Electronic Journal of Differential Equations, Vol. 2002(2002), No. 17, pp. 1–31.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ON SOME ANALYTICAL INDEX FORMULAS RELATED TO OPERATOR-VALUED SYMBOLS

GRIGORI ROZENBLUM

Abstract. For several classes of pseudodifferential operators with operator-valued symbol analytic index formulas are found. The common feature is that usual index formulas are not valid for these operators. Applications are given to pseudodifferential operators on singular manifolds.

1. Introduction

Analytical index formulas play an important part in the study of topological characteristics of elliptic operators. They complement index formulas expressed in topological and algebraical terms, and often enter in these formulas as an ingredient. For elliptic pseudodifferential operators on compact manifolds, such formulas were found by Fedosov [6]; for topologically simple manifolds, i.e. having trivial Todd class, they reduce to the co-homological Atiyah-Singer formula. Later, analytical index formulas for elliptic boundary value problems were obtained in [7]. These formulas have a common feature: they involve an integral, with integrand containing analytical expressions for the classical characteristic classes entering into the co-homological formulas.

In 90-s a systematic study started of topological characteristics of operators on singular manifolds - [20, 21, 22, 27, 28, 16, 17, 31, 9, 37] etc. Even before these papers had appeared, it became clear that analysis of operators on singular manifolds must involve many-level symbolic structure, where the leading symbol of the operator is the same as in the regular case, but here a hierarchy of operator-valued symbols arises, responsible for the singularities (see [19, 23, 32, 5], and later [24, 33, 34, 35, 29, 30, 4], etc.). Each of these symbols contributes to the index formulas. In some, topologically simple, cases, such contributions can be separated, and thus the problem arises of calculation of the index for pseudodifferential operators with operator-valued symbol. However, even one-dimensional examples show that the usual formulas, originating from the scalar or matrix situation, may be unsuitable in the operator-valued case. Consider the simplest situation. Let A be the Toeplitz operator on the real line $\mathbb{R}$, with symbol $a(x)$, i.e. it acts in the Hardy

2000 *Mathematics Subject Classification.* 35S05, 47G30, 58J20, 58J40.
Key words and phrases. Pseudodifferential operators, index, cyclic cohomology, singular manifolds.

Submitted October 8, 2001. Published February 18, 2002.

space $H^2(\mathbb{R})$ by the formula

$$Au = Pau$$

where $P : L_2 \to H^2$ is the Riesz projection. Under the condition that the symbol $a(x)$ is smooth, invertible and stabilises to 1 at infinity, the operator A is Fredholm and its index equals

$$\operatorname{ind} A = -(2\pi i)^{-1} \int a(x)^{-1} a'(x) dx.$$

If we consider a Toeplitz operator in the space of *vector*-functions, so that the symbol is a matrix, the same formula for the index holds, with a natural modification:

$$\operatorname{ind} A = -(2\pi i)^{-1} \int \operatorname{tr}(a(x)^{-1} a'(x)) dx, \tag{1.1}$$

where tr is the usual matrix trace. However, when we move to an even more general case, the one of Toeplitz operators acting in the space of functions on $\mathbb{R}^1$ with values in an infinite-dimensional Hilbert space, so that $a(x)$ is an operator in this space, the formula (1.1) makes sense only under the condition that $a^{-1}a'$ belongs to the trace class. If this is not the case, (1.1) makes no sense, so even if the Toeplitz operator happens to be Fredholm, one needs another formula for the index to be found (and justified).

A similar situation was considered by Connes [3, Sect. III. 2α]. It was noticed that (1.1) (in the matrix case) requires certain smoothness of the symbol $a(x)$ with respect to x variable, e.g. $a \in H^{1/2}_{\text{loc}}$. On the other hand, for the Toeplitz operator to be Fredholm it is sufficient that the symbol is continuous and has equal limit values at $\pm\infty$. The cyclic cohomology technics was used to find a series of formulas assuming less and less smoothness of the symbol.

Earlier, the same approach was used by B.Plamenevsky and the author in [22] for the above problem of finding analytical index formulas for Toeplitz operators with operator-valued symbols. This became an important step in the study of topological characteristics of pseudodifferential operators with isolated singularities in symbols. The cyclic cohomology methods were used there as well.

In the present paper we consider a class of pseudodifferential operators with operator symbols and find analytical index formulas for elliptic operators in this class. The expressions have different form, depending on the quality of the symbol, and are derived by means of cyclic co-homology approach. Some operators arising in analysis on singular manifolds fit into the abstract scheme. Applying our general approach, we find index formulas for Toeplitz operators with operator-valued symbols and, extending results of [8, 28], of cone Mellin operators. In the last section we consider edge pseudodifferential operators arising in analysis on manifolds with edge-type singularities. Our abstract approach to the index formulas requires less structure from the operator symbols compared with the traditional one (see, e.g., [32, 33, 34, 31, 9]), therefore we present here a new version of the edge calculus.

The problem of regularising formulas of the type (1.1) was attacked from different points of view also in [20, 10, 16, 17, 9, 37]. In all these papers, some specific information on the nature of the symbol a was essentially used - actually, the fact that it is a parameter dependent pseudodifferential operator on a compact manifold. Our approach is more abstract, it does not use any special form of the symbol but rather describes its properties in the terms of Shatten classes. The pseudodifferential calculus for such operator-valued symbols was first proposed by the author in [26] and then used in [28]. Here we present this calculus more systematically.

The results presented in the paper were obtained as a part of the Swedish Royal Academy of Sciences project. The author is grateful to the Departments of Mathematics of the University of Nantes and the University of Potsdam (the group of Partial Differential Equations and Complex Analysis), where in 1999 a part of the results were obtained and a useful discussion took place, and, especially, to Professors D.Robert, B.-W. Schulze, and B.Fedosov.

2. The algebraic scheme

In this section we describe the abstract setting enabling one to derive new index formulas from the existing ones. We recall some constructions from the K-theory for operator algebras and cyclic cohomologies; proofs and details can be found in [1, 14, 2, 3].

Let $\mathfrak{S}$ be a Banach *-algebra with unit, $M(\mathfrak{S})$ be the set of matrices over $\mathfrak{S}$. The groups $\mathbf{K}_j(\mathfrak{S}), j = 0, 1$ are the usual K-groups in the theory of Banach *-algebras. Thus $\mathbf{K}_0(\mathfrak{S})$ is the group of equivalence classes of (formal differences of) projections with entries in $\mathfrak{S}$. The group $\mathbf{K}_1(\mathfrak{S})$ consists of equivalence classes of invertible matrices in $M(\mathfrak{S})$, i.e., elements in $GL(\mathfrak{S})$. If $\mathfrak{S}$ does not have a unit, one attaches it and thus replaces $M(\mathfrak{S})$ by $M(\mathfrak{S})^+$ in the latter definition. (We use boldface $\mathbf{K}$ in order to distinguish operator algebras K-groups from topological ones.) The notion of these $\mathbf{K}$-groups carries over to local *-subalgebras of $\mathfrak{S}$, i.e. subalgebras closed with respect to the holomorph functional calculus in $\mathfrak{S}$. Important here is the fact that the $\mathbf{K}$-groups of a dense local subalgebra in $\mathfrak{S}$ are isomorphic to the ones of $\mathfrak{S}$ (see, e.g., [1]).

The $\mathbf{K}$-cohomological group $\mathbf{K}^1(\mathfrak{S})$ consists of equivalence classes of 'quantisations', i.e. unital homomorphisms of the algebra of matrices over $\mathfrak{S}$ to the Calkin algebra in some Hilbert space $\mathfrak{H}$, or, what is equivalent, *-linear mappings $\tau : M(\mathfrak{S}) \to B(\mathfrak{H})$, multiplicative up to a compact error. Each element $[\tau] \in \mathbf{K}^1(\mathfrak{S})$ defines the index homomorphism $\operatorname{ind}_{[\tau]} : \mathbf{K}_1(\mathfrak{S}) \to \mathbb{Z}$, associating to the matrix $a \in GL(\mathfrak{S})$ the index of the operator $\tau(a)$. Thus we have the integer index coupling between $\mathbf{K}^1(\mathfrak{S})$ and $\mathbf{K}_1(\mathfrak{S})$: $[\tau] \times [a] = \operatorname{ind} \tau(a)$. Again, if $\mathfrak{S}$ is non-unital, the unit is attached.

For a normed *-algebra $\mathfrak{S}$, the group $C^k_\lambda(\mathfrak{S})$ of cyclic cochains consists of $(k+1)$-linear continuous functionals $\varphi(a_0, a_1, \dots, a_k)$, cyclic in the sense $\varphi(a_0, a_1, \dots, a_k) = (-1)^k \varphi(a_1, a_2, \dots, a_0)$. The Hochschild co-boundary operator $b : C^k_\lambda(\mathfrak{S}) \to C^{k+1}_\lambda(\mathfrak{S})$ generates, in a usual way, co-homology groups $HC^k_\lambda(\mathfrak{S})$.

There is also a coupling of $HC^{2k+1}_\lambda(\mathfrak{S})$ and $K_1(\mathfrak{S})$ (see [3, Ch.III.3]):

$$[\varphi] \times_k [a] = \gamma_k(\varphi \otimes \operatorname{tr})(a^{-1} - 1, a - 1, a^{-1} - 1, a - 1, \dots, a^{-1} - 1, a - 1), \quad (2.1)$$

where tr is the matrix trace and γ_k is the normalisation constant, chosen in [3] to be equal to $(2i)^{-1/2} 2^{-2k-1} \Gamma(k + \frac{3}{2})$, for functoriality reasons.

An important role in the paper is played by the suspension homomorphism $S : HC^k_\lambda(\mathfrak{S}) \to HC^{k+2}_\lambda(\mathfrak{S})$. This operation is *not*, in general, an isomorphism. In fact, it is a monomorphism, with range isomorphic to the kernel of the homomorphism I associating to every cyclic cocycle representing a class in HC^{k+2}_λ, the class of the same cocycle in the Hochschild cohomology group $H^{k+2}(\mathfrak{S})$ (see [3, III.1.γ]. The suspension homomorphism is consistent with the coupling (2.1):

$$[\varphi] \times_k [a] = S[\varphi] \times_{k+1} [a],\ [a] \in \mathbf{K}_1(\mathfrak{S}), [\varphi] \in H_\lambda^{2k+1}(\mathfrak{S}). \tag{2.2}$$

In this context, the problem of finding an analytic index formula for a given 'quantisation' $[\tau] \in \mathbf{K}^1(\mathfrak{S})$ consists in determining a proper element $[\varphi] = [\varphi^{[\tau]}]$ in the cohomology group of some order, $[\varphi] \in HC_\lambda^{2k+1}(\mathfrak{S})$ such that

$$[\tau] \times [a] = [\varphi] \times_k [a],\ [a] \in \mathbf{K}_1(\mathfrak{S}), \tag{2.3}$$

or even a cyclic cocycle $\varphi \in C_\lambda^{2k+1}(\mathfrak{S})$ such that (2.3) holds.

In [2, 3] such problem, for different situations was handled by constructing a Chern character Ch, the homomorphism from $\mathbf{K}$- co-homologies to cyclic co-homologies, so that $[\varphi] = Ch([\tau])$. However, it is not always possible to use this construction directly. The reason for this is that for $*$-algebras arising in concrete analytical problems, the cyclic co-homology groups are often not rich enough to carry the index classes one needs. For example, in a simple case, $\mathfrak{S} = C_0(\mathbb{R}^1)$ and τ being the Toeplitz quantisation, associating the Toeplitz operator with symbol $a(x)$ to the continuous (matrix-)function $a(x)$ stabilizing at infinity, the index is well defined on the $\mathbf{K}$-theoretical level, but there are no analytical index formulas, since all odd cyclic cohomology groups are trivial (see [12, 2]). This means that one has to chose some 'natural' dense local subalgebra $\mathfrak{S}_0 \subset \mathfrak{S}$, equipped with a norm, stronger than the initial norm in $\mathfrak{S}$, having rich enough cyclic co-homologies. On the level of $\mathbf{K}$-groups this substitution is not felt, since the natural inclusion $\iota : \mathfrak{S}_0 \to \mathfrak{S}$ generates isomorphism $\iota^* : \mathbf{K}^1(\mathfrak{S}) \to \mathbf{K}^1(\mathfrak{S}_0)$, but in co-homologies this may produce analytical index formulas. Moreover, the choice of the dimension $2k+1$ of the target cyclic cohomology group may depend on the properties of the subalgebra $\mathfrak{S}_0$. An example of this can be found in [3, II.2.α, III.6.β]. There, for the dense local subalgebra $\mathfrak{S}_1 = C_0^1(\mathbb{R}^1)$ in the C^*-algebra $\mathfrak{S} = C_0(\mathbb{R}^1)$, one associates to the Toeplitz quantisation $[\tau]$ the class $[\varphi_1^{[\tau]}] \in HC_\lambda^1(\mathfrak{S}_1)$ generated by the cocycle

$$\varphi_1^{[\tau]}(a_0, a_1) = -(2\pi i)^{-1} \int \operatorname{tr}(a_0 da_1). \tag{2.4}$$

In fact, coupling with this class gives the standard formula (1.1) for the index of the Toeplitz operator. However, the cocycle (2.4) is not defined on larger subalgebras in $\mathfrak{S}$, for example on $\mathfrak{S}_\gamma = C_0^\gamma(\mathbb{R}^1),\ 0 < \gamma < 1$, consisting of functions satisfying Lipshitz condition with exponent γ, and this prevents one from using (1.1) for calculating the index. To deal with this situation, it is proposed in [3] to consider the image of $[\varphi_1^{[\tau]}]$ in $HC_\lambda^{2l+1}(\mathfrak{S}_1)$ under l times iterated suspension homomorphism S, with properly chosen l. This produces cocycles $\varphi_{2l+1}^{[\tau]}$ on $\mathfrak{S}_1$, functionals in $2l+2$ variables, which give new analytical index formulas for the Toeplitz operators with differentiable symbols - see the formula on p. 209 in [3]. However, these cocycles, for $2l+1 > \gamma$ admit continuous extension to the algebra $\mathfrak{S}_\gamma$, thus defining elements in $HC_\lambda^{2l+1}(\mathfrak{S}_\gamma)$ and giving index formulas for less and less smooth functions.

