

We obtain the result by integrating the time variable from S to T. A density argument completes the proof. □

In order to apply Lemma 1, we have to prove the following results.

Preliminary results

In this subsection, we assume (1) and

$$\begin{aligned} (\mathbf{u}^0, \mathbf{u}^1) &\in \big(\mathbb{H}^2(\Omega) \cap \mathbb{H}^1_{\Gamma_0}(\Omega)\big) \times \mathbb{H}^1_{\Gamma_0}(\Omega)\,, \\ \sigma(\mathbf{u}^0)\nu + A\mathbf{u}^0 + B\mathbf{u}^1 &= 0\,, \quad \text{on } \Gamma_1\,, \end{aligned} \tag{13}$$

and consider the (strong) solution of (3). Let $\mathbf{h}$ be a vector field satisfying (5) and (6). For some positive constant β, define

$$M\mathbf{u} = 2(\mathbf{h}.\nabla)\mathbf{u} + \beta\mathbf{u}\,.$$

The value of β will be chosen later.

Lemma 2 *The strong solution* $\mathbf{u}$ *of* (3) *satisfies*

$$\begin{aligned} &\int_S^T \int_\Omega (\operatorname{div}(\mathbf{h}) - \beta)\,(|\mathbf{u}'|^2 - \sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}))\, d\mathbf{x}\, dt \\ &+2\int_S^T \int_\Omega \sigma_{i,j}(\mathbf{u}) h_{k,j} u_{i,k}\, d\mathbf{x}\, dt - \int_S^T \int_\Omega h_m \partial_m(a_{ijkl})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\, d\mathbf{x}\, dt \\ &\quad = \quad -\Big[\int_\Omega \mathbf{u}' M\mathbf{u} d\mathbf{x}\Big]_S^T + \int_S^T \int_\Gamma ((\sigma(\mathbf{u}).\nu)M\mathbf{u} - \mathbf{h}.\nu\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}))\, d\mathbf{x}\, dt\,. \\ &\qquad + \int_S^T \int_\Gamma \mathbf{h}.\nu|\mathbf{u}'|^2\, d\Gamma dt\,. \end{aligned}$$

Proof. We use the multipliers method (see [9], [15]). Thanks to the first equation in (3), we may write

$$\int_S^T \int_\Omega \mathbf{u}''.M\mathbf{u}\, d\mathbf{x}\, dt = \int_S^T \int_\Omega \operatorname{div}(\sigma(\mathbf{u})).M\mathbf{u}\, d\mathbf{x}\, dt\,. \tag{14}$$

Consider the left-hand side of the above equation.

$$\begin{aligned}\int_S^T\int_\Omega \mathbf{u}''.M\mathbf{u}\,d\mathbf{x}\,dt =& \Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T - \int_S^T\int_\Omega \mathbf{u}'.M\mathbf{u}'\,d\mathbf{x}\,dt\\ =& \Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T\\ &- 2\int_S^T\int_\Omega u_i' h_j u_{i,j}'\,d\mathbf{x}\,dt - \beta\int_S^T\int_\Omega |\mathbf{u}'|^2\,d\mathbf{x}\,dt\,.\end{aligned}$$

We have

$$\begin{aligned}2\int_S^T\int_\Omega u_i' h_j u_{i,j}'\,d\mathbf{x}\,dt &= \int_S^T\int_\Omega h_j\partial_j\left(|\mathbf{u}'|^2\right)\,d\mathbf{x}\,dt\\ &= \int_S^T\int_\Gamma \mathbf{h}.\nu\,|\mathbf{u}'|^2\,d\Gamma\,dt - \int_S^T\int_\Omega \operatorname{div}(\mathbf{h})|\mathbf{u}'|^2\,d\mathbf{x}\,dt\,.\end{aligned}$$

Hence

$$\begin{aligned}\int_S^T\int_\Omega \mathbf{u}''.M\mathbf{u}\,d\mathbf{x}\,dt =& \Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T - \int_S^T\int_\Gamma \mathbf{h}.\nu\,|\mathbf{u}'|^2\,d\Gamma\,dt\\ &+ \int_S^T\int_\Omega (\operatorname{div}(\mathbf{h})-\beta)\,|\mathbf{u}'|^2\,d\mathbf{x}\,dt\,.\end{aligned} \tag{15}$$

Now, we consider the right-hand side of (14):

$$\begin{aligned}\int_S^T\int_\Omega \operatorname{div}(\sigma(\mathbf{u})).M\mathbf{u}\,d\mathbf{x}\,dt =& 2\int_S^T\int_\Omega \sigma_{ij,j}(\mathbf{u})h_m u_{i,m}\,d\mathbf{x}\,dt\\ &+ \beta\int_S^T\int_\Omega \mathbf{u}.\operatorname{div}(\sigma(\mathbf{u}))\,d\mathbf{x}\,dt\,.\end{aligned}$$

We have

$$\begin{aligned}&\int_S^T\int_\Omega \sigma_{ij,j}(\mathbf{u})h_k u_{i,k}\,d\mathbf{x}\,dt\\ =&\quad \int_S^T\int_\Gamma \sigma_{ij}(\mathbf{u})\nu_j h_k u_{i,k}\,d\Gamma\,dt - \int_S^T\int_\Omega \sigma_{ij}(\mathbf{u})\partial_j(h_k u_{i,k})\,d\mathbf{x}\,dt\\ =&\quad \int_S^T\int_\Gamma \sigma(\mathbf{u})\nu.(\mathbf{h}.\nabla)\mathbf{u}\,d\Gamma\,dt - \int_S^T\int_\Omega \sigma_{ij}(\mathbf{u})h_{k,j}u_{i,k}\,d\mathbf{x}\,dt\\ &\quad - \int_S^T\int_\Omega \sigma_{ij}(\mathbf{u})h_k u_{i,jk}\,d\mathbf{x}\,dt\,,\end{aligned}$$

and

$$\begin{aligned}
&-\int_S^T\int_\Omega \sigma_{ij}(\mathbf{u})h_k u_{i,jk}\,d\mathbf{x}\,dt\\
&=-\frac{1}{2}\int_S^T\int_\Omega h_k\partial_k(\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u}))\,d\mathbf{x}\,dt+\frac{1}{2}\int_S^T\int_\Omega h_m\partial_m(a_{ijkl})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\,d\mathbf{x}\,dt\\
&=\frac{1}{2}\int_S^T\int_\Omega \operatorname{div}(\mathbf{h})\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\,d\mathbf{x}\,dt-\frac{1}{2}\int_S^T\int_\Gamma \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\,d\Gamma\,dt\\
&\quad+\frac{1}{2}\int_S^T\int_\Omega h_m\partial_m(a_{ijkl})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\,d\mathbf{x}\,dt\,.
\end{aligned}$$

Furthermore,

$$\int_S^T\int_\Omega \mathbf{u}.\operatorname{div}(\sigma(\mathbf{u}))\,d\mathbf{x}\,dt=\int_S^T\int_\Gamma(\sigma(\mathbf{u})\nu).\mathbf{u}\,d\Gamma\,dt-\int_S^T\int_\Omega\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\,d\mathbf{x}\,dt\,.$$

