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Nonlocal quasilinear damped differential inclusions *

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Abstract

In this paper we investigate the existence of mild solutions to second order initial value problems for a class of damped differential inclusions with nonlocal conditions. By using suitable fixed point theorems, we study the case when the multivalued map has convex and nonconvex values.

1 Introduction

The study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force in a Hilbert space, can be modelled by the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \Big(\alpha + \beta \int_0^L \big|\frac{\partial u}{\partial t}(\xi, t)\big|^2 d\xi\Big) \frac{\partial^2 u}{\partial x^2} + g\left(\frac{\partial u}{\partial t}\right) = 0, \tag{1.1}$$

where $\alpha, \beta, L > 0$, $u(t,x)$ is the deflection of the point x of the beam at the time t, g is a nondecreasing numerical function, and L is the length of the beam.

Equation (1.1) has its analogue in $\mathbb{R}^n$ and can be included in a general mathematical model

$$u'' + A^2 u + M(\|A^{1/2}u\|_H^2)Au + g(u') = 0, \tag{1.2}$$

where A is a linear operator in a Hilbert space H and M, g are real functions. Equation (1.1) was studied by Patcheu [19] and (1.2) was studied by Matos and Pereira [15]. These equations are special cases of the following second order damped nonlinear differential equation in an abstract space

$$\begin{gathered} u'' + Au + Bu' = f(t,u), \\ u(0) = u_0, \quad u'(0) = u_1, \end{gathered}$$

where A and B are linear operators.

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In this paper, we study the existence of mild solutions, defined on a compact real interval J, for second order Initial Value Problems (IVP), for damped differential inclusions, with nonlocal conditions, of the form

$$y'' - Ay \in By' + F(t, y), \quad t \in J := [0, b], \tag{1.3}$$

$$y(0) + f(y) = y_0, \quad y'(0) = \eta, \tag{1.4}$$

where $F : J \times E \to \mathcal{P}(E)$ is a multivalued map, $f \in C(C(J, E), E)$, A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a Banach space $E = (E, \|\cdot\|)$, B is a bounded linear operator on E and $y_0, \eta \in E$.

The study of IVP with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. Some authors have paid attention to the research of IVP with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [1], Byszewski [3], [4], Ntouyas [18], and Ntouyas and Tsamatos [16], [17]. IVP ,for second order semilinear equations with nonlocal conditions, was studied by Ntouyas and Tsamatos [17], and Ntouyas [18].

Here, we study existence results on compact intervals, when the multivalued F has convex or nonconvex values. In the first case, a fixed point theorem due to Martelli [14] is used and, in the later, a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [6] is applied.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\},$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [22]).

$L^1(J, E)$ denotes the linear space of equivalence classes of measurable functions $y : J \to E$ such that $\int_0^b \|y(s)\| \, ds < \infty$.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to \mathcal{P}(E)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_0)$, there exists an open neighbourhood U of x_0 such that $G(U) \subseteq B$. G is said to be completely semicontinuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely semicontinuous with nonempty compact values, then G is u.s.c. if and only if

G has a closed graph (i.e. $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

$$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}, \quad P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\},$$
$$P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}, \quad P_c(X) = \{Y \in P(X) : Y \text{ convex}\}.$$

A multivalued map $G : J \to P_{cl}(X)$ is said to be *measurable* if for each $x \in X$ the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\},$$

is measurable. Other equivalent definitions of the measurability for multivalued maps can be found in [12]. For more details on multivalued maps and for the proofs of the known results cited in this section we refer the interesting reader to the books of Deimling [7] and Hu and Papageorgiou [12].

An upper semicontinuous map $G : X \to \mathcal{P}(E)$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to the book of Banas and Goebel [2]. We remark that a completely semicontinuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps we refer to the books of Deimling [7], Gorniewicz [10] and Hu and Papageorgiou [12].

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

(i) $C(0) = I$ (I is the identity operator in E)

(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$

(iii) the map $t \mapsto C(t)y$ is strongly continuous for each $y \in E$

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

It is known [21] that if A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$, of bounded linear operators, then there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|}, t \in \mathbb{R} \quad \text{and} \quad \|S(t_1) - S(t_2)\| \leq M \left| \int_{t_2}^{t_1} e^{\omega|s|}\ ds \right|, t_1, t_2 \in \mathbb{R}.$$

For a strongly continuous cosine family if $X = \{x \in E : C(t)x$ is once continuously differentiable on $\mathbb{R}\}$, then $S(t)E \subset X$ for $t \in \mathbb{R}$, $S(t)X \subset D(A)$ for $t \in \mathbb{R}$, $(d/dt)C(t)x = AS(t)x$ for $x \in X$ and $t \in \mathbb{R}$, and $(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax$ for $x \in D(A)$ and $t \in \mathbb{R}$.