One must note here, that, although the suspension homomorphism S in cyclic co-homology groups is defined uniquely, in a canonical way, it can be realised in different ways on cocycle level. Several types of methods representing suspension of cocycles were proposed in [2, 3]. It may happen that these methods applied to one and the same cocycle produce cocycles in the smaller algebra of which not all can be extended to the larger algebra. This, in particular, happens in [3] and [22] where different methods applied to the same initial one-dimensional cocycle produce quite different suspended cocycles, with different properties.

Now let us describe this situation in a more abstract setting.

Let $\mathfrak{S}$ be a Banach *-algebra and

$$\mathfrak{S}_1 \supset \mathfrak{S}_2 \supset \dots \supset \mathfrak{S}_m \supset \dots \tag{2.5}$$

be a sequence of local *-subalgebras, being Banach spaces with respect to the norms $\|\cdot\|_m$ such that embeddings in (2.5) are dense. Let $[\tau]$ be an element in $\mathbf{K}^1(\mathfrak{S})$ and $[\tau]_m$ the corresponding element in $\mathbf{K}^1(\mathfrak{S}_m)$ obtained by restriction of τ. Suppose that for some m and k, the index class for $[\tau]_m$ is found in $HC^{2k+1}_\lambda(\mathfrak{S}_m)$, i.e some cyclic cocycle $\varphi \in C^{2k+1}(\mathfrak{S}_m)$ such that

$$[\tau]_m \times [a] = [\varphi] \times_k [a],\ a \in \mathrm{GL}(\mathfrak{S}_m). \tag{2.6}$$

Consider the sequence of suspended classes:

$$[\varphi]_l = S^l[\varphi] \in HC^{2l+2k+1}_\lambda(\mathfrak{S}_m). \tag{2.7}$$

Now assume that for some l, in the cohomology class $[\varphi]_l$, one can choose an element $\varphi_l \in C^{2l+2k+1}_\lambda(\mathfrak{S}_m)$ which, as a multi-linear functional, admits continuous extension $\overline{\varphi_l}$ onto $\mathfrak{S}_1$, thus defining a class $[\overline{\varphi_l}] \in HC^{2l+2k+1}_\lambda(\mathfrak{S}_1)$.

Proposition 2.1. *In the above situation, for $a \in GL(\mathfrak{S}_1)$,*

$$[\tau]_1 \times [a] = [\overline{\varphi_l}] \times_{k+l} [a]. \tag{2.8}$$

Proof. Due to the properties of the suspension homomorphism (2.2) the equality (2.8) holds for $a \in \mathrm{GL}(\mathfrak{S}_m)$. Now, it remains to remember that $\mathfrak{S}_m$ is dense in $\mathfrak{S}_1$ and both parts in (2.8) are continuous in the norm of $\mathfrak{S}_1$.•

3. Operator-valued symbols

In this paper we deal only with operators acting on functions defined on the Euclidean space. For this situation, we describe here algebras of operator-valued symbols and develop the corresponding calculus of pseudodifferential operators.

In the literature, starting, probably, from [15], there exist several versions of operator-valued pseudodifferential calculi, each adopted to some particular, more or less general, situation (see, e.g., [32, 29, 4]). Each time, one has to establish some abstract setting, modelling the most obvious (and sought for) application - the operator-valued symbol being a pseudodifferential operator of a proper class in 'transversal' variables. For particular cases, this 'proper class' may consist of usual pseudodifferential operators, Wiener-Hopf operators, Mellin operators, with, probably, attachment of trace and co-trace ones, operators on singular manifolds, etc. Each time, in the calculus, the problem arises, of finding a convenient description for the property of improvement of the symbol under the differentiation in co-variables.

Let us, in the simplest case, in $L_2(\mathbb{R}^n) = L_2(\mathbb{R}^m \times \mathbb{R}^k)$, consider the pseudodifferential operator $a(x, D_x)$ with a symbol $a(x,\xi) = a(y, z, \eta, \zeta)$, zero order homogeneous and smooth in ξ, $\xi \neq 0$, which we treat as an operator in $L_2(\mathbb{R}^m, L_2(\mathbb{R}^k))$ with operator valued symbol $\mathbf{a}(y, \eta) = a(y, z, \eta, D_z)$. Then the differentiation in $\eta, \eta \neq 0$, produces the operator symbol $\partial_\eta \mathbf{a}$ of order -1, next η-differentiation gives the symbol $\partial^2_\eta \mathbf{a}$ of order -2, etc. We refer to this effect by saying that the quality of the operator symbol is improved under η-differentiation. (Strictly speaking, this improvement, actually, may take place not under each differentiation, in the case when the order of the transversal operator is already low from the very beginning, like, say, for $a(x,\xi) = \psi(x)|\eta|^l|\xi|^{-l}$.) Usually, in concrete situations, this property is described by introducing proper scales of 'smooth' spaces, like, as in the leading example, weighted Sobolev spaces in $\mathbb{R}^k$, and describing the spaces where the differentiated operator symbol acts. Such approach is used, in particular, in [32, 33, 34, 31, 9, 4] etc. This, however, requires a rather detailed analysis of action of 'transversal operators' $a(y,z,\eta,D_z)$ in these scales and becomes fairly troublesome in singular cases. At the same time, these extra spaces are in no way reflected in index formulas and are superfluous in this context. Thus, it seems to be useful to introduce a calculus of pseudodifferential operators not using extra spaces but at the same time possessing the above improvement property. Our approach is based on describing the property of improvement of operator valued symbol under differentiation not by improvement of smoothness but by improvement of compactness. So, in the above example, suppose that the symbol a has compact support in z variable. Then, if the differential order γ of the operator is negative, the operator symbol $\mathbf{a}(y,\eta)$ is a compact operator, and its singular numbers $s_j(\mathbf{a}(y,\eta))$ decay as $O(j^{\gamma/k})$. Each differentiation in η variable, lowering the differential order, leads to improvement of the decay rate of these s-numbers; after N differentiations, the s-numbers of the differentiated symbol decay as $O(j^{(\gamma-N)/k})$. At the same time, the decay rate as $|\eta| \to \infty$ of the operator norm of the differentiated symbol also improves under the differentiation. This justifies the introduction of classes of symbols in the abstract situation.

So, let $\mathfrak{K}$ be a Hilbert space. By $\mathfrak{s}_p = \mathfrak{s}_p(\mathfrak{K})$, $0 < p < \infty$ we denote the Shatten class of operators T in $\mathfrak{K}$ for which the sequence of singular numbers (s-numbers) $s_j(T) = (\lambda_j(T^*T))^{1/2}$ belongs to l_p. The most important are the trace class $\mathfrak{s}_1$ and the Hilbert-Schmidt class $\mathfrak{s}_2$. The l_p-norm $|T|_p$ of this sequence defines for $p \ge 1$ a norm in $\mathfrak{s}_p$, otherwise, it is a quasi-norm. The norm property enables one to integrate families of $\mathfrak{s}_p$-operators for $p \ge 1$: if $T(y,\eta)$ is a family of operators in $\mathfrak{s}_p$, and $|T(y,\eta)|_p \le f(y)g(\eta)$ then $|\int T(y,\eta)dy|_p \le (\int f(y)^p dy)^{1/p} g(\eta)$.

In the definition below, as well as in the formulations, N is some sufficiently large integer. We do not specify the particular choice of N in each case, as long as it is of no importance.

Definition 3.1. Let $\gamma \le 0, q > 0$. The class $\mathcal{S}_q^\gamma = \mathcal{S}_q^\gamma(\mathbb{R}^m \times \mathbb{R}^{m'}, \mathfrak{K})$ consists of functions $\mathbf{a}(y,\eta)$, $(y,\eta) \in \mathbb{R}^m \times \mathbb{R}^{m'}$, such that for any (y,η), $\mathbf{a}(y,\eta)$ is a bounded operator in $\mathfrak{K}$ and, moreover,

$$\|D_\eta^\alpha D_y^\beta \mathbf{a}(y,\eta)\| \le C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|+\gamma}, \tag{3.1}$$

$$|D_\eta^\alpha D_y^\beta \mathbf{a}(y,\eta)|_{\frac{q}{-\gamma+|\alpha|}} \le C_{\alpha,\beta}. \tag{3.2}$$

for $|\alpha|, |\beta| \le N$.

Note here that for the case when M is a k-dimensional compact manifold and $a(y,z,\eta,\zeta)$ is a classical pseudodifferential symbol of order less than γ on $\mathbb{R}^m \times M$, the operator valued symbol $\mathbf{a}(y,\eta) = a(y,z,\eta,D_z)$ acting in $\mathfrak{K} = L_2(M)$ belongs to $\mathcal{S}_k^\gamma$ for any N. A more involved example arises in the study of operators with discontinuous symbols.

Suppose that the symbol $a(y,z,\eta,\zeta)$ has compact support in z, order $\gamma \le 0$ positively homogeneous in (η,ζ) (with a certain smoothening near the point $(\eta,\zeta) = 0$), but near the subspace $z = 0$ it is positively homogeneous of order γ in z variable, thus having a singularity at the subspace $z = 0$. The operator symbol $\mathbf{a}(y, \eta)$ is a bounded operator in $\mathfrak{K} = L_2(\mathbb{R}^k)$, differentiation in η lowers the homogeneity order in (η, ζ), but the singularity in z prevents it from acting into usual Sobolev spaces (it is here the need for weighted Sobolev spaces arises). However, in the terms of the Definition 3.1, the properties of the operator symbol are easily described: it belongs to $\mathcal{S}_q^\gamma$ for any $q > k$. This example will be the basic one in considerations in Sect. 7.

The interpolation inequality $|\mathbf{a}|_q^q \le |\mathbf{a}|_p^p \|\mathbf{a}\|^{q-p}$ for $p < q$ implies that for $-\gamma + |\alpha| - q > 0$ the derivatives in (3.1), (3.2) belong to trace class and for $-\gamma + |\alpha| - q > m$ the integral of its trace class norm with respect to η converges. The same holds for any $\mathfrak{s}_p$ - norm, provided $|\alpha|$ is big enough. On the other hand, since

$$|\mathbf{ab}|_{(p^{-1}+q^{-1})^{-1}} \le |\mathbf{a}|_p |\mathbf{b}|_q, \tag{3.3}$$

the product of symbols $\mathbf{a} \in \mathcal{S}_q^\gamma$ and $\mathbf{b} \in \mathcal{S}_q^\delta$ belongs to $\mathcal{S}_q^{\gamma+\delta}$.

For a symbol in $\mathbf{a} \in \mathcal{S}_q^\gamma$ and a function $f(\lambda)$ analytical in a sufficiently large domain in the complex plain, the symbol $f(\mathbf{a})$ can be defined by means of the usual analytical functional calculus for bounded operators. One can check directly that for any such f, the symbol $f(\mathbf{a})$ belongs to $\mathcal{S}_q^0$; if, additionally, $f(0) = 0$, then $f(\mathbf{a}) \in \mathcal{S}_q^\gamma$, moreover, if $f(0) = f'(0) = \cdots = f^{(\nu)}(0) = 0$ then $f(\mathbf{a}) \in \mathcal{S}_q^{(\nu+1)\gamma}$. Thus, $\mathcal{S}_q^\gamma$ becomes a local $*$-subalgebra in the algebra of bounded continuous operator-valued functions on $\mathbb{R}^m \times \mathbb{R}^{m'}$.

We are going to sketch the operator-valued version of the usual pseudodifferential calculus. The main difference of this calculus from the usual one is the notion of 'negligible' operators. In the scalar case, one considers as negligible the infinitely smoothing operators. In our case, we take trace class operators as negligible, and it is up to a trace class error, that the classical relations of the pseudodifferential calculus will be shown to hold. This is sufficient for the needs of index theory.

Having a symbol $\mathbf{a}(y, y', \eta) \in \mathcal{S}_q^\gamma(\mathbb{R}^{2m} \times \mathbb{R}^m, \mathfrak{K})$, we define the pseudodifferential operator with this symbol as

$$(OPS(\mathbf{a})u)(y) = (\mathbf{a}(y, y', D_y)u)(y) = (2\pi)^{-m} \int\int e^{i(y-y')\eta} \mathbf{a}(y, y', \eta) u(y') d\eta dy', \tag{3.4}$$

where $u(y)$ is a function on $\mathbb{R}^m$ with values in $\mathfrak{K}$. In particular, if $\mathbf{a}$ does not depend on y', this is the usual formula involving the Fourier transform:

$$\mathbf{a}(y, D_y)u = OPS(\mathbf{a}) = \mathcal{F}^{-1}\mathbf{a}(y, \eta)\mathcal{F}u, \tag{3.5}$$

Without any changes, on the base of (3.1), the standard reasoning applied in the scalar case to give precise meaning to (3.4), (3.5) defines the action of the operator $\mathbf{a}(y, D_y)$ on rapidly decaying smooth functions u and establishes its boundedness in L_2. We are going to show now is that the property (3.2) produces trace class estimates.

The following proposition gives a sufficient condition for a pseudodifferential operator to belong to trace class.