Hence

$$\begin{aligned}
&\int_S^T\int_\Omega \operatorname{div}(\sigma(\mathbf{u})).M\mathbf{u}\,d\mathbf{x}\,dt\\
&\qquad=\int_S^T\int_\Omega(\operatorname{div}(\mathbf{h})-\beta)\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\,d\mathbf{x}\,dt-2\int_S^T\int_\Omega\sigma_{ij}(\mathbf{u})h_{k,j}u_{i,k}\,d\mathbf{x}\,dt\\
&\qquad\quad+\int_S^T\int_\Omega h_m\partial_m(a_{ijkl})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\,d\mathbf{x}\,dt\\
&\qquad\quad+\int_S^T\int_\Gamma\left((\sigma(\mathbf{u})\nu).M\mathbf{u}-\mathbf{h}.\nu\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\right)\,d\Gamma dt\,.
\end{aligned}\tag{16}$$

We deduce the desired result from (14), (15) and (16). □

Lemma 3 *There exist $\beta>0$ and $C_1>0$ such that*

$$\begin{aligned}
C_1\int_S^T E\,dt\quad\le\quad&-\left[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\right]_S^T\\
&+\int_S^T\int_{\Gamma_0}\mathbf{h}.\nu\left(\mu|\partial_\nu\mathbf{u}_T|^2+(2\mu+\lambda)|\partial_\nu u_\nu|^2\right)\,d\Gamma\,dt\\
&+\int_S^T\int_{\Gamma_1}\left(\Big(\frac{C_1}{2}-\beta\Big)A|\mathbf{u}|^2-\beta B\mathbf{u}.\mathbf{u}'+\mathbf{h}.\nu\,|\mathbf{u}'|^2\right)\,d\Gamma\,dt\\
&-\int_S^T\int_{\Gamma_1}\left(2(A\mathbf{u}+B\mathbf{u}').(\mathbf{h}.\nabla)\mathbf{u}+\mathbf{h}.\nu\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\right)\,d\Gamma\,dt\,.
\end{aligned}$$

Proof. Lemma 2, (5) and (6) give

$$\begin{aligned}
&\int_S^T \int_\Omega \big((\mathrm{div}(\mathbf{h}) - \beta)\big(|\mathbf{u}'|^2 - \sigma(\mathbf{u})\colon \varepsilon(\mathbf{u})\big) + (2\alpha_{\mathbf{h}} - \gamma_{\mathbf{h}})\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u})\big)\ d\mathbf{x}\,dt \\
&\quad \le -\Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T + \int_S^T \int_{\Gamma_1} \mathbf{h}.\nu\,|\mathbf{u}'|^2\,d\Gamma\,dt \\
&\quad\quad + \int_S^T \int_\Gamma ((\sigma(\mathbf{u})\nu).M\mathbf{u} - \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}))\ d\Gamma\,dt\,.
\end{aligned}$$

Since (6) is satisfied, we can choose $\beta > 0$ such that

$$\beta < \mathrm{div}(\mathbf{h}) < 2\alpha_{\mathbf{h}} + \beta - \gamma_{\mathbf{h}}\,, \quad \text{in } \Omega\,, \tag{17}$$

and choose $C_1 > 0$ such that

$$\begin{aligned}
C_1 \int_S^T E\,dt \le &-\Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T \\
&+ \int_S^T \int_{\Gamma_1} \Big(\frac{C_1}{2} A|\mathbf{u}|^2 + \mathbf{h}.\nu\,|\mathbf{u}'|^2\Big)\,d\Gamma\,dt \\
&+ \int_S^T \int_\Gamma ((\sigma(\mathbf{u})\nu).M\mathbf{u} - \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}))\ d\Gamma\,dt\,.
\end{aligned} \tag{18}$$

We write $\mathbf{u} = \mathbf{u}_T + u_\nu \nu$ and $\mathbf{h} = \mathbf{h}_T + h_\nu \nu$ on Γ. With Dirichlet boundary condition, we get on Γ_0

$$M\mathbf{u} = 2(\mathbf{h}.\nabla)\mathbf{u} = 2\mathbf{h}.\nu\,\partial_\nu \mathbf{u}\,, \quad \varepsilon_T(\mathbf{u}) = 0\,, \quad 2\varepsilon_S(\mathbf{u}) = \partial_\nu \mathbf{u}_T\,.$$

and, with our notations, using Remark 3, we get

$$\begin{gathered}
(\sigma(\mathbf{u})\nu).M\mathbf{u} = 2\mathbf{h}.\nu\big(\overline{\sigma_S(\mathbf{u})}\partial_\nu \mathbf{u}_T + \sigma_\nu(\mathbf{u})\partial_\nu \mathbf{u}_\nu\big)\,, \quad \text{on } \Gamma_0\,, \\
\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}) = \overline{\sigma_S(\mathbf{u})}\partial_\nu \mathbf{u}_T + \sigma_\nu(\mathbf{u})\partial_\nu \mathbf{u}_\nu\,, \quad \text{on } \Gamma_0\,, \\
(\sigma(\mathbf{u})\nu).M\mathbf{u} - \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}) = \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u})\,, \quad \text{on } \Gamma_0\,.
\end{gathered}$$

Hence

$$\int_{\Gamma_0} ((\sigma(\mathbf{u})\nu).M\mathbf{u} - \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u}))\ d\Gamma = \int_{\Gamma_0} \mathbf{h}.\nu\,\sigma(\mathbf{u})\colon \varepsilon(\mathbf{u})\,d\Gamma\,. \tag{19}$$

Using the boundary conditon on Γ_1, we get

$$\int_{\Gamma_1} (\sigma(\mathbf{u})\nu).M\mathbf{u}\,d\Gamma = -\int_{\Gamma_1} 2(A\mathbf{u} + B\mathbf{u}').(\mathbf{h}.\nabla)\mathbf{u}\,d\Gamma - \beta \int_{\Gamma_1} \mathbf{u}.(A\mathbf{u} + B\mathbf{u}')\,d\Gamma\,. \tag{20}$$

We deduce the result from (18), (19) and (20). □

Lemma 4 *There exists $C_2 > 0$ such that*

$$\Big|\int_\Omega \mathbf{u}'(t).M\mathbf{u}(t)\,d\mathbf{x}\Big| \le C_2 E(\mathbf{u}, t)\,, \quad \forall t \ge 0\,.$$

Proof. Given $t \geq 0$, for every $\eta > 0$, we can write

$$\Big| \int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big| \leq \frac{\eta}{2}\|\mathbf{u}'\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2\eta}\|M\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)}\,.$$

We have

$$\begin{aligned}
\|M\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)} &= \int_\Omega (|2(\mathbf{h}.\nabla)\mathbf{u}|^2 + \beta^2|\mathbf{u}|^2 + 4\beta\mathbf{u}.(\mathbf{h}.\nabla)\mathbf{u})\,d\mathbf{x}\\
&= \int_\Omega (|2(\mathbf{h}.\nabla)\mathbf{u}|^2 + \beta^2|\mathbf{u}|^2 + 2\beta\nabla(|\mathbf{u}|^2)\mathbf{h})\,d\mathbf{x}\\
&= \int_\Omega (|2(\mathbf{h}.\nabla)\mathbf{u}|^2 + \beta(\beta - 2\,\mathrm{div}(\mathbf{h}))|\mathbf{u}|^2)\,d\mathbf{x} + 2\beta\int_{\Gamma_1}\mathbf{h}.\nu\,|\mathbf{u}|^2\,d\Gamma\,.
\end{aligned}$$