For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [9], Heikkila and Lakshmikantham [11], Fattorini [8], and to the papers of Travis and Webb [20], [21].

3 Existence result: The convex case

Assume in this section that $F : J \times E \to \mathcal{P}(E)$ is a bounded, closed and convex valued multivalued map.

Definition 3.1 A function $y \in C(J, E)$ is called a mild solution of (1.3)–(1.4) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J, $y(0) + f(y) = y_0, y'(0) = \eta$ and

$$\begin{aligned} y(t) &= (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)v(s)ds. \end{aligned}$$

We will need the following assumptions:

(H1) A is the infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$, and $M = \sup\{\|C(t)\|; t \in J\}$;

(H2) $F : J \times E \to BCC(E); (t, y) \mapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$, and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \Big\{g \in L^1(J, E) : \ g(t) \in F(t, y(t)) \text{ for a.e. } t \in J\Big\}$$

is nonempty;

(H3) there exists a constant Q, with $QM(1 + b\|B\|) < 1$, such that

$$\|f(y)\| \le Q\|y\| \quad \text{for each } y \in C(J, E);$$

(H4) $\|F(t, y)\| := \sup\{\|v\| \in F(t, y)\} \le p(t)\psi(\|y\|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{ds}{s + \psi(s)},$$

where

$$c = \frac{1}{1 - QM(1 + b\|B\|)}[M(1 + b\|B\|)\|y_0\| + bM\|\eta\|]$$

and $\widehat{m}(t) = \max\{M\|B\|, bMp(t)\}$;

(H5) for each bounded set $B \subset C(J,E)$ and $t \in J$ the set

$$\Big\{(C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds : g \in S_{F,B}\Big\}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Remark 3.1 (i) If $\dim E < \infty$ then, for each $y \in C(J,E)$, $S_{F,y} \neq \emptyset$ (see Lasota and Opial [13]).

(ii) If $\dim E = \infty$ then $S_{F,y}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) := \inf\{\|v\| : v \in F(t,y)\},$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [12]).
Also, if $\dim E = \infty$, in order to get meausurable selections for the multifunction $t \mapsto F(t, y(t))$, we can suppose that F is measurable with respect to $\mathcal{L} \otimes \mathcal{B}$, where $\mathcal{L}$ and $\mathcal{B}$ are the Lebesque and Borel σ-fields on J and E respectively.

(iii) Assumption (H4) is satisfied for example if F satisfies the standard domination

$$\|F(t,y)\| \le p(t)(1 + \|y\|), \quad p \in L^1,\ t \in J,\ y \in E.$$

(iv) If we assume that $C(t),\ t \in J$ is completely continuous then (H5) is satisfied.

The following lemmas are crucial in the proof of our main theorem.

Lemma 3.1 ([13]) *Let I be a compact real interval and X be a Banach space. Moreover, let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC(C(I,X)),\ y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 3.2 ([14]) *Let X be a Banach space and $N : X \to BCC(X)$ be an upper semicontinuous and condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

The following theorem is our main result in this article.

Theorem 3.1 *Let f be a continuous and convex function. Assume that (H1)-(H5) hold. Then the IVP (1.3)-(1.4) has at least one mild solution.*

Proof. We transform (1.3)-(1.4) into a fixed point problem. Consider the multivalued map $N : C(J,E) \to \mathcal{P}(C(J,E))$, defined by

$$\begin{aligned} N(y) &:= \Big\{h \in C(J,E) : h(t) = (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds \ \ g \in S_{F,y}\Big\}, \end{aligned}$$

where $S_{F,y} = \big\{g \in L^1(J,E) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in J\big\}$.

Remark 3.2 It is clear that the fixed points of N are mild solutions to IVP (1.3)-(1.4).