Proposition 3.2. *Let the operator-valued symbol* $\mathbf{a}(y, y', \eta)$ *in* $\mathbb{R}^{2m} \times \mathbb{R}^m$ *be smooth with respect to* y, y'*, let all* y, y'*-derivatives* $D_y^\beta D_{y'}^{\beta'} \mathbf{a}$ *up to some (sufficiently large) order* N *be trace class operators with trace class norm bounded uniformly in* y, y'*. Suppose that* $g(y), h(y) = O((1+|y|)^{-2m})$*. Then the operator* $h\mathbf{a}(y, y', D_y)g$ *belongs to* $\mathfrak{s}_1(L_2(\mathbb{R}^m; \mathfrak{K}))$*.*

Proof. Suppose first that the functions h, g have compact support in some unit cubes Q, Q'. Take smooth functions f, f' compactly supported in concentric cubes with twice as large side such that $hf = h, gf' = g$. We can represent our operator $hf\mathbf{a}(y, y', D_y)f'g = h\mathbf{a}(y, y', D_y)g$ in the form

$$hf\mathbf{a}(y, y', D_y)f'g = (2\pi)^{-2m} \int\int e^{iy\zeta+iy'\zeta'} h\mathbf{a}_{\zeta,\zeta'}(D_y)gd\zeta d\zeta' g, \tag{3.6}$$

where $\mathbf{a}_{\zeta,\zeta'}(\eta) = \int\int e^{-i(y\zeta+y'\zeta')}f(y)\mathbf{a}(y, y', \eta)f'(y')dydy'$. The conditions imposed on the symbol $\mathbf{a}$ guarantee that the symbol $\mathbf{a}_{\zeta,\zeta'}(\eta)$ is a trace class operator for all η, ζ, ζ', its trace class norm is in L_1 with respect to η variable and decays rapidly at infinity in ζ, ζ'. We will use this to prove that for all ζ, ζ' the operator $h\mathbf{a}_{\zeta,\zeta'}(D_y)g$ belongs to the trace class and its trace class norm decreases sufficiently fast as ζ, ζ' tend to ∞. In order to do this, we factorize this operator into the product of two Hilbert-Schmidt operators with rapidly decreasing Hilbert-Schmidt norm. Recall that for a pseudodifferential operator with operator-valued symbol $\mathbf{k}(y, \eta)$, one has $|\mathbf{k}(y, D_y)|_2^2 = (2\pi)^{-m} \int\int |\mathbf{k}(y, \eta)|_2^2 dyd\eta$, and, similarly for an operator with symbol $\mathbf{k}(y', \eta)$. Represent the symbol $\mathbf{a}_{\zeta,\zeta'}(\eta)$ as the product $\mathbf{b}_{\zeta,\zeta'}(\eta)\mathbf{c}_{\zeta,\zeta'}(\eta)$ where $\mathbf{b}_{\zeta,\zeta'}(\eta) = |\mathbf{a}_{\zeta,\zeta'}(\eta)|^{1/2}$. The symbol $h(y)\mathbf{b}_{\zeta,\zeta'}(\eta)$ belongs to the Hilbert-Schmidt class $\mathfrak{s}_2(\mathfrak{K})$ at any point (y, η), the Hilbert-Schmidt norm belongs to L_2 in (y, η) variables and decays fast as as (ζ, ζ') tend to infinity. Therefore, the operator $h(y)\mathbf{b}_{\zeta,\zeta'}(D_y)$ belongs to the Hilbert-Schmidt class, with norm fast decaying in (ζ, ζ'). The same reasoning takes care of $\mathbf{c}_{\zeta,\zeta'}g$. Thus the trace class norm of the integrand on the right-hand side in (3.6) decays fast in (ζ, ζ'), and this, after integration in ζ, ζ', establishes the required property of $h\mathbf{a}(y, y', D_y)g$. Note here, that the trace norm of the operator $h\mathbf{a}(y, y', D_y)g$ is estimated by the L_2-norms of the functions h, g over the cubes Q, Q'. To dispose of the condition of h, g to have compact support, we take a covering of the space by a lattice of unit cubes Q_j and define h_j, g_j as restrictions of h, g to the corresponding cube. Then the reasoning above can be applied to each of the operators $h_j\mathbf{a}(y, y', D_y)g_{j'}$, and the series of trace class norms of these operators converges. □

Remark 3.3. Note that we do not impose on the operator-valued symbol any smoothness conditions in η variable. This proves to be useful later, especially, in Sect.7. A somewhat unusual presence of both functions g, h (instead of just one of them, as one might expect comparing with the scalar theory) is explained by the fact that without smoothness conditions with respect to η, our pseudodifferential operators are not necessarily pseudo-local in any reasonable sense.

Remark 3.4. *A special case where Proposition 3.2 can be used for establishing trace class properties is the one of the symbol* $\mathbf{a}$ *decaying sufficiently fast in* y, y'*, together with derivatives, without factors* g, h*. In fact, consider* $\mathbf{a} = (1 + |y|^2)^{-N}\mathbf{b}(1 + |y'|^2)^{-N}$*, with* N *large enough, and apply Proposition 3.2 to the symbol* $\mathbf{b}$*.*

If symbols belong to the classes $\mathcal{S}_q^\gamma$, the usual properties and formulas in the pseudodifferential calculus hold, with our modification of the notion of negligible operators.

Theorem 3.5 (Pseudo-locality)**.** *Let the symbol* $\mathbf{a}(y, y', \eta)$ *belong to* $\mathcal{S}_q^\gamma(\mathbb{R}^{2n} \times \mathbb{R}^n)$ *for some* $q > 0, \gamma \leq 0$*, let* h, g *be bounded functions with disjoint supports, at least one of them being compactly supported. Then (for* N *large enough) the operator*

$h\mathbf{a}(y,y',D)g$ belongs to $\mathfrak{s}_1(L_2(\mathbb{R}^m;\mathfrak{K}))$, moreover,

$$|h\mathbf{a}(y,y',D)g|_{\mathfrak{s}_1}\le C||g||_\infty||h||_\infty(1+d^{-N})\max\{C_{\alpha,\beta};|\alpha|,|\beta|\le N\},$$

where $C_{\alpha,\beta}$ are constants in (3.1), (3.2) and $d=\operatorname{dist}(\operatorname{supp}(g),\operatorname{supp}(h))$.

Proof. First, let h have compact support. Take two more bounded functions $h',g'\in C^\infty$ with disjoint supports such that $\operatorname{supp}h'$ is compact, $hh'=h,\ gg'=g$. Again represent the operator in question in the form

$$h\mathbf{a}(y,y',D_y)g=(2\pi)^{-m}\int e^{iy\zeta}h(y)\mathbf{a}_\zeta(y',D)g(y')d\zeta, \tag{3.7}$$

where $\mathbf{a}_\zeta(y',\eta)=\int e^{iy\zeta}h'(y)\mathbf{a}(y,y',\eta)g'(y')dy$. We will show that the integrand in (3.7) belongs to the trace class and its trace norm is integrable with respect to ζ. We have

$$(\mathbf{a}_\zeta(y',D)u)(y)=(2\pi)^{-m}\int\int e^{i\eta(y-y')}h'(y)\mathbf{a}_\zeta(y',\eta)g'(y')u(y')dy'd\eta. \tag{3.8}$$

The first order partial differential operator $L=L(D_\eta)=-i|y-y'|^{-2}(y-y')D_\eta$ has the property $Le^{i\eta(y-y')}=e^{i\eta(y-y')}$, so we can insert L^N into (3.8) for any N. After integration by parts (first formal, but then justified in the usual way), we obtain that (3.7) equals

$$(2\pi)^{-m}\int\int e^{i\eta(y-y')}h'(y)|y-y'|^{-2N}((y-y')D_\eta)^N\mathbf{a}_\zeta(y',\eta)g'(y')u(y')dy'd\eta.$$

Since the supports of h',g' are disjoint, the function

$$h'(y)|y-y'|^{-2N}((y-y')D_\eta)^N\mathbf{a}_\zeta(y',\eta)g'(y')$$

is smooth with respect to y,y'. By choosing N large enough, we can, using (3.1), (3.2), arrange it to belong to trace class and have trace class norm decaying fast in y',η,ζ, together with as many derivatives as we wish. Now, according to Proposition 3.2 (see Remark 3.4), this implies that the trace class norm of the operator (3.8) decays fast in ζ, and the result follows, together with the estimate.

The same reasoning works if not h but g has a compact support, one just makes the representation similar to (3.7), making Fourier transform in y' variable. □

The usual formula expressing the symbol of the composition of operators via the symbols of the factors also holds in the operator-valued situation.

Theorem 3.6. *Let the symbols $\mathbf{a}(y,\eta),\mathbf{b}(y,\eta)$ belong to $\mathcal{S}^\gamma_q(\mathbb{R}^{2m}\times\mathbb{R}^m)$ for some $q>0,\gamma\le 0$ and $h(y)=O((1+|y|)^{-m-1})$. Then, for N large enough, the operator $hOPS(\mathbf{a})OPS(\mathbf{b})\ -hOPS(\mathbf{c}_N)$ belongs to trace class, where, as usual,*

$$\mathbf{c}_N=\mathbf{a}\circ_N\mathbf{b}=\sum_{|\alpha|<N}(\alpha!)^{-1}\partial^\alpha_\eta\mathbf{a}D^\alpha_y\mathbf{b}. \tag{3.9}$$

Proof. We follow the standard way of proving the composition formula, however the remainder term will be estimated by means of Proposition 3.2.

Suppose first that h has a compact support in a unit cube Q. Take a function $g\in C_0^\infty$ which is equal to 1 in the concentric cube with side 2 and vanishes outside the concentric cube with side 3. Set $\mathbf{b}=g\mathbf{b}+(1-g)\mathbf{b}=\mathbf{b}'+\mathbf{b}''$. For the symbol $\mathbf{b}''$, we have $h\mathbf{a}\circ_N\mathbf{b}''=0$, at the same time, $hOPS(\mathbf{a})OPS(\mathbf{b}'')$ is trace class due to the pseudo-locality property. Thus, $hOPS(\mathbf{a})OPS(\mathbf{b}'')-hOPS(\mathbf{a}\circ_N\mathbf{b}'')$ belongs to $\mathfrak{s}_1$, with trace class norm controlled by the L_∞ norm of h in Q. Next, since $h\mathbf{a}=hg\mathbf{a}$, we can assume that $\mathbf{a}$ has a compact support in y.

We represent the operator $\mathbf{b}'(y,D)$ as the integral similar to (3.6):

$$\mathbf{b}'(y',D) = \int e^{izy'}\mathbf{b}'_\zeta(D)dz,$$

where

$$\mathbf{b}'_\zeta(\eta) = (2\pi)^{-m}\int e^{-iy'\zeta}\mathbf{b}'(y',\eta)dy'. \tag{3.10}$$

Then the difference $hOPS(\mathbf{a})OPS(\mathbf{b}') - hOPS(\mathbf{c}_N)$ can be written as

$$\int h(\mathbf{a}(y,D)e^{izy'}\mathbf{b}'_\zeta(D) - \sum_{|\alpha|<N}(\alpha!)^{-1}\mathbf{a}_\alpha(y,D)\mathbf{b}^\alpha_{\zeta'}(D))d\zeta, \tag{3.11}$$

where $\mathbf{b}'^\alpha$ is defined by the formula similar to (3.10), with $D^\alpha_{y'}\mathbf{b}'$ instead of $\mathbf{b}'$ and $\mathbf{a}_\alpha = \partial^\alpha_\eta\mathbf{a}$. We will show that for N large enough, the integrand in (3.11) is a trace class operator, with trace norm decaying sufficiently fast as $\zeta\to\infty$. To do this, we write the action of the operator $\mathbf{a}(y,D)e^{izy'}\mathbf{b}'_\zeta(D_{y'})$ on some function u by means of the Fourier transform, as in (3.4):

$$(\mathbf{a}(y,D)e^{i\zeta y'}\mathbf{b}'_\zeta(D_{y'})u)(y) = \int \mathbf{a}(y,\eta)e^{i\eta(y-y')}e^{i\zeta y'}\mathbf{b}'_\zeta(\eta)\hat{u}(\eta)d\eta. \tag{3.12}$$

Now, as it is usually done in the scalar case, we write a finite section of the Taylor expansion of $\mathbf{a}(y,\eta)$ at the point $(y,\eta-\zeta)$ in powers of ζ, with a remainder term. Taking into account that $\zeta^\alpha\mathbf{b}_\zeta(\eta) = \mathbf{b}^\alpha_\zeta$, only the remainder term survives in (3.11) and one can express the integrand in (3.11) as a pseudodifferential operator with symbol

$$e^{i\zeta y}\sum_{|\alpha|=N}\int_0^1 \zeta^\alpha\mathbf{a}_\alpha(y,(\eta-\zeta)+t\zeta){\mathbf{b}'_\zeta}^\alpha(\eta)dt. \tag{3.13}$$

Now, if N is large enough, $\mathbf{a}_\alpha(y,(\eta-\zeta)+t\zeta)$ is trace class, with trace class norm fast decaying in η variable; at the same time, smoothness of the symbol $\mathbf{b}'$ guarantees fast norm decay of ${\mathbf{b}'}^\alpha_\zeta$ in ζ variable (these estimates just repeat the scalar ones in [13]). This, according to Proposition 3.2, leads to a fast trace norm decay in ζ variable of the integrand in (3.13), and therefore (3.11) is a trace class operator, again with trace norm controlled by L_∞-norm of h. To deal with general h, represent it as a sum of functions h_j supported in disjoint cubes, apply the above reasoning to each h_j and sum the resulting estimates. □

A version of Theorem 3.6 will be used, where the function h is not present, but instead of this, as $y\to\infty$, the symbol $\mathbf{a}$ tends, sufficiently fast, to a symbol $\mathbf{a}_0\in\mathcal{S}^\gamma_q$ not depending on η: there exists a (smooth) function $h(y) = O((1+|y|)^{-m-1})$ such that $h(y)^{-1}(\mathbf{a}(y,\eta)-\mathbf{a}_0(y))\in\mathcal{S}^\gamma_q$. In the course of the paper, it is in this sense we will mean that the symbol *stabilises at infinity*.

Corollary 3.7. *Suppose that the conditions of the Theorem 3.6 are fulfilled, and, additionally,* $\mathbf{a}$ *stabilises at infinity. Then*

$$OPS(\mathbf{a})OPS(\mathbf{b}) - OPS(\mathbf{c}_N)\in\mathfrak{s}_1(L_2(\mathbb{R}^m;\mathfrak{K})).$$

Proof. The symbol $\mathbf{a}'(y,\eta) = h(y)^{-1}(\mathbf{a}(y,\eta) - \mathbf{a}_0(y))$ satisfies the conditions of Theorem 3.6, and thus the difference $h(y)(OPS(\mathbf{a}')OPS(\mathbf{b}) - OPS(\mathbf{c}'_N))$ belongs to the trace class where $\mathbf{c}'_N$ is constructed similarly to $\mathbf{c}_N$, with $\mathbf{a}$ replaced by $\mathbf{a}'$. As for the remaining term, generated by $\mathbf{a}_0$, it makes no contribution into the remainder term in the composition formula. □

We note here, although we do not use it in the present paper, that in the same way, the usual formulas for the change of variables in a pseudodifferential operator and for the adjoint operator are carried over to the operator-valued case in our trace class setting, again, with essential use of the Proposition 3.2. Taking into account the pseudo-locality property, this enables one to introduce pseudodifferential operators with operator-valued symbols on compact manifolds.

Now we introduce the notion of ellipticity for our operators.

Definition 3.8. The symbol $\mathbf{a}(y,\eta) \in \mathcal{S}^0_q(\mathbb{R}^m \times \mathbb{R}^m)$ stabilizing in y at infinity is called elliptic if for $|y| + |\eta|$ large, $\mathbf{a}(y,\eta)$ is invertible and $\|\mathbf{a}^{-1}\| \le C$.