Setting $R = \sup_{\overline{\Omega}}|\mathbf{h}|$, from (17), we get

$$\|M\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)} \leq 4R^2\int_\Omega |\nabla\mathbf{u}|^2\,d\mathbf{x} + 2\beta\int_{\Gamma_1}\mathbf{h}.\nu\,|\mathbf{u}|^2\,d\Gamma\,.$$

With Korn's inequality, we can find the smallest positive real number R_1 (depending on $\mathbf{h}$ and β) such that for all $\mathbf{v} \in \mathbb{H}^1_{\Gamma_0}(\Gamma)$,

$$4R_1^2\Big(\int_\Omega \sigma(\mathbf{v})\colon\varepsilon(\mathbf{v})\,d\mathbf{x} + \int_{\Gamma_1} A|\mathbf{v}|^2\,d\Gamma\Big) \geq 4R^2\int_\Omega |\nabla\mathbf{v}|^2\,d\mathbf{x} + 2\beta\int_{\Gamma_1}\mathbf{h}.\nu\,|\mathbf{v}|^2\,d\Gamma\,.$$

It follows that

$$\Big|\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big| \leq \frac{\eta}{2}\|\mathbf{u}'\|^2_{\mathbb{L}^2(\Omega)} + \frac{2R_1^2}{\eta}\Big(\int_\Omega \sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\,d\mathbf{x} + \int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma\Big)\,.$$

The choice $\eta = 2R_1$ gives the result with $C_2 = 2R_1$. □

Lemma 5 *There exists $C_3 > 0$ such that, for every η in $(0,1)$,*

$$\int_S^T\int_{\Gamma_1}|\mathbf{u}|^2\,d\Gamma\,dt \leq \frac{C_3}{\eta}E(\mathbf{u},S) + \eta\int_S^T E(\mathbf{u},t)\,dt\,,\quad 0 \leq S < T < +\infty\,.$$

Proof. We proceed as in [9]. We define $\mathbf{z}$, depending on t, as follows:

$$\begin{gathered}
\mathrm{div}(\sigma(\mathbf{z})) = 0\,,\quad \text{in } \Omega\,,\\
\mathbf{z} = \mathbf{u}\,,\quad \text{on } \Gamma\,.
\end{gathered}$$

We have

$$\int_\Omega \mathbf{z}.\,\mathrm{div}(\sigma(\mathbf{v}))\,d\mathbf{x} = \int_{\Gamma_1}\mathbf{u}.(\sigma(\mathbf{v})\nu)\,d\Gamma\,,\quad \forall\mathbf{v} \in \mathbb{H}^2(\Omega)\cap\mathbb{H}^1_0(\Omega)\,.$$

Using the definition of the energy functional and Proposition 3, we can find some positive constants c_1, c_2, c_1', c_2' such that

$$\begin{gathered}
\int_\Omega |\mathbf{z}|^2\,d\mathbf{x} \leq c_1\int_{\Gamma_1}|\mathbf{u}|^2\,d\Gamma \leq c_2 E\,;\\
\int_\Omega |\mathbf{z}'|^2\,d\mathbf{x} \leq c_1'\int_{\Gamma_1}|\mathbf{u}'|^2\,d\Gamma \leq c_2'(-E')\,.
\end{gathered}$$

Furthermore, we have

$$\int_\Omega \sigma(\mathbf{z})\colon \varepsilon(\mathbf{u}-\mathbf{z})\,d\mathbf{x} = -\int_\Omega (\mathbf{u}-\mathbf{z}).\operatorname{div}(\sigma(\mathbf{z}))\,d\mathbf{x} + \int_{\Gamma_1} (\mathbf{u}-\mathbf{z}).(\sigma(\mathbf{z})\nu)\,d\Gamma = 0\,.$$

Then

$$\int_\Omega \sigma(\mathbf{z})\colon \varepsilon(\mathbf{u})\,d\mathbf{x} = \int_\Omega \sigma(\mathbf{z})\colon \varepsilon(\mathbf{z})\,d\mathbf{x} \geq 0\,.$$

From (3), we deduce

$$\begin{aligned} 0 &= \int_\Omega \mathbf{z}.(\mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u}))\,d\mathbf{x}\,dt \\ &= \int_\Omega \mathbf{z}.\mathbf{u}''\,d\mathbf{x} + \int_\Omega \sigma(\mathbf{z})\colon \varepsilon(\mathbf{u})\,d\mathbf{x} - \int_{\Gamma_1} \mathbf{z}.(\sigma(\mathbf{u})\nu)\,d\Gamma \\ &= \int_\Omega \mathbf{z}.\mathbf{u}''d\mathbf{x} + \int_\Omega \sigma(\mathbf{z})\colon \varepsilon(\mathbf{u})\,d\mathbf{x} + \int_{\Gamma_1} \mathbf{u}.(A\mathbf{u}+B\mathbf{u}')\,d\Gamma\,. \end{aligned}$$

Hence

$$\int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma \leq -\int_\Omega \mathbf{z}.\mathbf{u}''\,d\mathbf{x} - \int_{\Gamma_1} B\mathbf{u}.\mathbf{u}'\,d\Gamma\,.$$

For $0 < S < T < \infty$, we obtain

$$\int_S^T \int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma\,dt \leq -\Big[\int_\Omega \mathbf{z}.\mathbf{u}'\,d\mathbf{x}\Big]_S^T + \int_S^T \int_\Omega \mathbf{z}'.\mathbf{u}'\,d\mathbf{x}\,dt - \int_S^T \int_{\Gamma_1} B\mathbf{u}.\mathbf{u}'\,d\Gamma\,dt\,.$$

Let C be a positive constant, large enough. Using the Cauchy-Schwarz inequality and the estimates obtained above, we can write for every $\theta > 0$,

$$\begin{aligned} \int_S^T \int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma\,dt \quad &\leq \quad CE(\mathbf{u},S) + C\int_S^T (-E'(\mathbf{u},t))^{1/2}(E(\mathbf{u},t))^{1/2}\,dt \\ &\quad\quad +B\int_S^T \int_{\Gamma_1} |\mathbf{u}|\,|\mathbf{u}'|\,d\Gamma\,dt \\ &\leq \quad CE(\mathbf{u},S) + \frac{C\theta}{2}\int_S^T E(\mathbf{u},t)\,dt + \frac{C}{2\theta}E(\mathbf{u},S) \\ &\quad\quad +\frac{1}{2}\int_S^T \int_{\Gamma_1} A|\mathbf{u}|^2\,d\Gamma\,dt + \frac{B^2}{2A}\int_S^T \int_{\Gamma_1} |\mathbf{u}'|^2\,d\Gamma\,dt\,. \end{aligned}$$

From Proposition 3, we get

$$\int_S^T \int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,dt \leq \left(\frac{2C}{A} + \frac{C}{A\theta} + \frac{B}{A^2}\right) E(\mathbf{u},S) + \frac{C\theta}{A}\int_S^T E(\mathbf{u},t)\,dt\,.$$

We now choose $\theta = \frac{A\eta}{C}$ and obtain the desired result. $\square$

Proof of Theorem 1

We assume (1) and that $\mathbf{h}$ satisfies (5) and (6). From Lemma 5, it suffices to consider the case $\mathrm{meas}(\Gamma_0) \neq 0$. We first suppose (13) and we consider the (strong) solution $\mathbf{u}$ of (3).