We shall show that N is completely semicontinuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in C(J,E)$.
Indeed, if $h_1,\ h_2$ belong to Ny then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g_i(s)ds, \quad i = 1,2. \end{aligned}$$

Let $0 \le \alpha \le 1$. Then for each $t \in J$ we have

$$\begin{aligned} &(\alpha h_1 + (1-\alpha)h_2)(t) \\ &= (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta + \int_0^t C(t-s)By(s)ds \\ &\quad + \int_0^t S(t-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values) it follows that $\alpha h_1 + (1-\alpha)h_2 \in N(y)$.

Step 2: N is bounded on bounded sets of $C(J,E)$.
Indeed, it is enough to show that for each $r > 0$ there exists a positive constant ℓ such that for each $h \in Ny, y \in B_r := \{y \in C(J,E) : \|y\|_\infty \le r\}$, one has $\|h\|_\infty \le \ell$. If $h \in N(y)$ then there exists $g \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned} h(t) &= (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds. \end{aligned}$$

By (H3) and (H4), we have, for each $t \in J$, that

$$\begin{aligned}
\|h(t)\| &\leq (\|C(t)\| + \|S(t)\|\|B\|)(\|y_0\| + Q\|y(t)\|) + \|S(t)\|\|\eta\| \\
&\quad + \int_0^t \|C(t-s)\|\|B\|\|y(s)\|ds + \int_0^t \|S(t-s)g(s)\|ds \\
&\leq (M + bM\|B\|)(\|y_0\| + Qr) + bM\|\eta\| \\
&\quad + M\|B\|br + bM \cdot \sup_{y\in[0,r]} \psi(y)\Big(\int_0^t p(s)ds\Big).
\end{aligned}$$

Then for each $h \in N(B_r)$ we have

$$\begin{aligned}
\|h\|_\infty &\leq (M + \|bM\|B\|)(\|y_0\| + Qr) + bM\|\eta\| \\
&\quad + M\|B\|br + bM \sup_{t\in J}\Big(\int_0^t p(s)ds\Big) \max_{y\in B_r} \sup_{y\in[0,r]} \psi(y) := \ell.
\end{aligned}$$

Step 3: N sends bounded sets of $C(J,E)$ into equicontinuous sets.
Let $t_1, t_2 \in J, t_1 < t_2$ and B_r be as before. For each $y \in B_r$ and $h \in Ny$, there exists $g \in S_{F,y}$ such that

$$\begin{aligned}
h(t) &= (C(t) - S(t)B)(y_0 - f(y)) + S(t)\eta \\
&\quad + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|h(t_2) - h(t_1)\| \\
&\leq \|C(t_2) - C(t_1)\|\|y_0 - f(y)\| + \|S(t_2)B - S(t_1)B\|\|y_0 - f(y)\| \\
&\quad + \|S(t_2) - S(t_1)\|\|\eta\| + \Big\|\int_0^{t_1} [C(t_2-s)By(s) - C(t_1-s)By(s)]ds\Big\| \\
&\quad + \Big\|\int_{t_1}^{t_2} C(t_2-s)By(s)ds\Big\| + \Big\|\int_0^{t_1} [S(t_2-s) - S(t_1-s)]g(s)ds\Big\| \\
&\quad + \Big\|\int_{t_1}^{t_2} S(t_2-s)g(s)ds\Big\| \\
&\leq \|C(t_2) - C(t_1)\|\|y_0 - f(y)\| + \|S(t_2)B - S(t_1)B\|\|y_0 - f(y)\| \\
&\quad + \|S(t_2) - S(t_1)\|\|\eta\| + \Big\|\int_0^{t_1} [C(t_2-s)By(s) - C(t_1-s)By(s)]ds\Big\| \\
&\quad + \Big\|\int_{t_1}^{t_2} C(t_2-s)By(s)ds\Big\| \\
&\quad + \sup_{t\in J} p(t) \sup_{y\in[0,r]} \psi(y)\Big\|\int_0^{t_1} [S(t_2-s) - S(t_1-s)]g(s)ds\Big\| \\
&\quad + Mb(t_2 - t_1) \sup_{y\in[0,r]} \psi(y)\Big(\int_0^t p(s)ds\Big).
\end{aligned}$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$. As a consequence of Step 2, Step 3 and (H5), together with the Ascoli-Arzelá theorem, we conclude that N is completely continuous and, therefore, a condensing map.