For small $|\eta| + |y|$, the symbol $\mathbf{a}(y,\eta)^{-1}$ is not necessarily defined. As usual, one often needs a regularising symbol defined everywhere and coinciding with $\mathbf{a}^{-1}$ for large η. This can also be done in our calculus, however the cut-off and gluing operations, used freely in the standard situation, are not so harmless now: even the multiplication by a nice function of η variable may throw us out of the class $\mathcal{S}^0_q$. Therefore we have to be rather delicate when operating with cut-offs.

Proposition 3.9. *Suppose that the symbol* $\mathbf{a} \in \mathcal{S}^0_q(\mathbb{R}^m \times \mathbb{R}^m)$ *is elliptic. Then there exists a symbol* $\mathbf{r}_0(y,\eta) \in \mathcal{S}^0_q(\mathbb{R}^m \times \mathbb{R}^m)$*, such that* $\mathbf{r}_0(y,\eta) = \mathbf{a}(y,\eta)^{-1}$ *for large* $|y|^2 + |\eta|^2$ *and the symbols* $\mathbf{r}_0\mathbf{a} - 1$ *and* $\mathbf{a}\mathbf{r}_0 - 1$ *belong to* $\mathcal{S}^{-1}_q$.

Proof. Suppose that $\mathbf{a}$ is invertible for $|y|^2+|\eta|^2 \ge R^2$. The inequalities of the form (3.1), (3.2) hold for $\mathbf{a}(y,\eta)^{-1}$ for such η. Thus we have to take care of small $|y|^2+|\eta|^2$ only. Fix some η_0, $|\eta_0| \ge R$. Due to (3.2), the symbol $\mathbf{s}(y,\eta) = 1-\mathbf{a}(y,\eta_0)^{-1}\mathbf{a}(y,\eta)$ belongs to $\mathfrak{s}_q$ for $|\eta| \le R$, with $\mathfrak{s}_q$–norms bounded uniformly. Set

$$\mathbf{r}'_0(y,\eta) = \mathbf{a}(y,\eta_0)^{-1}\exp(\mathbf{s}(y,\eta) + \mathbf{s}(y,\eta)^2/2 + \cdots + \mathbf{s}(y,\eta)^N/N), \tag{3.14}$$

where the expression under the exponent is the starting section of the Taylor series for $-\log(1-\mathbf{s})$. From (3.14) it follows that $\mathbf{r}'_0$ belongs to $\mathcal{S}^0_q$, is invertible, and, moreover, $\mathbf{r}'_0(y,\eta) - \mathbf{a}^{-1}(y,\eta) \in \mathfrak{s}_{\frac{q}{N}}$ for $|y|^2+|\eta|^2 \ge R^2$. Now take a cut-off function $\chi \in C_0^\infty(\{|\rho| < 2R\})$ which equals 1 for $|\rho| \le R$ and set

$$\mathbf{r}_0(y,\eta) = \chi((|y|^2+|\eta|^2)^{1/2})\mathbf{r}'_0(y,\eta) + (1-\chi((|y|^2+|\eta|^2)^{1/2}))\mathbf{a}(y,\eta)^{-1}.$$

According to our construction, $\mathbf{r}_0\mathbf{a} - 1$ and $\mathbf{a}\mathbf{r}_0 - 1$ have compact support. They *do not* improve their properties under η–differentiation, since the cut-off function prevents this, but they already belong to $\mathfrak{s}_{\frac{q}{N}}$ for all (y,η), together with all derivatives, and therefore (3.2) holds, for given N. □

Remark 3.10. *Proposition 3.9 illustrates usefulness of our introduction of symbol classes with only a finite number of derivatives subject to estimates of the form (3.1), (3.2). Even if for the symbol* $\mathbf{a}$ *in Definition 3.8, estimates (3.1), (3.2) hold for all* α, β*, they hold only for derivatives of order up to* N *for our regularizer* $\mathbf{r}_0$.

The notion of ellipticity is justified by the following construction of a more exact regularizer, inverting the given pseudodifferential operator up to a trace class error.

Theorem 3.11. *Let the symbol* $\mathbf{a} \in \mathcal{S}_q^0$ *stabilize in* y *at infinity and be elliptic. Then there exists a symbol* $\mathbf{r}(y,\eta) \in \mathcal{S}_q^0$ *such that*

$$OPS(\mathbf{a})OPS(\mathbf{r}) - 1,\ OPS(\mathbf{r})OPS(\mathbf{a}) - 1 \in \mathfrak{s}_1(L_2(\mathbb{R}^m, \mathfrak{K})). \tag{3.15}$$

Proof. Taking into account Theorem 3.6, Corollary 3.7 and just established properties of our symbol calculus, the construction follows the usual one. There is, however, an important difference. As usual, one sets, for N large enough,

$$\mathbf{r} = [\mathbf{r}_0 \circ_N \sum_{j=0}^{N} (1 - \mathbf{a} \circ_N \mathbf{r}_0)^{\circ_N{}^j}]_N, \tag{3.16}$$

where $\circ_N$ denotes the composition rule (3.9) for symbols, and the expression $[\cdot]_N$ means that one leaves only the terms containing derivatives of the $\mathbf{a}, \mathbf{r}_0$ up to the order N. However, in the scalar calculus, the number N determining the quantity of terms retained in the composition formula depends only on the dimension m of the space while in our operator-valued calculus it depends additionally on the number q involved in the definition of the symbol class. □

Note that the explicit expression (3.16) guarantees, just like in the usual situation, that the symbols $[\mathbf{a} \circ_N \mathbf{r} - 1]_N$, $[\mathbf{r} \circ_N \mathbf{a} - 1]_N$ have compact support in (y, η) - variables.

Remark 3.12. Analysing the proofs in this section, one can note that the conditions in the definition 3.1 may be relaxed, without changing the properties of the above pseudodifferential calculus. In fact, the 'better-than-trace-class' properties of the η-derivatives of the symbols are nowhere used. What is actually used is that the trace class norm of these derivatives decays fast, for derivatives of sufficiently high order. More exactly, it is sufficient to require (3.2) only for such α that $q/(-\gamma + |\alpha|) \le 1$. If $|\alpha| > \gamma + q$, (3.2) can be replaced by

$$|D_\eta^\alpha D_y^\beta \mathbf{a}(y,\eta)|_1 \le C_{\alpha,\beta}(1 + |\eta|)^{q+\gamma-|\alpha|},\ \ |\alpha| > \gamma + q. \tag{3.17}$$

In this form, the conditions are much easier to check, since one does not need any criteria for an operator to be 'better-than-trace-class'.

4. Preliminary index formulas and $\mathbf{K}_1$-theoretical invariants.

As it follows from Theorem 3.11 in a usual way, a pseudodifferential operator with elliptic symbol in the class $\mathcal{S}_q^0$ is Fredholm. In fact, it is already well known for a long time (see, e.g., [15]) that this is the case even for a much wider class of operator-valued symbols. Under our conditions, we will be able to investigate what the index of such operators can depend on. Note, first of all, that due to Theorem 3.11, the index of the operator is preserved under homotopy in the class of elliptic symbols. However, the notion of elliptic symbol does not have (at least direct) $\mathbf{K}_1$-theoretical meaning: it does not define an invertible element in a local $*$-algebra. This problem does not arise in the usual pseudodifferential calculus since, at least for classical poly-homogeneous symbols, the notion of the principal symbol of an operator saves the game. In the operator case, the homogeneous symbols are *not* interesting for applications, and therefore there is no natural notion of the leading symbol. For non-homogeneous symbols, in the scalar (matrix) case, the index was studied by L. Hörmander ([11]). There, a procedure was used of approximating a non-homogeneous symbol by homogeneous ones. A possibility of applying analytical index formulas based on (4.2) in the topological study is indicated in [11] as well.

We start by establishing an analytical index formulas for elliptic symbols in our classes. Here, under an analytical formula we mean one which involves an expression containing integrals of some finite combinations of the symbol, its regularizer and their derivatives. The first formula is rather rough, preliminary, and it will be improved later. This is the abstract operator-valued version of the 'algebraic index formula' obtained for the matrix situation in [6] and later for some concrete operator symbols in [9].

In what follows, the symbols are supposed to belong to classes $\mathcal{S}_q^0$. The definition of these classes involves a certain finite number N of derivatives. In our constructions, this N may vary from stage to stage. It is supposed that from the very beginning, N is chosen large enough, so on all later stages, it is still sufficiently large, so that the results of Sect.3 hold.

Proposition 4.1. *Let $\mathbf{a}(y,\eta) \in \mathcal{S}_q^0(\mathbb{R}^m \times \mathbb{R}^m; \mathfrak{K})$ be an elliptic operator symbol stabilizing in y at infinity in the sense of Sect. 3, $\mathbf{r}(y,\eta)$ is the regularizer constructed in Theorem 3.11, A, R be the corresponding operators in $L_2(\mathbb{R}^m, \mathfrak{K})$. Then, for M large enough,*

$$\operatorname{ind} A = \frac{1}{(2\pi)^m} \int\limits_{\mathbb{R}^m \times \mathbb{R}^m} \operatorname{tr}[(\mathbf{a} \circ_M \mathbf{r} - \mathbf{r} \circ_M \mathbf{a})]_M dy d\eta, \tag{4.1}$$

where tr *denotes the trace in the Hilbert space $\mathfrak{K}$.*

Proof. We modify the reasoning in [6] to fit into the operator-valued situation. In the classical Calderon formula

$$\operatorname{ind} A = \operatorname{Tr}(AR - RA) \tag{4.2}$$

we calculate the right-hand side in the terms of the symbols $\mathbf{a}, \mathbf{r}$. Introduce the regularized trace for the product of two pseudodifferential operators. For symbols $\mathbf{a}, \mathbf{b} \in \mathcal{S}_q^\gamma$, stabilising at infinity, $A = OPS(\mathbf{a})$, $B = OPS(\mathbf{b})$ and fixed M, we set

$$\operatorname{Tr}_M(AB) = \operatorname{Tr}(AB - OPS([\mathbf{a} \circ_M \mathbf{b}]_M)). \tag{4.3}$$

As it follows from Theorem 3.6, for M large enough, (4.3) is well defined and finite. Next we include $\mathbf{a}, \mathbf{b}$ in the families of symbols depending analytically on a parameter.

Fix a positive self-adjoint operator $Z \in \mathfrak{s}_q(\mathfrak{K})$ and introduce the families of symbols

$$\mathbf{a}_\kappa(y,\eta) = (1 + |y|^2)^{-\kappa} \mathbf{a}(y,\eta)((1 + |\eta|^2)^{1/2} + Z^{-1})^{-\kappa}, \Re\kappa \geq 0$$

and, similarly, $\mathbf{b}_\kappa(y,\eta)$. For any fixed $\kappa, \Re\kappa \geq 0$, the symbols $\mathbf{a}_\kappa, \mathbf{b}_\kappa$ belong to $\mathcal{S}_q^{-\Re\kappa}(\mathbb{R}^m \times \mathbb{R}^m; \mathfrak{K})$ and, additionally, decay as $|y|^{-2\Re\kappa}$, together with derivatives, as $y \to \infty$. According to Proposition 3.2, Remark 3.4, the operators $A_\kappa = OPS(\mathbf{a}_\kappa)$, $B_\kappa = OPS(\mathbf{b}_\kappa)$, depending on κ analytically, belong to the trace class as soon as $\Re\kappa$ is large enough. Therefore, for such κ the usual equality

$$\operatorname{Tr} A_\kappa B_\kappa = \operatorname{Tr} B_\kappa A_\kappa \tag{4.4}$$

holds. Next, again due to Proposition 3.2, the operators $C_\kappa = OPS([\mathbf{a}_\kappa \circ_M \mathbf{b}_\kappa]_M)$, $D_\kappa = OPS([\mathbf{b}_\kappa \circ_M \mathbf{a}_\kappa]_M)$ belong to trace class for $\Re\kappa$ large enough. Calculating the trace of the operator C_k in the usual way, we come to the expression

$$\operatorname{Tr} C_\kappa = (2\pi)^{-m} \int\limits_{\mathbb{R}^m\times\mathbb{R}^m} \sum_{|\alpha|\le M} \frac{1}{\alpha!} \operatorname{tr}(\partial_\eta^\alpha \mathbf{a}_\kappa D_y^\alpha \mathbf{b}_\kappa) dy d\eta. \tag{4.5}$$

For $\Re\kappa$ large enough, the operators under the trace sign in (4.5) belong to trace class and therefore can be commuted, preserving the trace of their product. After this, exactly like in [6], we can, by integration by parts, move y-derivatives in the integrand in (4.5) from $\mathbf{b}$ to $\mathbf{a}$ and η-derivatives – from $\mathbf{a}$ to $\mathbf{b}$. This gives $\operatorname{Tr} C_\kappa = \operatorname{Tr} D_\kappa$. Together with (4.4), this produces

$$\operatorname{Tr}_M(A_\kappa B_\kappa) - \operatorname{Tr}_M(B_\kappa A_\kappa) = 0, \quad \Re\kappa >> 0. \tag{4.6}$$

Since both parts in (4.6) are analytical for $\Re\kappa > 0$ and continuous for $\Re\kappa \ge 0$, this implies $\operatorname{Tr}_M(AB) - \operatorname{Tr}_M(BA) = 0$. Setting now $B = R$ and using (4.2) we come to (4.1). ☐

Analysing this preliminary index formula, we find out now, on which characteristics of the symbol the index actually depends.

Proposition 4.2. *Let $\mathbf{a}, \mathbf{a}' \in \mathcal{S}_q^0$ be elliptic symbols stabilizing at infinity, A, A' be corresponding pseudodifferential operators in $L_2(\mathbb{R}^m, \mathfrak{K})$ and suppose that for some $R \ge 0$, both symbols $\mathbf{a}(y,\eta), \mathbf{a}'(y,\eta)$ are invertible for $|y|^2 + |\eta|^2 \ge R^2$. Let the symbols $\mathbf{a}, \mathbf{a}'$ coincide on the sphere $S_R = \{(y,\eta) : |y|^2 + |\eta|^2 = R^2\}$, Then* $\operatorname{ind} A = \operatorname{ind} A'$.