The energy functional is non-increasing (Proposition 3). From Lemma 4, we deduce

$$-\Big[\int_\Omega \mathbf{u}'.M\mathbf{u}\,d\mathbf{x}\Big]_S^T \leq 2C_2E(\mathbf{u},S)\,.$$

Since $\mathbf{h}.\nu \leq 0$ on Γ_0, Lemma 3 gives

$$\begin{aligned}C_1\int_S^T E\,dt \quad\leq\quad & 2C_2E(\mathbf{u},S)\\ &+\int_S^T\int_{\Gamma_1}\Big(\big(\frac{C_1}{2}-\beta\big)A|\mathbf{u}|^2-\beta B\mathbf{u}.\mathbf{u}'+\mathbf{h}.\nu\,|\mathbf{u}'|^2\Big)\,d\Gamma\,dt\\ &-\int_S^T\int_{\Gamma_1}\left(2(A\mathbf{u}+B\mathbf{u}').(\mathbf{h}.\nabla)\mathbf{u}+\mathbf{h}.\nu\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\right)\,d\Gamma\,dt\,.\end{aligned}$$

There exists $c>0$ such that

$$\Big|\int_S^T\int_{\Gamma_1}\beta\mathbf{u}.\mathbf{u}'\,d\Gamma\,dt\Big| \leq \int_S^T\int_{\Gamma_1}|\mathbf{u}|^2\,d\Gamma\,dt + c\int_S^T\int_{\Gamma_1}|\mathbf{u}'|^2\,d\Gamma\,dt\,.$$

Hence, using Proposition 3, we can find $C_4>0$ and $C_5>0$ such that

$$\begin{aligned}C_1\int_S^T E\,dt \leq & C_4E(\mathbf{u},S)+C_5\int_S^T\int_{\Gamma_1}|\mathbf{u}|^2\,d\Gamma\,dt\\ &-\int_S^T\int_{\Gamma_1}\left(2(A\mathbf{u}+B\mathbf{u}').(\mathbf{h}.\nabla)\mathbf{u}+\mathbf{h}.\nu\,\sigma(\mathbf{u})\colon\varepsilon(\mathbf{u})\right)\,d\Gamma\,dt\,.\end{aligned}$$

From (2), Remarks 1 and 3, we get

$$\begin{aligned}C_1\int_S^T E\,dt \leq & C_4E(\mathbf{u},S)+C_5\int_S^T\int_{\Gamma_1}|\mathbf{u}|^2\,d\Gamma\,dt\\ &-\int_S^T\int_{\Gamma_1}2(A\mathbf{u}+B\mathbf{u}').(\mathbf{h}.\nabla)\mathbf{u}\,d\Gamma\,dt \qquad (21)\\ &-k\alpha\int_S^T\int_{\Gamma_1}\left(\varepsilon_T(\mathbf{u})\colon\varepsilon_T(\mathbf{u})+2|\varepsilon_S(\mathbf{u})|^2+|\varepsilon_\nu(\mathbf{u})|^2\right)\,d\Gamma\,dt\,.\end{aligned}$$

Now, we estimate two integrals which appear on the right hand side (second line of the above formula).

Estimation of $\mathcal{I}_1=\int_S^T\int_{\Gamma_1}2A\mathbf{u}.(\mathbf{h}.\nabla)\mathbf{u}\,d\Gamma\,dt$.
We denote by C some positive constant which is independent of $\mathbf{u}$ and large enough. We have $\mathbf{u}.(\mathbf{h}.\nabla)\mathbf{u}=\frac{1}{2}\mathbf{h}.\nabla(|\mathbf{u}|^2)$. Setting $\mathbf{u}=\mathbf{u}_T+u_\nu\nu$ and $\mathbf{h}=\mathbf{h}_T+h_\nu\nu$ on Γ_1, we use (7) and get

$$\mathbf{u}.(\mathbf{h}.\nabla)\mathbf{u}=\frac{1}{2}\nabla_T(|\mathbf{u}|^2).\mathbf{h}_T+h_\nu\mathbf{u}_T.(\partial_\nu\mathbf{u}_T)+h_\nu u_\nu(\partial_\nu u_\nu)\,,\quad \text{on } \Gamma_1\,.$$

Since

$$\int_{\Gamma_1} A\nabla_T(|\mathbf{u}|^2).\mathbf{h}_T\,d\Gamma = -\int_{\Gamma_1} A|\mathbf{u}|^2 \operatorname*{div}_T(\mathbf{h}_T)\,d\Gamma\,,$$

hence

$$\Big|\int_{\Gamma_1} A\nabla_T(|\mathbf{u}|^2).\mathbf{h}_T\,d\Gamma\Big| \le C\int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,. \tag{22}$$

Using $\varepsilon_S(\mathbf{u})$ (see subsection 1.2), we can write

$$h_\nu\,\mathbf{u}_T.\partial_\nu\mathbf{u}_T = h_\nu\,\mathbf{u}_T.(2\varepsilon_S(\mathbf{u}) + (\partial_T\nu)\mathbf{u}_T - \nabla_T u_\nu)\,, \quad \text{on } \Gamma_1\,.$$

Let θ be some positive number. We have

$$\Big|\int_{\Gamma_1} 4Ah_\nu\,\mathbf{u}_T.\varepsilon_S(\mathbf{u})\,d\Gamma\Big| \le \theta\int_{\Gamma_1} |\varepsilon_S(\mathbf{u})|^2\,d\Gamma + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}_T|^2\,d\Gamma\,. \tag{23}$$

Since h_ν and $\partial_T\nu$ are bounded, we get

$$\Big|\int_{\Gamma_1} 2Ah_\nu\,\mathbf{u}_T.(\partial_T\nu)\mathbf{u}_T\,d\Gamma\Big| \le C\int_{\Gamma_1} |\mathbf{u}_T|^2\,d\Gamma\,. \tag{24}$$

Now, observe

$$\begin{aligned}\int_{\Gamma_1} h_\nu\,\mathbf{u}_T.\nabla_T u_\nu\,d\Gamma &= -\int_{\Gamma_1} u_\nu \operatorname*{div}_T(h_\nu\mathbf{u}_T)\,d\Gamma\\ &= -\int_{\Gamma_1} h_\nu\,u_\nu \operatorname*{div}_T(\mathbf{u}_T)\,d\Gamma - \int_{\Gamma_1} u_\nu\nabla_T h_\nu.\mathbf{u}_T\,d\Gamma\,.\end{aligned}$$

Proposition 1 implies

$$\Big|\int_{\Gamma_1} 2Ah_\nu\,u_\nu \operatorname*{div}_T(\mathbf{u}_T)\,d\Gamma\Big| \le \theta\|\mathbf{u}_T\|_1^2 + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,.$$