Step 4: N has a closed graph.
Let $y_n \to y^*$, $h_n \in Ny_n$, and $h_n \to h^*$. We shall prove that $h^* \in Ny^*$. The formula $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= (C(t) - S(t)B)(y_0 - f(y_n)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By_n(s)ds + \int_0^t S(t-s)g_n(s)ds. \end{aligned}$$

We have to prove that there exists $g^* \in S_{F,y^*}$ such that

$$\begin{aligned} h^*(t) &= (C(t) - S(t)B)(y_0 - f(y^*)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By^*(s)ds + \int_0^t S(t-s)g^*(s)ds. \end{aligned}$$

Consider the linear bounded operator $\Gamma : L^1(J,E) \to C(J,E)$, defined by

$$(\Gamma g)(t) := \int_0^t S(t-s)g(s)ds.$$

Clearly we have that

$$\begin{aligned} &\Big\| \Big(h_n - [C(t) - S(t)B](y_0 - f(y_n) - S(t)\eta) - \int_0^t C(t-s)By_n(s) \Big) \\ &\quad \Big(h^* - [C(t) - S(t)B](y_0 - f(y^*) - S(t)\eta) - \int_0^t C(t-s)By^*(s) \Big) \Big\| \to 0, \end{aligned}$$

as $n \to \infty$. From Lemma 3.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Since $y_n \to y^*$, it follows, from Lemma 3.1, that

$$\begin{aligned} &h^* - [C(t) - S(t)B](y_0 - f(y^*) - S(t)\eta) - \int_0^t C(t-s)By^*(s) \\ &\qquad = \int_0^t S(t-s)g^*(s)ds \end{aligned}$$

for some $g^* \in S_{F,y^*}$.

Step 5: The set $\Omega := \{y \in C(J,E) : \lambda y \in N(y),\ \text{for some } \lambda > 1\}$ is bounded. Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus, there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}(C(t) - S(t)B)(y_0 - f(y)) + \lambda^{-1}S(t)\eta \\ &\quad + \lambda^{-1}\int_0^t C(t-s)By(s)ds + \lambda^{-1}\int_0^t S(t-s)g(s)ds,\ t \in J. \end{aligned}$$

The above formula implies (by (H3) and (H4)) that, for each $t \in J$, we have

$$\begin{aligned}\|y(t)\| &\le (M + bM\|B\|)(\|y_0\| + Q\|y(t)\|) + bM\|\eta\| \\ &\quad + M\|B\| \int_0^t \|y(s)\|ds + Mb \int_0^t p(s)\psi(\|y(s)\|)ds,\end{aligned}$$

or

$$\begin{aligned}[1 - QM(1 + b\|B\|)]\|y(t)\| &\le (M + bM\|B\|)\|y_0\| + bM\|\eta\| \\ &\quad + M\|B\| \int_0^t \|y(s)\|ds + Mb \int_0^t p(s)\psi(\|y(s)\|)ds,\end{aligned}$$

and

$$\begin{aligned}\|y(t)\| &\le \frac{1}{1 - QM(1 + b\|B\|)}\Big\{(M + bM\|B\|)\|y_0\| + bM\|\eta\| \\ &\quad + M\|B\| \int_0^t \|y(s)\|ds + Mb \int_0^t p(s)\psi(\|y(s)\|)ds\Big\}, \quad t \in J.\end{aligned}$$

Let us denote the right-hand side of the above inequality as $v(t)$. Then we have

$$v(0) = \frac{1}{1 - QM(1 + b\|B\|)}[M(1 + b\|B\|)\|y_0\| + bM\|\eta\|],$$
$$\|y(t)\| \le v(t), \quad t \in J,$$

and $v'(t) = M\|B\|\|y(t)\| + bMp(t)\psi(\|y(t)\|)$, $t \in J$. Using the increasing character of ψ we get

$$v'(t) \le M\|B\|v(t) + bMp(t)\psi(v(t)) \le \widehat{m}(t)[v(t) + \psi(v(t))], \quad t \in J.$$

The above inequality implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \psi(s)} \le \int_0^b \widehat{m}(s)ds < \int_{v(0)}^{\infty} \frac{ds}{s + \psi(s)}.$$

Consequently, there exists a constant d such that $v(t) \le d$, $t \in J$, and hence $\|y\|_\infty \le d$, where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 3.2 we deduce that N has a fixed point which is a mild solution of (1.3)-(1.4). ♢

4 Existence Result: The nonconvex case

In this section we consider problem (1.3)-(1.4) with a nonconvex valued right hand side.

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 4.1 A multivalued operator $N : X \to P_{cl}(X)$ is called:

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \quad \text{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Moreover, N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $\operatorname{Fix} N$.