Proof. We start by performing a special homotopy of the symbol $\mathbf{a}'$. Since $\mathbf{a}(y,\eta) = \mathbf{a}'(y,\eta)$ on S_R, and their η- derivatives belong to $\mathfrak{s}_q$, the difference $\mathbf{a}(y,\eta) - \mathbf{a}'(y,\eta)$ belongs to $\mathfrak{s}_q$ for all (y,η) (this, however, does not imply $\mathbf{a} - \mathbf{a}' \in \mathcal{S}_q^{-1}$ since $\mathbf{a} - \mathbf{a}'$ does not necessarily decay as η tends to infinity.) Let $\mathbf{r}_0'$ be the rough regularizer for $\mathbf{a}'$ existing according to Proposition 3.9. Therefore, we have $\mathbf{a}\mathbf{r}_0' = 1 + \mathbf{s}, \mathbf{s}(y,\eta) \in \mathfrak{s}_q$. Consider the family

$$\mathbf{a}_t = \exp(t(\mathbf{s} - \mathbf{s}^2/2 - \cdots - (-1)^N \mathbf{s}^N/N))\mathbf{a}', \tag{4.7}$$

where, similarly to (3.14), the starting section of the Taylor series for $\log(1+\mathbf{s})$ is present under the exponent. In (4.7), for $t = 0$, we have $\mathbf{a}_0 = \mathbf{a}'$, and for $t = 1$, $\mathbf{a}_1 = \mathbf{a} + \mathbf{w}, \mathbf{w}(y,\eta) \in \mathfrak{s}_{\frac{q}{N}}$. All symbols $\mathbf{a}_t$ are elliptic (since the exponent of everything is invertible) and thus

$$\operatorname{ind} OPS(\mathbf{a}_1) = \operatorname{ind} OPS(\mathbf{a}').$$

Since $\mathbf{s} = 0$ on the sphere S_R, it follows from (4.7) that $\mathbf{a}_1 = \mathbf{a}$ on this sphere, moreover, $\mathbf{a}_1$ is invertible outside it.

Now take a cut-off function $\chi \in C_0^\infty$ which equals 1 inside S_R and has sufficiently small support, so that for $(y,\eta) \in \operatorname{supp} \nabla\chi$, the inequality $\|\mathbf{a}_1(y,\eta) - \mathbf{a}(y,\eta)\| \le \frac{1}{2}\|\mathbf{a}(y,\eta)\|^{-1}$ holds. Define the new symbol $\mathbf{a}_2 = \chi\mathbf{a} + (1-\chi)\mathbf{a}_1$. The symbol $\mathbf{a}_2$ belongs to $\mathcal{S}_q^0$ and is elliptic. The difference $\mathbf{a}_1 - \mathbf{a}_2 = \chi(\mathbf{a}_1 - \mathbf{a})$ has compact support and belongs to $\mathfrak{s}_{\frac{q}{N}}$ for all (y,η), therefore $\mathbf{a}_1 - \mathbf{b} \in \mathcal{S}_q^{-N}$. This all gives

$$\operatorname{ind}(OPS(\mathbf{a}_2)) = \operatorname{ind}(OPS(\mathbf{a}_1)) = \operatorname{ind}(OPS(\mathbf{a}')).$$

Moreover, $\mathbf{a}_2$ is invertible outside S_R and coincides with $\mathbf{a}$ inside S_R.

Finally, we construct regularising symbols $\mathbf{r}, \mathbf{r}_2$ for $\mathbf{a}, \mathbf{a}_2$ as in Theorem 3.11, in such way that $\mathbf{a} \circ_M \mathbf{r} - 1$, $\mathbf{a}_2 \circ_M \mathbf{r}_2 - 1$, $\mathbf{r} \circ_M \mathbf{a} - 1$, $\mathbf{r}_2 \circ_M \mathbf{a}_2 - 1$, vanish outside this ball.

Then the integrand in formulas (4.1) written for both $\mathbf{a}$ and $\mathbf{a}_2$ vanishes outside the sphere S_R and is the same inside S_R, thus $\operatorname{ind}(OPS(\mathbf{a}_2)) = \operatorname{ind}(OPS(\mathbf{a}))$. $\square$

Now we will see that the index is the same for two symbols if, in the conditions of Proposition 4.2, we replace equality of symbols on the sphere by their homotopy.

Proposition 4.3. *Denote by $\mathcal{E}_q = \mathcal{E}_q(S_R)$ the class of norm-continuous invertible operator-valued functions on the sphere S_R having first order η-derivatives in $\mathfrak{s}_q$. Let $\mathbf{a}, \mathbf{a}' \in \mathcal{S}^0_q$ be elliptic symbols stabilizing at infinity and invertible for $|y|^2 + |\eta|^2 \geq R^2$. Suppose that the restrictions $\mathbf{b}$ and $\mathbf{b}'$ of these symbols to the sphere S_R are homotopic in $\mathcal{E}_q$. Then* $\operatorname{ind} OPS(\mathbf{a}) = \operatorname{ind} OPS(\mathbf{a}')$.

Proof. The situation is reduced to the one in Proposition 4.2. First, performing a standard smoothing, we can assume that the given homotopy $\mathbf{b}_t$, $\mathbf{b}_0 = \mathbf{b}, \mathbf{b}_1 = \mathbf{b}'$, consists of functions possessing η-derivatives in $\mathfrak{s}_q$ and bounded y-derivatives up to some high enough order, additionally, it depends smoothly on the parameter of homotopy t. Then, by replacing $\mathbf{a}(y, \eta)$ by $\mathbf{a}(y, \eta_0)^{-1}\mathbf{a}(y, \eta)$ and similarly with $\mathbf{a}', \mathbf{b}_t$, we reduce the problem to the one where all symbols differ by terms in $\mathfrak{s}_q$ from the unit one. Next, applying the homotopy as in (4.7), we further arrive at the situation when all symbols differ from the unit operator by terms in $\mathfrak{s}_{\frac{q}{N}}$. After this preparatory reduction, we construct the final homotopy. Take a real function $\rho(\lambda)$, smooth on $(0, \infty)$, supported in $(\frac{1}{2}, 2)$, $0 \leq \rho(\ell) \leq 1$, such th at $\rho(1) = 1$. Denote $s = ((|y|^2 + |\eta|^2)^{\frac{1}{2}}R)^{-1}$ and define the homotopy in the following way.

$$\mathbf{a}_t(y, \eta) = \mathbf{a}(y, \eta)\mathbf{b}((Ry, R\eta)/s)^{-1}\mathbf{b}_{t\rho(s)}((Ry, R\eta)/s). \tag{4.8}$$

It is clear, that, for N large enough, the symbol $\mathbf{a}_t$ belongs to $\mathcal{S}^0_q$, is invertible as long as $\mathbf{a}$ is, in particular, outside S_R, coincides with $\mathbf{a}$ for $t = 0$, while for $t = 1$, $(y, \eta) \in S_R$, we have $\mathbf{a}_1(y, \eta) = \mathbf{b}'(y, \eta)$. Now we are in conditions of Proposition 4.2, and therefore indices coincide. $\square$

As a result of our considerations, we can see that the index of the pseudodifferential operator with operator-valued symbol depends only on the class of the symbol in $\mathbf{K}_1(\mathcal{E}_q(S_R))$.

Theorem 4.4. *The index of a pseudodifferential operator defines a homomorphism*

$$\mathrm{IND} : \mathbf{K}_1(\mathcal{E}(S_R)) \to \mathbb{Z}. \tag{4.9}$$

Proof. We will show that for any element $\mathbf{v} \in \mathcal{E}_q$, there exists an elliptic symbol $\mathbf{b} \in \mathcal{S}^0_q$ invertible outside and on the sphere S_R, such that the restriction $\mathbf{w}$ of $\mathbf{b}$ to the sphere is homotopic to $\mathbf{v}$ in $\mathcal{E}_q$. Provided such extension exists, Proposition 4.3 guarantees that the index of the operator with the above symbol $\mathbf{b}$ does not depend on the choice of the symbol $\mathbf{b}$ and therefore depends only on the homotopy class of the initial symbol $\mathbf{v}$. The homomorphism property and invariance under stabilisation are obvious.

So, for the given $\mathbf{v}$, choose, for any y, $|y| \leq R$, a $\eta_0(y)$, $|y|^2 + |\eta_0(y)|^2 = R^2$, depending smoothly on y. The function $\mathbf{v}_0(y, \eta) = \mathbf{v}(y, \eta_0(y))$ does not in fact depend on η, is smooth in y, $|y| \leq R$ and invertible. It therefore admits a smooth bounded invertible continuation $\mathbf{b}_0$ to the whole $\mathbb{R}^{2m}$, again not depending on η. The operator with this symbol is just a multiplication operator and therefore it has zero index. Consider now $\mathbf{u}(y, \eta) = \mathbf{v}(y, \eta)\mathbf{v}_0(y, \eta)^{-1} \in \mathcal{E}_q$. Due to the definition of the class $\mathcal{E}_q$, the symbol $\mathbf{u}$ has the form $\mathbf{1} + \mathbf{k}$ with some continuous once differentiable function $\mathbf{k} \in \mathcal{S}^0_q$ having values in $\mathfrak{s}_q$. Performing the homotopy as in (4.7), we reduce the situation to the case $\mathbf{k}(y, \eta) \in \mathfrak{s}_{q/N}$, with prescribed

N. Next, smoothen $\mathbf{k}$, to get a function $\mathbf{z}$ on S_R, with the prescribed number of η- derivatives in $\mathfrak{s}_{q/N}$. Let $\mathbf{c}$ be the extension of this function $\mathbf{z}$ to the whole $\mathbb{R}^{2m}\setminus\{0\}$ by homogeneity of order 0. Finally, the required symbol $\mathbf{b}$ is constructed as $\mathbf{b}(y,\eta)=(\mathbf{1}+(1-\chi(|y|^2+|\eta|^2))\mathbf{c})\mathbf{b}_0$ with a smooth cut-off function $\chi\in C_0^\infty$, $\chi=1$ near the origin. □

5. Reduction of index formulas

The analytical index formula (4.1) has a preliminary character; it involves higher order derivatives of the symbol and its regularizer. Moreover, it does not reflect the algebraic nature of the index. In fact, (4.1) contains integration over the ball, while we already know (see Theorem 4.4) that the index depends only on the homotopy class of the symbol on the (large enough) sphere. In other words, (4.1) does not correspond to a homomorphism (4.9) from $\mathbf{K}_1$ for the symbol algebra to $\mathbb{Z}$. Thus a reduction of the formula is needed.

The starting point in this reduction is the result of Fedosov [6] establishing the formula of required nature for the case of the space $\mathfrak{K}$ of finite dimension.

Theorem 5.1. *Let the Hilbert space $\mathfrak{K}$ have finite dimension. Then (4.1) takes the form*

$$\operatorname{ind} A = c_m\int_{S_R}\operatorname{tr}((\mathbf{a}^{-1}d\mathbf{a})^{2m-1}),\, c_m=-\frac{(m-1)!}{(2\pi i)^m(2m-1)!},\tag{5.1}$$

where, in the integrand, taking to power is understood in the sense of exterior product.

Taking into account Theorem 4.4, we can consider (5.1) not as the expression for the index of operator but as a functional on symbols defined on the sphere. To use the strategy outlined in Sect.2, we represent (5.1) by means of a proper cyclic cocycle in a local C^*-algebra.

We define several algebras where the cocycles will reside. All of them consist of operator-valued functions on the sphere $S=S_R$, continuous in the norm operator topology on the (now, infinite-dimensional) Hilbert space $\mathfrak{K}$. Moreover, when dealing with derivatives of the symbols, we suppose that they are continued zero order homogeneously in η in some neighbourhood of S, and it is for this continuation we consider η-derivatives.

First, the largest is the C^*-algebra $\mathfrak{B}$ of all continuous operator-valued functions on S. The closed ideal $\mathfrak{C}$ in $\mathfrak{B}$ is formed by functions with values being compact operators in $\mathfrak{K}$. Next, for $1\le q<\infty$, we define the subalgebra $\mathfrak{S}_q$ consisting of once differentiable functions with η-derivative belonging to the class $\mathfrak{s}_q(\mathfrak{K})$. An ideal $\mathfrak{S}_q^0$ in $\mathfrak{S}_q$ is formed by the functions having values in $\mathfrak{s}_q(\mathfrak{K})$. The smallest subalgebra $\mathfrak{S}_0^0$ consists of functions with finite rank values, with rank uniformly bounded over S. It is clear that $\mathfrak{S}_q,\mathfrak{S}_q^0$ are local C^*-algebras, moreover, $\mathfrak{S}_q^0$ are dense in $\mathfrak{C}$ while $\mathfrak{S}_q$ are dense in the C^*-algebra of functions in $\mathfrak{B}$ having compact η-variation.

Now we introduce our initial cyclic cocycle.

For $\mathbf{a}_0,\mathbf{a}_1,\dots,\mathbf{a}_{2m-1}\in\mathfrak{S}_0^0$ we set

$$\tau_{2m-1}(\mathbf{a}_0,\mathbf{a}_1,\dots,\mathbf{a}_{2m-1})=(-1)^{m-1}c_m\int_S\operatorname{tr}(\mathbf{a}_0d\mathbf{a}_1\dots d\mathbf{a}_{2m-1}).\tag{5.2}$$

The trace in (5.2) always exists, since at least one factor under the trace sign is a finite rank operator. The fact that the functional τ_{2m-1} is cyclic follows from the cyclic property of the trace and the fact that

$$\mathrm{tr}(\mathbf{a}_0 d\mathbf{a}_1 \dots d\mathbf{a}_{2m-1}) + \mathrm{tr}(d\mathbf{a}_0\mathbf{a}_1 \dots d\mathbf{a}_{2m-1}) = d\,\mathrm{tr}(\mathbf{a}_0\mathbf{a}_1 \dots d\mathbf{a}_{2m-1}),$$

the latter being thus an exact form on S. The Hochshild cocycle property is also checked directly.