Hence

$$\Big|\int_{\Gamma_1} 2Ah_\nu\,\mathbf{u}_T.\nabla_T u_\nu\,d\Gamma\Big| \le \theta\int_{\Gamma_1} \varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T)\,d\Gamma + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,. \tag{25}$$

We can also write

$$\Big|\int_{\Gamma_1} 2Ah_\nu\,u_\nu(\partial_\nu u_\nu)\,d\Gamma\Big| \le \theta\int_{\Gamma_1} |\partial_\nu u_\nu|^2\,d\Gamma + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,. \tag{26}$$

Finally, (22)–(26) give

$$\begin{aligned}|\mathcal{I}_1| \le{}&\theta\int_S^T\int_{\Gamma_1} \big(|\partial_\nu u_\nu|^2 + \varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + |\varepsilon_S(\mathbf{u})|^2\big)\,d\Gamma\,dt\\ &+ \frac{C}{\theta}\int_S^T\int_{\Gamma_1} |\mathbf{u}|^2\,d\Gamma\,dt\,.\end{aligned} \tag{27}$$

We emphasize that, in (27), θ is a positive number to be chosen later and C is a positive constant which does not depend on $\mathbf{u}$.

Estimation of $\mathcal{I}_2 = \int_S^T \int_{\Gamma_1} 2B\mathbf{u}'.(\mathbf{h}.\nabla)\mathbf{u}\, d\Gamma\, dt$.
Here, we use (8) and get

$$\begin{aligned}\mathbf{u}'.(\mathbf{h}.\nabla)\mathbf{u} =& \overline{\mathbf{u}'_T}(\partial_T \mathbf{u}_T)\mathbf{h}_T + u_\nu \overline{\mathbf{u}'_T}(\partial_T \nu)\mathbf{h}_T + h_\nu \overline{\mathbf{u}'_T}\partial_\nu \mathbf{u}_T \\ &+ u'_\nu(\partial_T u_\nu)\mathbf{h}_T - u'_\nu \overline{\mathbf{u}_T}(\partial_T \nu)\mathbf{h}_T + u'_\nu(\partial_\nu u_\nu)h_\nu\,, \quad \text{on } \Gamma_1\,.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\mathbf{u}'.(\mathbf{h}.\nabla)\mathbf{u} =& \mathbf{u}'_T.(\partial_T \mathbf{u}_T)\mathbf{h}_T + (u_\nu \mathbf{u}'_T - u'_\nu \mathbf{u}_T).(\partial_T \nu)\mathbf{h}_T \\ &+ u'_\nu \nabla_T u_\nu.\mathbf{h}_T + h_\nu(\mathbf{u}'_T.\partial_\nu \mathbf{u}_T + u'_\nu(\partial_\nu u_\nu))\,, \quad \text{on } \Gamma_1\,.\end{aligned}$$

Since $\mathbf{h}$ and $\partial_T \nu$ are bounded, we get

$$\Big|\int_{\Gamma_1} 2B\mathbf{u}'_T.(\partial_T \mathbf{u}_T)\mathbf{h}_T\, d\Gamma\Big| \le \frac{\theta}{2}\|\mathbf{u}_T\|_1^2 + \frac{2C}{\theta}\int_{\Gamma_1} |\mathbf{u}'_T|^2\, d\Gamma\,, \tag{28}$$

$$\Big|\int_{\Gamma_1} 2B(u_\nu \mathbf{u}'_T - u'_\nu \mathbf{u}_T).(\partial_T \nu)\mathbf{h}_T\, d\Gamma\Big| \le C\Big(\int_{\Gamma_1} |\mathbf{u}|^2\, d\Gamma + \int_{\Gamma_1} |\mathbf{u}'|^2\, d\Gamma\Big)\,. \tag{29}$$

Under the assumptions about Ω, we observe that Γ_1 is a compact manifold of dimension 2. So, we can build a finite number of local maps $(U_1, \phi_1), \dots, (U_k, \phi_k)$ and an associated partition of unity $(\vartheta_1, \dots, \vartheta_k)$. We have

$$\int_{\Gamma_1} 2Bu'_\nu \nabla_T u_\nu.\mathbf{h}_T\, d\Gamma = \sum_{j=1}^{\ell} \int_{U_j} 2B\vartheta_j u'_\nu \nabla_T u_\nu.\mathbf{h}_T\, d\Gamma\,.$$

Consider one of the ℓ terms of the previous sum. Omitting the index j, we denote $\int_U 2B\vartheta u'_\nu \nabla_T u_\nu.\mathbf{h}_T\, d\Gamma$. Using the notation introduced in section 1, we write $\mathbf{h}_T = h^1 a_1 + h^2 a_2$. Setting $|g| = |\det(g)|$, $W = \phi^{-1}(U)$, we get

$$\begin{aligned}&\int_U 2B\vartheta u'_\nu \nabla_T u_\nu.\mathbf{h}_T\, d\Gamma \\ &= \int_W 2B(\vartheta\circ\phi)(u'_\nu\circ\phi)\Big(\frac{\partial(u_\nu\circ\phi)}{\partial\xi_1}h^1 + \frac{\partial(u_\nu\circ\phi)}{\partial\xi_2}h^2\Big)|g|^{1/2}\, d\xi_1 d\xi_2\,.\end{aligned}$$

Since $\vartheta\circ\phi$ is continuous and compactly supported, $v_\nu = u_\nu\circ\phi$ belongs to $\mathrm{H}^{1/2}(W)$ and $\|v_\nu\|_{\mathrm{H}^{1/2}(W)} \le C\|u_\nu\|_{\mathrm{H}^{1/2}(U)}$. Let us define two subsets of W

$$W^+ = \{(\xi_1,\xi_2)\in W \,/\, h^1(\xi_1,\xi_2) > 0\}\,, \quad W^- = \{(\xi_1,\xi_2)\in W \,/\, h^1(\xi_1,\xi_2) < 0\}\,.$$

We have

$$\begin{aligned}\int_W 2B(\vartheta\circ\phi)v'_\nu \frac{\partial v_\nu}{\partial\xi_1}h^1|g|^{1/2}\, d\xi_1 d\xi_2 =& \int_{W^+} 2B(\vartheta\circ\phi)v'_\nu \frac{\partial v_\nu}{\partial\xi_1}h^1|g|^{1/2}\, d\xi_1 d\xi_2 \\ &+ \int_{W^-} 2B(\vartheta\circ\phi)v'_\nu \frac{\partial v_\nu}{\partial\xi_1}h^1|g|^{1/2}\, d\xi_1 d\xi_2\,.\end{aligned}$$

Setting $\psi = \big((\vartheta\circ\phi)h^1|g|^{1/2}\big)^{1/2}$, in W^+, we have

$$\begin{aligned}
&\int_{W^+} 2B(\vartheta\circ\phi)v'_\nu \frac{\partial v_\nu}{\partial\xi_1}h^1|g|^{1/2}\,d\xi_1 d\xi_2\\
&= \int_{W^+} 2B\psi^2 v'_\nu \frac{\partial v_\nu}{\partial\xi_1}\,d\xi_1 d\xi_2\\
&= \int_{W^+} 2B\psi v'_\nu \frac{\partial(\psi v_\nu)}{\partial\xi_1}\,d\xi_1 d\xi_2 - \int_{W^+} B\psi\left(|v_\nu|^2\right)'\frac{\partial\psi}{\partial\xi_1}\,d\xi_1 d\xi_2\,.
\end{aligned}$$