Our considerations are based on the following fixed point theorem for contraction multivalued operators, given by Covitz and Nadler in 1970 [6] (see also Deimling [7, Thm. 11.1]).

Lemma 4.1 *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Theorem 4.1 *Assume that:*

(A1) A is an infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$ with $\|C(t)\|_{B(E)} \le M$;

(A2) $F : J \times E \to P_{cl}(E)$ has the property that $F(\cdot, u) : J \to P_{cl}(E)$ is measurable for each $u \in E$;

(A3) there exists $l \in L^1(J, \mathbb{R})$ such that

$$H_d(F(t,u), F(t,\overline{u})) \le l(t)\|u - \overline{u}\|, \quad \textit{for ea } \ t \in J \quad \textit{and } u, \overline{u} \in E,$$

and

$$d(0, F(t,0)) \le l(t), \quad \textit{for almost each} \quad t \in J.$$

(A4) $\|f(y) - f(\overline{y})\| \le c\|y - \overline{y}\|$, for each $t \in J$ and $y, \overline{y} \in C(J,E)$, where c is a nonnegative constant.

Then IVP (1.3)-(1.4) has at least one mild solution on J, provided

$$cM(1 + b\|B\|) + M\|B\|b + \frac{M}{\tau} < 1.$$

Proof. Transform (1.3)-(1.4) into a fixed point problem. Consider the multivalued operator $N : C(J,E) \to \mathcal{P}(C(J,E))$, defined by

$$\begin{aligned} N(y) \quad := \quad & \Big\{ h \in C(J,E) : h(t) = [C(t) - S(t)B](y_0 - f(y)) + S(t)\eta \\ & + \int_0^t C(t-s)By(s)\,ds + \int_0^t S(t-s)v(s)\,ds, \Big\} \end{aligned}$$

where $v \in S_{F,y} = \Big\{ v \in L^1(J,E) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J \Big\}$.

Remark 4.1 (i) It is clear that the fixed points of N are solutions to (1.3)-(1.4).
(ii) For each $y \in C(J,E)$, the set $S_{F,y}$ is nonempty since, by (A2), F has a measurable selection [5, Theorem III.6].

We shall show that N satisfies the assumptions of Lemma 4.1. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(C(J,E)$ for each $y \in C(J,E)$.
Indeed, let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \to \tilde{y}$ in $C(J,E)$. Then $\tilde{y} \in C(J,E)$ and

$$\begin{aligned} y_n(t) &\in [C(t)-S(t)B](y_0 - f(y)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By(s)\,ds + \int_0^t S(t-s)F(s,y(s))\,ds, \ t \in J. \end{aligned}$$

Using the closedness property of the values of F and the second part of (A3) we can prove that $\int_0^t C(t-s)By(s)\,ds + \int_0^t S(t-s)F(s,y(s))\,ds$ is closed, for each $t \in J$. Then $y_n(t) \to \tilde{y}(t)$ in

$$[C(t)-S(t)B](y_0-f(y))+S(t)\eta+\int_0^t C(t-s)By(s)\,ds+\int_0^t S(t-s)F(s,y(s))\,ds,$$

$t \in J$. So, $\tilde{y} \in N(y)$.

Step 2: $H_d(N(y_1),N(y_2)) \leq \gamma\|y_1 - y_2\|$ for each $y_1, y_2 \in C(J,E)$ (where $\gamma < 1$).
Let $y_1, y_2 \in C(J,E)$ and $h_1 \in N(y_1)$. Then, there exists $g_1(t) \in F(t, y_1(t))$ such that

$$\begin{aligned} h_1(t) &= [C(t)-S(t)B](y_0 - f(y_1)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By_1(s)\,ds + \int_0^t S(t-s)g_1(s)\,ds, \quad t \in J. \end{aligned}$$

From (A3), it follows that

$$H_d(F(t,y_1(t)),F(t,y_2(t))) \leq l(t)\|y_1 - y_2\|.$$

Hence, there is $w \in F(t,y_2(t))$ such that $\|g_1(t) - w\| \leq l(t)\|y_1 - y_2\|$, $t \in J$. Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \leq l(t)\|y_1 - y_2\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,y_2(t))$ is measurable [5, Prop. III.4]), there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t,y_2(t))$ and

$$\|g_1(t) - g_2(t)\| \leq l(t)\|y_1 - y_2\| \ \text{ for each } t \in J.$$

Let us define, for each $t \in J$,

$$\begin{aligned} h_2(t) &= [C(t) - S(t)B](y_0 - f(y_2)) + S(t)\eta \\ &\quad + \int_0^t C(t-s)By_2(s)\,ds + \int_0^t S(t-s)g_2(s)\,ds, \quad t \in J. \end{aligned}$$