Next, the cocycle (5.2) extends to the algebra $\mathfrak{S}_0$ obtained by attaching a unit to $\mathfrak{S}_0^0$: for $\tilde{\mathbf{a}}_j = \lambda_j\mathbf{1} + \mathbf{a}_j, \lambda_j \in \mathbb{C}, \mathbf{a}_j \in \mathfrak{S}_0^0$, we have

$$\tau_m(\tilde{\mathbf{a}}_0, \dots, \tilde{\mathbf{a}}_{2m-1}) = \tau_m(\mathbf{a}_0, \dots, \mathbf{a}_{2m-1}). \tag{5.3}$$

Proposition 5.2. *The cocycle τ_m in (5.3) extends by continuity to the algebra $\mathfrak{S}_q^0$, for any $q < 2m-1$. Moreover, for an element $\tilde{\mathbf{a}} \in GL(\mathfrak{S}_q^0)$,*

$$\mathrm{IND}([\tilde{\mathbf{a}}]) = \tau_m(\tilde{\mathbf{a}}^{-1} - 1, \tilde{\mathbf{a}} - 1, \dots, \tilde{\mathbf{a}} - 1). \tag{5.4}$$

Proof. Due to compactness of the sphere, a continuous operator-valued function with values in the ideal $\mathfrak{s}_q$ can be approximated in the metric of $\mathfrak{S}_q^0$ by finite rank functions, moreover, having, for all (y, η), range in the same finite-dimensional subspace in $\mathfrak{K}$. We approximate in this way all symbols $\mathbf{a}_j$, and due to the inequality (3.3), the functional τ_m depends on finite rank symbols continuously in the sense of $\mathfrak{S}_q^0$ and this enables us to extend the functional to the whole of $\mathfrak{S}_q^0$. As for the formula (5.4), it follows from the above continuity and Theorem 4.4. □

The next step consists in finding an index formula for the algebra $\mathfrak{S}_q$, $q < 2m-1$. Take some $\mathbf{a} \in GL(\mathfrak{S}_q)$. For any y, $|y| \le 1$, fix some $\eta_0(y)$ so that $|y|^2 + |\eta_0(y)|^2 = 1$, so that $\eta_0(y)$ depends continuously on y. For the symbol $\mathbf{a}_0(y, \eta) = \mathbf{a}(y, \eta_0(y))$, the index vanishes, since, after the natural continuation to the whole of $\mathbb{R}^{2m}$, it defines an invertible multiplication operator. Thus, for

$$\tilde{\mathbf{a}}(y, \eta) = \mathbf{a}(y, \eta)\mathbf{a}_0^{-1}(y, \eta), \tag{5.5}$$

we have $\mathrm{IND}([\tilde{\mathbf{a}}]) = \mathrm{IND}([\mathbf{a}])$. This gives us

Proposition 5.3. *For $\mathbf{a} \in GL(\mathfrak{S}_q)$,*

$$\mathrm{IND}([\mathbf{a}]) = \tau_m(\tilde{\mathbf{a}}^{-1} - 1, \tilde{\mathbf{a}} - 1, \dots, \tilde{\mathbf{a}} - 1), \tag{5.6}$$

with $\tilde{\mathbf{a}}$ defined in (5.5).

Now we apply the strategy depicted in Sect.2, to construct index formulas for even wider classes of symbols. For doing this, we will use a specific algebraical realisation of the periodicity homomorphism in cyclic co-homologies, introduced in [2, 3] (see also [14]).

For an algebra $\mathfrak{S}$, not necessarily with unit, the universal graded differential algebra $\Omega^*(\mathfrak{S})$ is defined in the following way. Denote by $\tilde{\mathfrak{S}}$ the algebra obtained by adjoining a unit $\mathbf{1}$ to $\mathfrak{S}$. For each $n \in \mathbb{N}$, $n \le 1$, let Ω^n be the linear space

$$\Omega^n = \Omega^n(\mathfrak{S}) = \tilde{\mathfrak{S}} \otimes_{\mathfrak{S}} \mathfrak{S}^{\otimes n};\ \Omega = \oplus\Omega^n.$$

The differential $d : \Omega^n \to \Omega^{n+1}$ is given by

$$d((\mathbf{a}_0 + \lambda\mathbf{1})\otimes\mathbf{a}_1\otimes \dots \otimes\mathbf{a}_n) = \lambda\mathbf{1}\otimes\mathbf{a}_0\otimes \dots \otimes\mathbf{a}_n \in \Omega^{n+1}$$

By construction, one has $d^2 = 0$. One defines a right $\mathfrak{S}-$module structure on Ω^n by setting

$$(\tilde{\mathbf{a}}_0\otimes\mathbf{a}_1\otimes \dots \otimes\mathbf{a}_n)\mathbf{a} = \sum_{j=0}^{n}(-1)^{n-j}\tilde{\mathbf{a}}_0\otimes \dots \otimes\mathbf{a}_j\mathbf{a}_{j+1}\otimes \dots \otimes\mathbf{a}.$$

This right action of $\mathfrak{S}$ extends to a unital right action of $\tilde{\mathfrak{S}}$. Then the product $\Omega^i \times \Omega^j \to \Omega^{i+j}$ is defined by

$$\omega(\tilde{\mathbf{b}}_0 \otimes \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_j) = (\omega \tilde{\mathbf{b}}_0) \otimes \mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_j, \ \omega \in \Omega^i.$$

This product satisfies

$$\tilde{\mathbf{a}}_0 d\mathbf{a}_1 \dots d\mathbf{a}_n = \tilde{\mathbf{a}}_0 \otimes \mathbf{a}_1 \otimes \dots \otimes \mathbf{a}_n, \ \mathbf{a}_j \in \mathfrak{S}$$

and gives Ω the structure of a graded differential algebra. This algebra is universal in the sense that any homomorphism ρ of $\mathfrak{S}$ into some differential graded algebra (Ω', d') extends to a homomorphism of (Ω, d) to (Ω', d') respecting the product of differentials.

We make this construction concrete, taking as $\mathfrak{S}$ the algebra $\mathfrak{S}_q^0$ and as differential algebra $\Omega'(\mathfrak{S}_q)$ the algebra of operator-valued differential forms on S^{2m-1}

$$\omega' = \tilde{\mathbf{a}}_0 d'\mathbf{a}_1 \dots d'\mathbf{a}_j, \ j = 0, 1, \dots 2m-1,$$

where in the product, for the terms of the form

$$d'\mathbf{a} = \sum (\mathbf{a}^\nu dy_\nu + \mathbf{a}'^\nu d\eta_\nu),$$

the usual product of operators and the exterior product of differentials is used. We take identity as the homomorphism ρ involved in the definition of the universality property. We will omit the prime symbol in the sequel.

According to [3, Proposition 4, Ch.III.1], any cyclic cocycle $\tau \in C_\lambda^n(\mathfrak{S})$ of dimension n can be represented as

$$\tau(\mathbf{a}_0, \dots \mathbf{a}_n) = \hat{\tau}(\mathbf{a}_0 d\mathbf{a}_1 \dots \mathbf{a}_n),$$

where $\hat{\tau}$ is a closed graded trace of dimension n on $\Omega(\mathfrak{S})$. In our particular case, this representation is generated by

$$\hat{\tau}(\omega) = (-1)^{m-1} c_m \int_S \operatorname{tr} \omega, \ \omega = \tilde{\mathbf{a}}_0 d\mathbf{a}_1 \dots d\mathbf{a}_{2m-1}.$$

For $q \le 2m$, $\hat{\tau}$ is, in fact, a graded closed trace of dimension $2m-1$ on $\Omega(\mathfrak{S}_q^0)$. Moreover, for $q \le 2m-1$ the trace $\hat{\tau}$, together with the cocycle τ, extends to the unitalisation $\mathfrak{S}_q$ of $\mathfrak{S}_q^0$.

We consider the representation of the homomorphism S on the cocycle level, in the terms of the above model. For the algebra of complex numbers $\mathbb{C}$, we consider the graded differential algebra $\Omega(\mathfrak{S}) \otimes \Omega(\mathbb{C})$, with elements having the form

$$(\mathbf{a}_0 \otimes w_0) d(\mathbf{a}_1 \otimes w_1) \dots d(\mathbf{a}_n \otimes w_n), w_0, \dots, w_n \in \tilde{\mathbb{C}},$$

with differential

$$d(\mathbf{a} \otimes w) = (d\mathbf{a}) \otimes w + \mathbf{a} \otimes dw. \tag{5.7}$$

For a cyclic cocycle $\tau \in C_\lambda^n(\mathfrak{S})$ and cyclic cocycle $\sigma \in C_\lambda^{P1}(\mathbb{C})$, following [3], we define the cup product $\tau \sharp \sigma \in C_\lambda^{n+p}(\mathfrak{S} \otimes \mathbb{C}) = C_\lambda^{n+p}(\mathfrak{S})$ by setting

$$\tau \sharp \sigma(\mathbf{a}_0, \dots, \mathbf{a}_{n+p}) = (\hat{\tau} \otimes \hat{\sigma})((\mathbf{a}_0 \otimes e) d(\mathbf{a}_1 \otimes e) \dots d(\mathbf{a}_{n+p} \otimes e)), \tag{5.8}$$

Here, e is the unit in $\mathbb{C}$, i.e. the element $1 + 0\mathbf{1} \in \tilde{\mathbb{C}}$, $\hat{\tau}, \hat{\sigma}$ are graded closed traces of degree, respectively, n and p in $\Omega(\mathfrak{S}), \Omega(\mathbb{C})$ representing τ, σ and thus only terms of bidegree (n, p) survive in (5.8). It is shown in [3] that $\tau \sharp \sigma$ is a cyclic cocycle. In particular, take $\sigma = \sigma_1 \in C_\lambda^2(\mathbb{C})$, $\sigma_1(e, e, e) = 1$. Cup product with σ_1 generates the homomorphism S in cyclic co-homologies.

Now we consider iterations of S. For an even integer $p = 2l$, we consider $\sigma_l = \sigma_1^{\sharp l}$ where $\sharp l$ denotes taking to the power l in the sense of $\sharp$ operation. According to [3, Corollary 13, Ch.III.1], $\sigma_l(e, e, \dots, e) = l!$. To the cocycle σ_l, there corresponds the graded closed trace $\hat{\sigma}_l$ of degree p on $\Omega^*(\mathbb{C})$, moreover

$$\hat{\sigma}_l(ede \dots de) = l!.$$

Cup multiplication with the cocycle σ_l generates the iterated homomorphism S^l in cyclic cohomologies of the algebra $\mathfrak{S}$. We will study the structure of $S^l\tau$, for $\tau \in C^n_\lambda(\mathfrak{S})$. According to (5.8) and (5.7),

$$\begin{aligned} S^l\tau(\mathbf{a}_0, \mathbf{a}_1 \dots \mathbf{a}_{n+2l}) &= \widehat{S^l\tau}((\mathbf{a}_0 \otimes e)d(\mathbf{a}_1 \otimes e) \dots d(\mathbf{a}_{n+p} \otimes e)) \\ &= (\hat{\tau} \otimes \hat{\sigma})((\mathbf{a}_0 \otimes e)(d\mathbf{a}_1 \otimes e + \mathbf{a}_1 \otimes de) \dots (d\mathbf{a}_{n+p} \otimes e + \mathbf{a}_{n+p} \otimes de). \end{aligned} \quad (5.9)$$

Since $\hat{\tau}$ is a graded trace of degree n and $\hat{\sigma}_l$ is a graded trace of degree $p = 2l$, only the terms of bidegree (n, p) contribute to (5.9). There are a lot of such terms, and each of them involves the value of $\hat{\tau}$ on a certain product of $\mathbf{a}_j$ and $d\mathbf{a}_j$, where exactly $n + 1$ factors are of the form $\mathbf{a}_j$, including $\mathbf{a}_0$, and the value of $\hat{\sigma}_l$ on the product of the elements e and de, with $n + 1$ factors e, including the first one, and p factors de. Quite a lot of these terms vanish. In fact, since e is an idempotent, $e = e^2$, we have

$$de = ede + dee, \quad edee = 0, \quad edede = dedee. \quad (5.10)$$

Therefore, if some term contains the product of an odd number of de surrounded by e, the corresponding term vanishes. Thus only those terms survive where each group of consecutive de in the product contains an even number of de. For any such product, using (5.10), we can rearrange the factors e, de and come to the expression $\hat{\sigma}_l(ede \dots de)$ which equals $l!$. This leaves us with the contribution to (5.9) involving $\mathbf{a}_j$ and $d\mathbf{a}_j$. In these terms, the variables $\mathbf{a}_j$ enter in a very special way. Since, for $j \neq 0$, $\mathbf{a}_j$ stand on the places where we had de in (5.9), and $d\mathbf{a}_j$ stand on the places where we had e, only those terms survive in (5.9), where an even number of variables $\mathbf{a}_j$ stand in succession, not counting $\mathbf{a}_0$. This gives us the following characterisation of $S^l\tau$.

Proposition 5.4. *The image $S^l\tau$ in C^{n+2l}_λ of a cyclic cocycle $\tau_n \in C^n_\lambda$ under the iterated homomorphism S^l equals*

$$\tau_{n+p} S^l\tau_n(\mathbf{a}_0, \dots, \mathbf{a}_{n+p}) = l! \sum_{\mu_j, \nu_j} \hat{\tau}(\mathbf{a}_0 \prod_j A_j B_j), \quad (5.11)$$

where the summation is performed over collections of μ_j, ν_j such that $1 = \nu_0 \leq \mu_1 < \nu_1 < \mu_2 < \dots < \mu_s \leq \nu_s = n+p+1$, $\nu_j - m_j$ are even, $A_j = \mathbf{a}_{\nu_{j-1}} \dots \mathbf{a}_{\mu_j - 1}$, $B_j = d\mathbf{a}_{\mu_j} \dots d\mathbf{a}_{\nu_j - 1}, \sum(\nu_j - \mu_j) = n$.

Getting an explicit analytical description of (5.11) might be quite troublesome. We, however, are interested only in the value of $S^l\tau$ on a very special collection of variables $\mathbf{a}_j$. In fact, when calculating index, according to (2.1), we evaluate $\tau_{2m-1+2l}(\mathbf{a}_0, \dots, \mathbf{a}_{2m-1+2l})$, for $\mathbf{a}_0 = \mathbf{a}^{-1}$, $\mathbf{a}_{2k-1} = (\mathbf{a} - 1)$, $\mathbf{a}_{2k} = (\mathbf{a}^{-1} - 1)$. This enables us to give the following analytical expression for the index.