Thus

$$\Big|\Big[\int_{W^+} B\psi|v_\nu|^2\frac{\partial\psi}{\partial\xi_1}\,d\xi_1 d\xi_2\Big]_S^T\Big| \le CE(\mathbf{u},S)\,.$$

We know that ψ is compactly supported in W, and $\psi = 0$, on ∂W^+. Define function

$$G = \psi v_\nu\,,\ \text{ in } W^+\times\mathbb{R}_+\,,\quad G = 0\,,\ \text{ in } (\mathbb{R}^2\setminus W^+)\times\mathbb{R}_+\,.$$

We have

$$\int_{W^+} 2B\psi v'_\nu \frac{\partial(\psi v_\nu)}{\partial\xi_1}\,d\xi_1 d\xi_2 = \int_{\mathbb{R}^2} 2BG'\frac{\partial G}{\partial\xi_1}\,d\xi_1 d\xi_2\,.$$

Let $\hat{G}$ be the Fourier transform of G, with respect to ξ_1. We write

$$\int_{W^+} 2B\psi v'_\nu \frac{\partial(\psi v_\nu)}{\partial\xi_1}\,d\xi_1 d\xi_2 = \int_{\mathbb{R}^2} 4Bi\pi\eta_1\hat{G}'\hat{G}\,d\eta_1 d\xi_2\,.$$

This implies

$$\int_S^T\int_{W^+} 2B\psi v'_\nu \frac{\partial(\psi v_\nu)}{\partial\xi_1}\,d\xi_1 d\xi_2 dt = \Big[\int_{\mathbb{R}^2} 2Bi\pi\eta_1|\hat{G}|^2\,d\eta_1 d\xi_2\Big]_S^T\,.$$

But

$$\Big|\int_{\mathbb{R}^2}\eta_1|\hat{G}|^2\,d\eta_1 d\xi_2\Big| \le C_1\|G\|^2_{\mathrm{H}^{1/2}(\mathbb{R}^2)} \le C_2\|u_\nu\|^2_{\mathrm{H}^{1/2}(\Gamma_1)}\,.$$

Hence, using the energy functional and Proposition 3, we get

$$\Big|\int_S^T\int_{W^+} 2B\psi v'_\nu \frac{\partial(\psi v_\nu)}{\partial\xi_1}\,d\xi_1 d\xi_2 dt\Big| \le CE(\mathbf{u},S)\,.$$

For the integral in W^-, we replace a_1 by $-a_1$, h^1 by $-h^1$, respectively and proceed as above. We can also get a similar result concerning the integral terms containing h^2.

Finally, we obtain

$$\Big|\int_S^T\int_{\Gamma_1} 2Bu'_\nu\nabla_T u_\nu.\mathbf{h}_T\,d\Gamma\,dt\Big| \le CE(\mathbf{u},S)\,. \tag{30}$$

Using $\varepsilon_S(\mathbf{u})$,

$$h_\nu\,\mathbf{u}'_T.\partial_\nu\mathbf{u}_T = h_\nu\,\mathbf{u}'_T.(2\varepsilon_S(\mathbf{u}) - (\nabla_T u_\nu) + \partial_T\nu\mathbf{u}_T)\,,\quad \text{on } \Gamma_1\,,$$

and, for (23) and (24), respectively,

$$\big|\int_{\Gamma_1} 4Bh_\nu\, \mathbf{u}'_T.\varepsilon_S(\mathbf{u})\, d\Gamma\big| \le \theta \int_{\Gamma_1} |\varepsilon_S(\mathbf{u})|^2\, d\Gamma + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}'_T|^2\, d\Gamma\,, \tag{31}$$

$$\big|\int_{\Gamma_1} 2Bh_\nu\, \mathbf{u}'_T.(\partial_T\nu)\mathbf{u}_T\, d\Gamma\big| \le C\int_{\Gamma_1} \left(|\mathbf{u}'_T|^2 + |\mathbf{u}_T|^2\right)\, d\Gamma\,. \tag{32}$$

Now compute

$$\begin{aligned} &\int_S^T \int_{\Gamma_1} h_\nu\, \mathbf{u}'_T.\nabla_T u_\nu\, d\Gamma\, dt \\ &= \Big[\int_{\Gamma_1} h_\nu\, \mathbf{u}_T.\nabla_T u_\nu\, d\Gamma\Big]_S^T - \int_S^T \int_{\Gamma_1} h_\nu\, \mathbf{u}_T.(\nabla_T u'_\nu)\, d\Gamma\, dt \\ &= \Big[\int_{\Gamma_1} h_\nu\, \mathbf{u}_T.\nabla_T u_\nu\, d\Gamma\Big]_S^T + \int_S^T \int_{\Gamma_1} u'_\nu \operatorname*{div}_T(k\mathbf{u}_T)\, d\Gamma\, dt\,. \end{aligned}$$

Since $\operatorname{div}_T(h_\nu \mathbf{u}_T) = h_\nu \operatorname{div}_T(\mathbf{u}_T) + \nabla_T h_\nu.\mathbf{u}_T$, Proposition 1 gives

$$\big|\int_{\Gamma_1} 2Bu'_\nu \operatorname*{div}_T(h_\nu\mathbf{u}_T)\, d\Gamma\big| \le \frac{\theta}{2}\|\mathbf{u}_T\|_1^2 + \frac{2C}{\theta}\int_{\Gamma_1} |\mathbf{u}'|^2\, d\Gamma\,. \tag{33}$$

Now consider $\int_{\Gamma_1} h_\nu\, \mathbf{u}_T.\nabla_T u_\nu\, d\Gamma$. Given $t > 0$, let ζ be in $\mathrm{H}^1(\Gamma_1)$ (notice that $\mathrm{H}^1(\Gamma_1) = \mathrm{H}^1_0(\Gamma_1)$) such that

$$\zeta - \Delta_T\zeta = \operatorname*{div}_T(\mathbf{u}_T)(t)\,.$$

Since $\operatorname{div}_T(\mathbf{u}_T)(t)$ belongs to $\mathrm{H}^{-1/2}(\Gamma_1)$, ζ satisfies

$$\begin{gathered} \|\zeta\|_{\mathrm{H}^1(\Gamma_1)} \le C\|\mathbf{u}_T(t)\|_{\mathrm{L}^2(\Gamma_1, T(\Gamma_1))}\,, \\ \zeta \in \mathrm{H}^{3/2}(\Gamma_1) \ \text{ and } \|\zeta\|_{\mathrm{H}^{3/2}(\Gamma_1)} \le C\|\mathbf{u}_T(t)\|_{\mathrm{H}^{1/2}(\Gamma_1, T(\Gamma_1))}\,. \end{gathered} \tag{34}$$