Then, we have

$$\begin{aligned} &\|h_1(t) - h_2(t)\| \\ &\leq (M + bM\|B\|)\|f(y_1) - f(y_2)\| + M\|B\| \int_0^t \|y_1(s) - y_2(s))\|ds \\ &\quad + M \int_0^t \|g_1(s) - g_2(s)\|\,ds \\ &\leq cM(1 + b\|B\|)\|y_1(t) - y_2(t)\| + M\|B\| \int_0^t \|y_1(s) - y_2(s)\|ds \\ &\quad + M \int_0^t l(s)\|y_1(s) - y_2(s)\|ds \\ &= cM(1 + b\|B\|)\|y_1(t) - y_2(t)\| + M\|B\| \int_0^t \|y_1(s) - y_2(s)\|ds \\ &\quad + M \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|\,ds \\ &\leq cM(1 + b\|B\|)e^{\tau L(t)}\|y_1 - y_2\|_{\mathcal{B}} + M\|B\|be^{\tau L(t)}\|y_1 - y_2\|_{\mathcal{B}} \\ &\quad + M\|y_1 - y_2\|_{\mathcal{B}} \int_0^t l(s)e^{\tau L(s)}ds \\ &\leq cM(1 + b\|B\|)e^{\tau L(t)}\|y_1 - y_2\|_{\mathcal{B}} + M\|B\|be^{\tau L(t)}\|y_1 - y_2\|_{\mathcal{B}} \\ &\quad + M\frac{\|y_1 - y_2\|_{\mathcal{B}}}{\tau}e^{\tau L(t)}, \end{aligned}$$

where $L(t) = \int_0^t l(s)ds$, τ is a positive constant, and $\|\cdot\|_{\mathcal{B}}$ is the Bielecki norm on $C(J, E)$, defined by

$$\|y\|_{\mathcal{B}} = \max_{t \in J}\{\|y(t)\|e^{-\tau L(t)}\}.$$

Then

$$\|h_1 - h_2\|_{\mathcal{B}} \leq \big[cM(1 + b\|B\|) + M\|B\|b + \frac{M}{\tau}\big]\|y_1 - y_2\|_{\mathcal{B}}.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \leq \big[cM(1 + b\|B\|) + M\|B\|b + \frac{M}{\tau}\big]\|y_1 - y_2\|_{\mathcal{B}}.$$

Since $cM(1+b\|B\|)+M\|B\|b+\frac{M}{\tau} < 1$, N is a contraction and thus, by Lemma 4.1, it has a fixed point y, which is a mild solution to (1.3)-(1.4). $\diamondsuit$

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The authors would like to thank Prof. P. Ch. Tsamatos for point out the invalidity of the growth condition imposed on f in conditon (H3). Consecuently, (H3) must be replaced by

(H3) There exists a constant Q such that

$$\|f(y)\| \leq Q \quad \text{for each } y \in C(J,E)$$

In (H4) the constant c must be replaced by

$$c = M(1 + b\|B\|)(\|y_0\| + Q) + bM\|\eta\|.$$

In Step 2 of the proof of Theorem 3.1, $... + Q\|y(t)\|)$ must be replaced by $.. + Q)$ and two lines below $... + Qr)$ must be replaced by $... + Q)$

In Step 5 of the proof Theorem 3.1 from "The above formula...." to 10 lines below "$\|y(t)\| \leq v(t), \quad t \in J,$" must be replaced by:
The above formula implies (by (H3) and (H4)) that, for each $t \in J$, we have

$$\begin{aligned} \|y(t)\| \quad &\leq \quad (M + bM\|B\|)(\|y_0\| + Q) + bM\|\eta\| \\ &\quad +M\|B\| \int_0^t \|y(s)\|ds + Mb \int_0^t p(s)\psi(\|y(s)\|)ds. \end{aligned}$$

Let us denote the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{gathered} v(0) = M(1 + b\|B\|)(\|y_0\| + Q) + bM\|\eta\|, \\ \|y(t)\| \leq v(t), \quad t \in J, \end{gathered}$$

End of addendum.