Theorem 5.5. *For $\mathbf{a} \in \mathcal{E}_q = GL(\mathfrak{S}_q)$, by $\alpha_{2l}(\mathbf{a})$, denote*

$$\alpha_{2l}(\mathbf{a}) = (l!)^{-1} \int_{S_R} \operatorname{tr} \left[(\frac{d}{dt})^l (\mathbf{b}^{-1}(1 - t\mathbf{c})^{-1} d\mathbf{b})^{2m-1} |_{t=0} \right], \quad (5.12)$$

and

$$\alpha'_{2l}(\mathbf{a}) = \operatorname{tr} \int_{S_R} (\mathbf{c} + \mathbf{b}^{-1} d\mathbf{b})^{2m-1+l}, \tag{5.13}$$

where $\mathbf{b}(y,\eta) = \mathbf{a}(y,\eta)\mathbf{a}(y,\eta_0(y))^{-1}$, $\mathbf{c} = (\mathbf{b}-1)(\mathbf{b}^{-1}-1)$ *and in (5.13) only the term of degree* $2m-1$ *is naturally preserved under integration.*

Then for $2l + 2m - 1 > q$ *the form in the integrand in (5.12) and the integral in (5.13) belong to trace class and*

$$\operatorname{IND}[\mathbf{a}] = c_{m,l}\alpha_{2l}(\mathbf{a}) = c_{m,l}\alpha'_{2l}(\mathbf{a}),\ c_{m,l} = -(2\pi i)^{-m}\frac{l!(m+l-1)!}{(2m+2l-1)!}. \tag{5.14}$$

Proof. Note first that, as it was done in the proof of Theorem 4.4, passing from $\mathbf{a} \in GL(\mathfrak{S}_q)$ to $\mathbf{b} \in \mathrm{GL}(\mathfrak{S}^0_q)$ does not change the index. We can therefore restrict ourselves to the case of $\mathbf{a} \in GL(\mathfrak{S}^0_q)$ and $\mathbf{b} = \mathbf{a}$. Our task now is to show that for our specific choice of variables, the expression (5.11) takes the form (5.12),(5.13). This, according to Proposition 5.4 and the relation (2.2) between index pairing and suspension in cyclic co-homology, will mean that for the symbol $\mathbf{a} \in GL(\mathfrak{S}^0_q) = \mathcal{E}_q$, $q \le 2m-1$, the index formula (5.13) holds. Then after showing that the functionals $\alpha_{2l}, \alpha'_{2l}$ depend continuously on $\mathbf{a} - 1 \in \mathfrak{S}^0_q$, $q \le 2m + 2l - 1$ we extend the index formula to $\mathcal{E}_q$ with such q, as in Proposition 2.1.

So, let us transform (5.11). For our particular choice of variables, each term A_j equals $\mathbf{c}^{\frac{(\nu_j - \mu_j)}{2}}$. This means that all terms in (5.11) can be obtained in the following way. Write down the expression $\mathbf{a}^{-1}d\mathbf{a}d(\mathbf{a}^{-1})\dots d\mathbf{a}$, with m factors $d\mathbf{a}$ and $m-1$ factors $d(\mathbf{a}^{-1})$. Before each $d\mathbf{a}$, $d(\mathbf{a}^{-1})$ insert several terms $\mathbf{c}$, so that there are l of them altogether. Summing all such products and taking into account that $d(\mathbf{a}^{-1}) = -\mathbf{a}^{-1}d\mathbf{a}\mathbf{a}^{-1}$ and that $\mathbf{c}$ and $\mathbf{a}$ commute, we come to the formula

$$\tau_{2l}(\mathbf{a}^{-1} - 1, \dots, \mathbf{a} - 1) = c_m \int_S \sum_{\sum \kappa_j = l} \prod_{j=1}^{2m-1} (\mathbf{c}^{\kappa_j}\mathbf{a}^{-1}d\mathbf{a}). \tag{5.15}$$

To describe (5.15) more explicitly, introduce an extra variable t and consider the expression depending on t:

$$\psi(t,\mathbf{a}) = c_m \int_S ((1 - t\mathbf{c})^{-1}\mathbf{a}^{-1}d\mathbf{a})^{2m-1}. \tag{5.16}$$

For t small enough, $1 - t\mathbf{c}$ is invertible, and therefore (5.16) can be rewritten as

$$\psi(t,\mathbf{a}) = c_m \int_S ((1 + t\mathbf{c} + t^2\mathbf{c}^2 + \dots)\mathbf{a}^{-1}d\mathbf{a})^{2m-1}. \tag{5.17}$$

Now we can see that (5.15) equals the coefficient at t^l in (5.17), and this gives us (5.12). Since in each term in the sum in (5.12), there are $2m-1$ factors $d\mathbf{a}$ belonging to $\mathfrak{s}_q$ and l factors $\mathbf{c}$ belonging to $\mathfrak{s}_{\frac{q}{2}}$, the form (5.12) extends by continuity to $GL(\mathfrak{S}^0_q)$, thus giving the index formula. As for (5.13), it is clear that the term of degree $2m-1$ in the integrand, the only one that survives under the integration, exactly equals the integrand in (5.15). □

6. Applications I. Toeplitz and cone operators

In this section we show how the results of Sect.5 enable one to derive, in an uniform way, index formulas for some concrete situations.

6.1. **Toeplitz operators.** We start by considering the case of Toeplitz operators on the line (or, what is equivalent, on the circle) with operator-valued symbols. Such operators form an important ingredient in the study of pseudodifferential operators on manifolds with cone- and edge-type singularities (see, e.g., [20, 22]). The results of this subsection present an abstract version of the analysis given in [22].

Let $\mathbf{b}(y)$ be a function on the real line $\mathbb{R}^1$, with values being operators in the Hilbert space $\mathfrak{K}$. We suppose that $\mathbf{b}(y) = 1 + \mathbf{k}(y)$ is differentiable and stabilises sufficiently fast at infinity:

$$\mathbf{k}(y),\mathbf{k}'(y) \in \mathfrak{s}_q(\mathfrak{K}); \|\mathbf{k}(y)\| = O((1+|y|)^{-q});\ ||\mathbf{k}'(y)||, |\mathbf{k}(y)|_q = O(1). \tag{6.1}$$

We consider the Toeplitz operator $T_\mathbf{b}$ in the Hardy space $H^2(\mathbb{R}^1, \mathfrak{K})$ acting as

$$T_\mathbf{b} u = P\mathbf{b}u,$$

where P is the Riesz projection $P : L_2 \to H^2$. This operator does not directly fit into the scheme of Sect. 3, however the considerations of Sect.5 can be easily adapted to it. In fact, consider the algebra $\mathfrak{P}^0_q$ consisting of symbols $\mathbf{k}$ satisfying (6.1); $\mathfrak{P}_q$ is obtained by attaching the unit to $\mathfrak{P}^0_q$. Additionally, for $q = 0$, $\mathfrak{P}^0_0$ consists of (uniformly) finite rank functions $\mathbf{k}(y)$ with compact support in $\mathbb{R}^1$. The classical formula (1.1) for the index of Toeplitz operator gives us the cyclic cocycle

$$\tau \in C^1_\lambda(\mathfrak{P}^0_0);\ \ \tau(\mathbf{k}_0, \mathbf{k}_1) = -\frac{1}{2\pi i}\int \operatorname{tr}(\mathbf{k}_0 d\mathbf{k}_1), \tag{6.2}$$

such that

$$\operatorname{ind}(T_\mathbf{b}) = \tau(\mathbf{b}^{-1} - 1, \mathbf{b} - 1)$$

for the invertible symbol $\mathbf{b} = \mathbf{1} + \mathbf{k},\ \mathbf{k} \in \mathfrak{P}_0$. This cocycle of dimension 1 has exactly the same form as the one in Sect. 5 for $m = 1$, the only (and non-essential) difference being non-compactness of the integration domain. Thus we can apply Theorem 5.5, having only to check that for the given q, the suspended cocycle extends continuously to the algebra $\mathfrak{P}_q$. Due to one-dimensionality of the problem, we can give an explicit expression to the cocycle (5.11). In fact, in our case $m = 1$ and we have

$$\tau_l(\mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_{2l+1}) = c_{1,l}\sum_{j=0}^{2l}\int \operatorname{tr}(\mathbf{k}_0 \dots \mathbf{k}_j d\mathbf{k}_{j+1}\mathbf{k}_{j+2}\dots \mathbf{k}_{2l+1}). \tag{6.3}$$

According to (3.3), the cocycle (6.3) is continuous on $\mathfrak{P}^0_q$ for $l > 2q$. This gives us the index formula for $T_\mathbf{b}$.

Theorem 6.1. *If $l > 2q$, $\mathbf{b}$ is an invertible symbol in $\mathfrak{P}_q$ then the index of $T_\mathbf{b}$ equals*

$$\operatorname{ind} T_\mathbf{b} = (2l+1)c_{1,l}\int \operatorname{tr}((\mathbf{b}^{-1} - 1)^l(\mathbf{b} - 1)^l \mathbf{b}^{-1} d\mathbf{b}). \tag{6.4}$$

In particular, when $\mathbf{b}(y)$ is a parameter dependent elliptic pseudodifferential operator on a compact k-dimensional manifold M, having the form $\mathbf{b}(y) = 1+\mathbf{k}(y)$, with $\mathbf{k}$ being an operator of negative order $-s$, the conditions of Theorem 6.1 are satisfied with any $q > k/s$. This was the situation considered in [22].

6.2. **Cone Mellin operators.** Cone Mellin operators (CMO) are involved into the local representation for singular pseudodifferential operators near conical points and edges. The were considered systematically in [19, 32, 33, 38] etc. The index formula for elliptic CMO was proved in [8], another approach to index formulas for CMO was proposed in [28]. Here we study CMO in a more abstract setting.

Let $\mathfrak{K}$ be a Hilbert space. In $L_2(\mathbb{R}_+, \mathfrak{K})$ we consider operators of the form

$$(Au)(t) = \frac{1}{2\pi i}\int_\Gamma dz \int_0^\infty (t/t_1)^z \mathbf{a}(t,z)u(t_1)\frac{dt_1}{t_1}, \tag{6.5}$$

where $\mathbf{a}(t,\zeta)$ is a function on $\mathbb{R}_+ \times \Gamma$ with values being bounded operators in $\mathfrak{K}$. The line Γ is any fixed vertical line $\Gamma = \Gamma_\beta = \{\Re z = \beta\}$ the choice of β determines the choice of the weighted L_2 space where the operator is considered, and the change $u(t) \mapsto t^\beta u(t)$ reduces the problem to the case $\beta = 0$ to which we therefore can restrict our study. The Mellin symbol $\mathbf{a}(t,z)$ is supposed to be a bounded operator in $\mathfrak{K}$ for all $(t,z) \in \mathbb{R}_+ \times \Gamma$. We say that it belongs to the class $\mathfrak{M}^\mu_q$, $\mu \geq 0$, if the following conditions are satisfied:

$$\|\partial_t^\alpha \partial_z^\nu \mathbf{a}(t,z)\| = O((1+|z|)^{-\nu+\mu}), \tag{6.6}$$

$$|\partial_t^\alpha \partial_z^\nu \mathbf{a}(t,z)|_{\frac{q}{\nu-\mu}} = O(1), \tag{6.7}$$

uniformly in t; for $t \in (0,c]$ and for $t \in [C,\infty)$ the symbol does not depend on t.

The change of variables $y = \log t$, $\eta = iz$, transforms CMO to a pseudodifferential operator considered in Sect.3 with operator-valued symbol in the class $\mathcal{S}^\mu_q$. This, in particular, means that for $\mu = 0$ the elliptic symbols, i.e., those for which, for (t,ζ) outside some compact in $\mathbb{R}_+ \times \Gamma$, $\mathbf{a}(t,z)$ is invertible, with uniformly bounded inverse, give Fredholm operators. The index for such operators can be found by any of formulas (5.12), (5.13), $m = 1$, with proper l. The explicit expression for the suspended cocycle is found in Sect. 6.1. In the co-ordinates (t,z) this gives

Proposition 6.2. *For the CMO with elliptic symbol $\alpha(t,z)$,*

$$\operatorname{ind} A = (2l+1)c_{1,l}\int_{\mathcal{L}} \operatorname{tr}[((\mathbf{b}(t,z)^{-1}-1)(\mathbf{b}(t,z)-1))^l \mathbf{b}(t,z)^{-1} d\mathbf{b}(t,z)], \tag{6.8}$$

where $\mathcal{L}$ is a contour in $\mathbb{R}_+ \times \Gamma$ such that on and outside it the symbol $\mathbf{a}$ is invertible, $\mathbf{b}(t,z) = \mathbf{a}(t,z)\mathbf{a}(t,z_0)^{-1}$, and z_0 is large enough, so that $\mathbf{a}(t,z_0)$ is invertible for all t.

One can give a more topological index formula for CMO.

Theorem 6.3. *Let $\mathbf{a}(t,z) \in \mathfrak{M}^0_q$ be an elliptic Mellin symbol and $\mathbf{r}(t,z)$ be the regularizer: $\mathbf{ar}-1, \mathbf{ra}-1$ belong to trace class for all $(t,z) \in \mathbb{R}_+ \times \Gamma$ and vanish outside some compact set. Define the Chern character of the symbol $\mathbf{a}$ as*

$$Ch(\operatorname{Ind}\mathbf{a}) = \operatorname{tr}((d\mathbf{r} + \mathbf{r}(d\mathbf{a})\mathbf{r})d\mathbf{a}). \tag{6.9}$$

Then

$$\operatorname{ind} A = \frac{1}{2\pi i}\int_{\mathbb{R}_+\times\Gamma} Ch(\operatorname{Ind}\mathbf{a}). \tag{6.10}$$

Proof. The sense of the formula (6.9) giving the analytic expression for the Chern character for the Fredholm family $\mathbf{a}$ is explained, e.g., in [8]. There, the index formula (6.10) was first proved for the particular case of $\mathbf{a}$ being a parameter dependent elliptic pseudodifferential operator on a compact manifold, using the detailed analysis of Mellin operators. Another proof of (6.10) was given in [28]; it was based on the K-theoretical analysis of the algebra of Mellin symbols. Now, having Proposition 6.2, the proof of (6.10) is quite short. In fact, under the homotopy of elliptic Mellin symbols, both parts of (6.8) and the left-hand side in (6.10) are invariant. The same holds for the right-hand side of (6.10), as an easy calculation shows. Now, having an elliptic symbol $\mathbf{a}$, we set

$$\mathbf{a}_s(t,z) = \exp(s(-\mathbf{c}(t,z) + \mathbf{c}(t,z)^2/2 - \cdots + (-1)^N \mathbf{c}(t,z)^N/N))\mathbf{a}(t,z), 0 \le \sigma \le 1, \tag{6.11}$$

where $\mathbf{c}(t,z) = \mathbf{b}(t,z) - 1$ and, similar to (4.7), the starting section of the Taylor expansion for $-s\log(1+\mathbf{c})$ is present under the exponent sign. The homotopy consists of elliptic symbols, all of them are invertible outside $\mathcal{L}$, moreover, for $s = 1$, the symbol $\mathbf{b}_1(t,z) = \mathbf{a}_1(t,z)\mathbf{a}_1(t,z_0)^{-1}$ differs from the identity by a trace class operator, provided $N > q$. Therefore, for the index of A the expression (6.8) with $l = 0$, holds, i.e. $\operatorname{ind} A = -\frac{1}{2\pi i}\int_{\mathcal{L}} \operatorname{tr}(\mathbf{b}_1^{-1} d\mathbf{b}_1)$. In the latter expression, one can now interchange trace and integration, and after applying Stokes formula, we arrive at (6.10). □

Comparing Theorem 6.3 with results of [8], we can see that the condition of analyticity of the symbol in z variable in no more needed. Moreover, $\mathbf{a}$ can be any operator-valued symbol, not necessarily a parameter dependent elliptic operator. In particular, it may be an operator on a compact singular manifold, which can give index formulas for operators on corners (see, e.g., [38, 35]).