Then we have

$$\begin{aligned} &\int_{\Gamma_1} h_\nu\, \mathbf{u}_T.\nabla_T u_\nu\, d\Gamma \\ &= -\int_{\Gamma_1} u_\nu \operatorname*{div}_T(h_\nu\mathbf{u}_T)\, d\Gamma \\ &= -\int_{\Gamma_1} u_\nu \nabla_T h_\nu.\mathbf{u}_T\, d\Gamma - \int_{\Gamma_1} u_\nu h_\nu \zeta\, d\Gamma + \int_{\Gamma_1} u_\nu h_\nu \Delta_T\zeta\, d\Gamma\,. \end{aligned}$$

First, thanks to (34), we have

$$\begin{gathered} \big|\int_{\Gamma_1} u_\nu \nabla_T h_\nu.\mathbf{u}_T\, d\Gamma\big| \le C\int_{\Gamma_1} |\mathbf{u}|^2\, d\Gamma\,, \\ \big|\int_{\Gamma_1} u_\nu h_\nu \zeta\, d\Gamma\big| \le C\int_{\Gamma_1} (|u_\nu|^2 + |\zeta|^2)\, d\Gamma \le C\int_{\Gamma_1} |\mathbf{u}|^2\, d\Gamma\,. \end{gathered}$$

Second

$$-\int_{\Gamma_1} u_\nu h_\nu \Delta_T \zeta \, d\Gamma = \int_{\Gamma_1} (-\Delta_T)^{1/4}(u_\nu h_\nu)(-\Delta_T)^{3/4}\zeta \, d\Gamma \,,$$

and, again with (34),

$$\Big|\int_{\Gamma_1} u_\nu h_\nu \Delta\zeta \, d\Gamma\Big| \le C\|u_\nu\|_{\mathrm{H}^{1/2}(\Gamma_1)}\|\zeta\|_{\mathrm{H}^{3/2}(\Gamma_1)} \le C\|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)} \,.$$

Hence

$$\Big|\int_{\Gamma_1} h_\nu \, \mathbf{u}_T.\nabla_T u_\nu \, d\Gamma\Big| \le C\Big(\int_{\Gamma_1} |\mathbf{u}|^2 \, d\Gamma + \|\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)}\Big) \,.$$

Using Poincaré's inequality and Korn's inequality, we finally get

$$\Big|\int_{\Gamma_1} 2Bh_\nu \, \mathbf{u}_T.\nabla_T u_\nu \, d\Gamma\Big| \le CE(\mathbf{u}, t) \,. \tag{35}$$

Observing that the energy functional is non-increasing and using (33), (35), we obtain

$$\begin{aligned} \Big|\int_S^T \int_{\Gamma_1} 2Bh_\nu \, \mathbf{u}'_T.\nabla_T u_\nu \, d\Gamma \, dt\Big| \le & CE(\mathbf{u}, S) + \theta \int_S^T \|\mathbf{u}_T\|_1^2 \, dt \\ & + \frac{C}{\theta}\int_S^T \int_{\Gamma_1} |\mathbf{u}'|^2 \, d\Gamma \, dt \,. \end{aligned} \tag{36}$$

Again, we use boundedness of $\mathbf{h}$ and get

$$\Big|\int_{\Gamma_1} 2Bh_\nu \, u'_\nu \partial_\nu u_\nu \, d\Gamma\Big| \le \theta \int_{\Gamma_1} |\partial_\nu u_\nu|^2 \, d\Gamma + \frac{C}{\theta}\int_{\Gamma_1} |\mathbf{u}'|^2 \, d\Gamma \,. \tag{37}$$

Finally, with (28)–(32), (36) and (37), we obtain

$$\begin{aligned} |\mathcal{I}_2| \le & \theta \int_S^T \int_{\Gamma_1} \big(\varepsilon_T(\mathbf{u}_T) \colon \varepsilon_T(\mathbf{u}_T) + |\varepsilon_S(\mathbf{u})|^2 + |\partial_\nu u_\nu|^2\big) \, d\Gamma \, dt \\ & + \frac{C}{\theta}\int_S^T \int_{\Gamma_1} (|\mathbf{u}|^2 + |\mathbf{u}'|^2) \, d\Gamma \, dt + CE(\mathbf{u}, S) \,. \end{aligned} \tag{38}$$

Again, we emphasize that, in (38), θ is a positive number to be chosen later and C is a positive constant which does not depend on $\mathbf{u}$.

Final segment of the proof.

From (21), (27), (38), we obtain two positive constants C_6 and C_7 such that

$$\begin{aligned} C_1 \int_S^T E \, dt \le & C_6 E(\mathbf{u}, S) + \frac{C_7}{\theta}\int_S^T \int_{\Gamma_1} (|\mathbf{u}|^2 + |\mathbf{u}'|^2) \, d\Gamma \, dt \\ & + 2\theta \int_S^T \int_{\Gamma_1} \big(\varepsilon_T(\mathbf{u}_T) \colon \varepsilon_T(\mathbf{u}_T) + |\varepsilon_S(\mathbf{u})|^2 + |\partial_\nu u_\nu|^2\big) \, d\Gamma \, dt \\ & - k\alpha \int_S^T \int_{\Gamma_1} \big(\varepsilon_T(\mathbf{u}) \colon \varepsilon_T(\mathbf{u}) + 2|\varepsilon_S(\mathbf{u})|^2 + |\partial_\nu u_\nu|^2\big) \, d\Gamma \, dt \,. \end{aligned} \tag{39}$$

From the relations $\varepsilon_T(\mathbf{u}) = \varepsilon_T(\mathbf{u}_T) + \varepsilon_T(u_\nu \nu) = \varepsilon_T(\mathbf{u}_T) + u_\nu \partial_T \nu$ on Γ, we have

$$\begin{aligned} &\varepsilon_T(\mathbf{u})\colon \varepsilon_T(\mathbf{u}) \\ &\quad = \quad \varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + 2\varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(u_\nu \nu) + \varepsilon_T(u_\nu \nu)\colon \varepsilon_T(u_\nu \nu)\,, \text{ on } \Gamma_1\,. \end{aligned}$$

Using $|\varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(u_\nu \nu)| \le \theta \varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + (4\theta)^{-1}\varepsilon_T(u_\nu \nu)\colon \varepsilon_T(u_\nu \nu)$, we get

$$\varepsilon_T(\mathbf{u})\colon \varepsilon_T(\mathbf{u}) \ge (1-2\theta)\varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + \big(1 - \frac{1}{2\theta}\big)\varepsilon_T(u_\nu \nu)\colon \varepsilon_T(u_\nu \nu)\,, \quad \text{on } \Gamma_1\,.$$

Since $\int_{\Gamma_1} \varepsilon_T(u_\nu \nu)\colon \varepsilon_T(u_\nu \nu)\, d\Gamma = \int_{\Gamma_1} |u_\nu|^2 (\partial_T \nu : \partial_T \nu)\, d\Gamma \le C \int_{\Gamma_1} |u_\nu|^2\, d\Gamma$, we deduce from (39) that there exists $C_8 > 0$ such that

$$\begin{aligned} &C_1 \int_S^T E\, dt \\ &\quad \le \quad C_6 E(\mathbf{u}, S) + \frac{C_8}{\theta} \int_S^T \int_{\Gamma_1} (|\mathbf{u}|^2 + |\mathbf{u}'|^2)\, d\Gamma\, dt \\ &\qquad + 2\theta \int_S^T \int_{\Gamma_1} \big(\varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + |\varepsilon_S(\mathbf{u})|^2 + |\varepsilon_\nu(\mathbf{u})|^2\big)\ d\Gamma\, dt \\ &\qquad - k\alpha \int_S^T \int_{\Gamma_1} \big((1-2\theta)\varepsilon_T(\mathbf{u}_T)\colon \varepsilon_T(\mathbf{u}_T) + 2|\varepsilon_S(\mathbf{u})|^2 + |\varepsilon_\nu(\mathbf{u})|^2\big)\ d\Gamma\, dt\,. \end{aligned}$$

Then, for $\theta > 0$ small enough, we can find a positive constant C_9 such that

$$C_1 \int_S^T E\, dt \le C_6 E(\mathbf{u}, S) + C_9 \int_S^T \int_{\Gamma_1} (|\mathbf{u}|^2 + |\mathbf{u}'|^2)\, d\Gamma\, dt\,.$$

From Proposition 3 and Lemma 5, there exists $C_{10} > 0$ such that, for every $\eta > 0$,

$$C_1 \int_S^T E\, dt \le \frac{C_{10}}{\eta} E(\mathbf{u}, S) + \eta \int_S^T E\, dt\,.$$

Hence, for η small enough, we get the theorem by applying Lemma 1 with $\omega = (C_1 - \eta)\eta / C_{10}$. Now, we can observe that all above constants do not depend on the strong solution $\mathbf{u}$ of (3). Hence, by a denseness argument, this result can be extended to a weak solution of (3). □

3.1 Proof of Theorem 2

We now show that Theorem 1 can be applied with the vector field

$$\mathbf{h}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0) + \rho \widetilde{\mathbf{h}}(\mathbf{x})\,,$$

where ρ is some positive constant and $\widetilde{\mathbf{h}} \in \big(\mathcal{C}^1(\overline{\Omega})\big)^3$ is such that

$$\widetilde{\mathbf{h}} = 0\,, \quad \text{on } \Gamma_0\,, \quad \widetilde{\mathbf{h}} = \nu\,, \quad \text{on } \Gamma_1\,.$$

Note that $\mathbf{h}$ satisfies (5). Indeed,

$$\mathbf{h}(\mathbf{x}).\nu(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_0).\nu(\mathbf{x}) \leq 0\,, \quad \text{if } \mathbf{x}\in\Gamma_0\,,$$
$$\mathbf{h}(\mathbf{x}).\nu(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_0).\nu(\mathbf{x}) + \rho > 0\,, \quad \text{if } \mathbf{x}\in\Gamma_1\,.$$

Also note that $\mathbf{h}$ satisfies (6). We will consider two cases.
First case: $\text{meas}(\Gamma_0)\neq 0$. Here we choose $\beta_{\mathbf{h}}=0$. Then

$$\int_\Omega \sigma_{ij}(\xi)h_{k,j}\xi_{i,k}\,d\mathbf{x} = \int_\Omega \sigma_{ij}(\xi)\xi_{i,j}\;d\mathbf{x} + \rho\int_\Omega \sigma_{ij}(\xi)\widetilde{h}_{k,j}\xi_{i,k}\,d\mathbf{x}\,.$$

Since $\xi\in\mathbb{H}^1_{\Gamma_0}(\Omega)$ and $\widetilde{\mathbf{h}}\in\big(\mathcal{C}^1(\overline{\Omega})\big)^3$, using Korn's inequality, we can find a constant $C(\widetilde{\mathbf{h}})>0$ such that

$$\Big|\int_\Omega \sigma_{ij}(\xi)\widetilde{h}_{k,j}\xi_{i,k}\,d\mathbf{x}\Big| \leq C(\widetilde{\mathbf{h}})\int_\Omega \sigma(\xi):\varepsilon(\xi)\,d\mathbf{x}\,,$$

and

$$\int_\Omega \sigma_{ij}(\xi)h_{k,j}\xi_{i,k}\,d\mathbf{x} \geq \big(1-\rho C(\widetilde{\mathbf{h}})\big)\int_\Omega \sigma(\xi):\varepsilon(\xi)\,d\mathbf{x}\,.$$

We choose $\alpha_{\mathbf{h}} = 1-\rho C(\widetilde{\mathbf{h}})$ and get $\alpha_{\mathbf{h}}>0$ for ρ small enough.
Second case: $\text{meas}(\Gamma_0)=0$. Since $\text{meas}(\Gamma_1)\neq 0$, the map

$$\xi\mapsto\Big(\int_\Omega \sigma(\xi):\varepsilon(\xi)\,d\mathbf{x} + \int_{\Gamma_1}|\xi|^2\,d\Gamma\Big)^{1/2}$$

defines an equivalent norm on $\mathbb{H}^1(\Omega)$ [15]. We have

$$\int_\Omega \sigma_{ij}(\xi)h_{k,j}\xi_{i,k}\,d\mathbf{x} = \int_\Omega \sigma_{ij}(\xi)\xi_{i,j}\,d\mathbf{x} + \rho\int_\Omega \sigma_{ij}(\xi)\widetilde{h}_{k,j}\xi_{i,k}\,d\mathbf{x}\,.$$

Since $\xi\in\mathbb{H}^1(\Omega)$ and $\widetilde{\mathbf{h}}\in\big(\mathcal{C}^1(\overline{\Omega})\big)^3$, we can find a constant $C(\widetilde{\mathbf{h}})>0$ such that

$$\Big|\int_\Omega \sigma_{ij}(\xi)\widetilde{h}_{k,j}\xi_{i,k}\,d\mathbf{x}\Big| \leq C(\widetilde{\mathbf{h}})\Big(\int_\Omega \sigma(\xi):\varepsilon(\xi)\,d\mathbf{x} + \int_{\Gamma_1}|\xi|^2\;d\Gamma\Big)\,,$$

and

$$\int_\Omega \sigma_{ij}(\xi)h_{k,j}\xi_{i,k}\,d\mathbf{x} \geq \big(1-\rho C(\widetilde{\mathbf{h}})\big)\int_\Omega \sigma(\xi):\varepsilon(\xi)\,d\mathbf{x} - C(\widetilde{\mathbf{h}})\int_{\Gamma_1}|\xi|^2\,d\Gamma\,.$$

Then we choose $\alpha_{\mathbf{h}} = 1-\rho C(\widetilde{\mathbf{h}})$, which is positive for ρ small enough, and $\beta_{\mathbf{h}} = -C(\widetilde{\mathbf{h}})$.

Now one can easily show that the other conditions in (6) are satisfied if

$$\rho < \min\Big(\frac{1}{C(\widetilde{\mathbf{h}})}, \frac{2-\gamma_{\mathbf{x}-\mathbf{x}_0}}{1+2C(\widetilde{\mathbf{h}})+\max_{\overline{\Omega}}(\operatorname{div}(\widetilde{\mathbf{h}}))-\min_{\overline{\Omega}}(\operatorname{div}(\widetilde{\mathbf{h}}))}, \frac{3}{|\min_{\overline{\Omega}}(\operatorname{div}(\widetilde{\mathbf{h}}))|}\Big)\,.$$

This completes the proof. □

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