7. Applications II. Edge operators

In this section we apply our abstract results to the case of edge pseudodifferential operators. Such operators arise in the study of pseudodifferential operators on singular manifolds, see [32, 33, 34, 38, 4, 29, 5], etc. The usual way to introduce such operators consists in prescribing an explicit representation, involving Mellin, Green operators and some others. The index formulas for model operators of edge type were obtained in [31] and in [9], on the base of such detailed edge calculus. Here we show that these formulas, as well as some new ones of the type found in Sect. 5, hold in a somewhat more general situation. We depict here a new version of calculus of edge operators where one avoids using Mellin or Green representation and weighted Sobolev spaces, thus defining operator symbols not by explicit formulas but rather by their properties. We just note here that the leading term in our calculus is the same as in the standard one. We present this calculus in just as general form as it is needed for illustrating our approach to index formulas. A more extended exposition of this version of edge calculus will be given elsewhere.

7.1. **Discontinuous symbols.** In the leading term, our edge operators will be glued together from usual pseudodifferential operators in the Euclidean space, with symbols having discontinuities at a subspace - see [19, 23, 25].

Let $a(x,\xi) = a(y,z,\eta,\zeta)$ be a (matrix) function in $\mathbb{R}^n \times \mathbb{R}^n = (\mathbb{R}^m \times \mathbb{R}^k) \times (\mathbb{R}^m \times \mathbb{R}^k)$, zero order positively homogeneous in $\xi = (\eta,\zeta)$ variables. We suppose that the function a has compact support in x variable and is smooth in all variables unless $\xi = 0$ or $z = 0$. At the subspace $z = 0$ the function a has a discontinuity: it has limits as z approaches 0, but these limits may depend on the direction of approach:

$$\Phi(y,\omega,\eta,\zeta) = \lim_{\rho\to 0} a(y,\rho\omega,\eta,\zeta), \tag{7.1}$$

moreover, (7.1) can be differentiated sufficiently many times in y, ω, η, ζ while in ρ variable the symbol a can be expanded by Taylor formula, with sufficiently many terms. (The reader may even suppose, for the sake of simplicity, that for small ρ, the function α does not depend on ρ at all.) Such functions we will call *discontinuous scalar symbols.*

To the symbol a we associate, in the usual way, the pseudodifferential operator in $\mathbb{R}^n$ acting as

$$\mathcal{A} = \mathcal{F}_0^{-1} a \mathcal{F}_{{}_0}, \tag{7.2}$$

where $\mathcal{F}_0$ is the Fourier transform in $\mathbb{R}^n$. It is often convenient to consider pseudodifferential operators in weighted L_2 spaces with weight $\rho^\sigma, \rho = |z|$. If $|\sigma| < k/2$, the rule (7.2) defines a bounded operator in the weighted space. For remaining σ, the definition requires a certain modification, see, e.g., [19, 25]; for the sake of simplicity, we restrict ourselves to the case $|\gamma| < k/2$.

The operator $\mathcal{A}$ is bounded in $L_2(\mathbb{R}^n)$ (with weight). We can represent it as a pseudodifferential operator with operator-valued symbol. Denote by $\mathfrak{K}_0$ the (weighted) space $L_2(\mathbb{R}^k)$ and set

$$\mathbf{a}(y, \eta) = \mathcal{F}^{-1} a(y, z, \eta, \zeta)\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform in $L_2(\mathbb{R}^k)$. This operator-valued symbol is a bounded operator for any (y, η), moreover it is differentiable in y, η for $\eta \neq 0$,

$$\partial_\eta^\beta \partial_y^\alpha \mathbf{a} = \mathcal{F}^{-1} \partial_\eta^\beta \partial_y^\alpha a \mathcal{F}.$$

This shows that $\partial_y^\alpha \partial_\eta^\beta \mathbf{a}$ is a pseudodifferential operator of order $-|\beta|$ in $\mathfrak{K}_0$. Since such operators, with symbol compactly supported in z, belong to $\mathfrak{s}_q$, $q > k/|\beta|$, (3.2) holds. Homogeneity implies (3.1).

The operator symbol $\mathbf{a}$, however, does not belong to our symbol class $\mathcal{S}_q^0$ since these estimates hold only for η outside some fixed neighbourhood of zero. At the point $\eta = 0$ the η-derivatives of a have singularities and thus (3.1), (3.2) fail. In order to satisfy them we have to introduce some corrections for the symbol.

Proposition 7.1. *For any discontinuous symbol a, there exists an operator symbol $\mathbf{b}(y, \eta) \in \mathcal{S}_q^0$ coinciding with $\mathbf{a}$ for η outside some neighbourhood of zero such that the difference $\mathcal{A} - OPS(\mathbf{b})$ belongs to the trace class, and, moreover, the norm of $\mathbf{a}(y, \eta) - \mathbf{b}(y, \eta)$ can be made arbitrarily small for all (y, η).*

Proof. Fix some (large enough) N and set

$$b_N(y, z, \eta, \zeta) = \psi(\delta^{-1}|\eta|)[a(y, z, 0, \zeta) + \sum_{|\alpha|=1}^{N} (\alpha!)^{-1} \eta^\alpha D_\eta^\alpha a(y, z, 0, \zeta)(1 - \psi(|\zeta|)] + (1 - \psi(\delta^{-1}|\eta|))a(y, z, \eta, \zeta). \tag{7.3}$$

Denote by $\mathbf{b}_N(y, \eta)$ the operator symbol corresponding to the scalar symbol b_N. It is N times differentiable and differs from $\mathbf{a}$ only for small η, therefore (3.1) is satisfied automatically and it is only for small η that we have to check (3.2). In the symbol b_N, the singularity of η-derivatives at $\zeta = 0$ is cut away, at the same time, as ζ goes to infinity, $\partial_\eta^\beta b_N$ decays as $|\zeta|^{-|\beta|}$ for $|\beta| \leq N$, which grants (3.2). The difference $\mathbf{a} - \mathbf{b}_N$ is generated by a bounded scalar symbol having a compact support in η and decaying as $|\zeta|^{-N-1}$ as $\zeta \to \infty$ (since for large ζ this symbol equals the remainder term in the Taylor expansion of a in η at the point $\eta = 0$). Therefore, for $N > k$, this operator symbol belongs to trace class (together with y-derivatives). We can now apply Proposition 3.2, and thus $\mathcal{A} - OPS(\mathbf{b}_N)$ belongs to trace class (note that at this place it is essential that Proposition 3.2 does not require any smoothness of the operator symbol in η.) Next we show that the symbol $\mathbf{a}$, although not differentiable at $\eta = 0$, is nevertheless norm-continuous at this point. Note that as $\eta \to 0$, the symbol $a(x, \eta, \zeta)$ converges to $a(x, 0, \zeta)$, but not uniformly in ζ: this non-uniformity takes place in the neighbourhood of the point $\zeta = 0$. Thus take a cut-off function $\psi(\tau) \in C_0^\infty([0, \infty))$ which equals one near zero. Then the symbol $a(x, \eta, \zeta)(1 - \psi(|\zeta|))$ converges as $\eta \to 0$ uniformly to $a(x, 0, \zeta)(1 - \psi(|\zeta|))$, together with all derivatives, which grants norm convergence of corresponding operator symbols. On the other hand, $(a(y, z, \eta, \zeta) - a(y, z, 0, \zeta))\psi(|\zeta|)$ converges as $\eta \to 0$ to zero in L_2 in z, ζ variables (uniformly in y) which implies Hilbert-Schmidt, and, therefore, norm convergence. As a result of this, we can achieve smallness of the norm of this latter difference due to norm continuity of $\mathbf{b}_N$, by choosing a small enough δ in (7.3). □

In the sequel, when talking about the operator symbol associated to the discontinuous scalar symbol a, we will mean the symbol $\mathbf{b}_N$, with N large enough, constructed as in Proposition 7.1. To simplify notations, it is this operator symbol that we now denote by $\mathbf{a} = OS(a)$. This does not determine $OS(a)$ in an unique way, but this ambiguity, due to Proposition 7.1, is not essential for index formulas.

The operator symbols obtained by the above construction possess, in addition to the general properties of the class $\mathcal{S}_q^0$, one more.

Proposition 7.2. *Let* $\mathbf{a}$ *be an operator symbol associated to some discontinuous scalar symbol* a. *Let* $\psi(\tau)$ *be a cut-off function on the semi-axis which equals 1 in the neighbourhood of zero. Then*

$$[D_\eta^\beta \mathbf{a}, \psi(|z|)] \in \mathcal{S}_q^{-|\beta|-1}, q > k, \tag{7.4}$$

for all $|\beta| < N'$, *where* N' *can be made arbitrarily large by choosing large enough* N *in (7.3).*

Proof. To show (7.4), it is sufficient to notice that the composition of the pseudodifferential operator with discontinuous symbol with the multiplication by a smooth function follows the same rules as for usual smooth symbols, since in this case no z-differentiation is involved. □

We denote the set of operator symbols in $\mathcal{S}_q^0$ satisfying additionally (7.4) by $\mathcal{L}_q^0$.

Up to the above ambiguity in the definition of $OS(a)$, the mapping $OS : a \mapsto \mathbf{a} \in \mathcal{L}_q^0$ is additive. It is, however, not multiplicative, even if one neglects compact operator symbols. We introduce here the class of operator symbols arising as the multiplicative error.

Definition 7.3. Let $\mathfrak{K}$ be the Hilbert space $L_2(\mathbf{K})$ (with weight), where $\mathbf{K}$ is a cone with base being a compact manifold. We say that the operator symbol $\mathbf{g}(y, \eta) \in \mathcal{L}_q^0(\mathbb{R}^m, \mathfrak{K})$ belongs to $\mathcal{I}_q^0$ if for any cut-off function ψ, as above,

$$(1 - \psi(|z|))D_\eta^\beta \mathbf{g} \in \mathcal{S}_q^{-|\beta|-1} \tag{7.5}$$

for all $|\beta| \le N'$.

It follows from Proposition 7.2 that one can commute terms in (7.5), moreover, that $\mathcal{I}_q^0$ is an ideal in $\mathcal{L}_q^0$. An example of operator symbol in $\mathcal{I}_q^0$, for $\mathbf{K} = \mathbb{R}^k$, is given by

$$\mathbf{g}(y, \eta) = \mathcal{F}_0^{-1} g(y, z, \xi) \mathcal{F}_0,$$

with function $g(y,z,\eta,\zeta) = |z|^{-\mu}(|\xi|^{-\mu}),\ \mu > 0$, smoothened as in (7.3). Symbols in this ideal present an abstract generalisation of singular Green operators in the traditional construction of the edge calculus (see, e.g., [32]).

Proposition 7.4. *Let a, b be discontinuous scalar symbols in $\mathbb{R}^m \times \mathbb{R}^k$, where $\mathbb{R}^k$ is considered as a cone with base S^{k-1}. Then*

$$OS(a)OS(b) - OS(ab) \in \mathcal{I}_q^0, \tag{7.6}$$

$$OS(a^*) - OS(a)^* \in \mathcal{I}_q^0. \tag{7.7}$$

Proof. Let ψ be a cut-off function as in Definition 7.3 and ψ' be another cut-off function such that $\psi\psi' = \psi'$. So we have

$$\begin{aligned}(1-\psi)(OS(a)OS(b) - OS(ab)) &= (1-\psi)(1-\psi')(OS(a)OS(b) - OS(ab))\\ = (OS((1-\psi)a)OS((1-\psi')b) - OS((1-\psi')ab)) &+ [OS((1-\psi)a), (1-\psi')]OS(b).\end{aligned} \tag{7.8}$$

In (7.8), the first term to the right belongs to $\mathcal{I}_q^0$, since the operators involved have smooth symbols; for the second term this holds due to Proposition 7.2. The relation (7.7) is checked in a similar way. □

Definition 7.5. *The class $\mathfrak{S}^0 = \mathfrak{S}^0(\mathbb{R}^m,\ L_2(\mathbb{R}^k))$ consists of elements $\mathbf{a} \in \mathcal{L}_q^0$ for which there exist a discontinuous scalar symbol a and a symbol $\mathbf{c} \in \mathcal{I}_q^0$ such that*

$$\mathbf{a} = OS(a) + \mathbf{c}. \tag{7.9}$$

In the sequel, we will refer to $OS(a)$ as the pseudodifferential part and $\mathbf{c}$ as the Green part of the symbol $\mathbf{a} \in \mathfrak{S}^0$. It follows from the above propositions that $\mathfrak{S}^0$ is a *- algebra (without unit).

Now, in order to treat cones with an arbitrary base, we introduce directional localisation. Let κ, κ' be smooth functions on the sphere S^{k-1}. For a discontinuous symbol a and corresponding operator symbol $\mathbf{a}$, we introduce the localised symbol $\mathbf{a}_{\kappa\kappa'} = \kappa(\omega)\mathbf{a}\kappa'(\omega), \omega = z/|z|$. Such operator symbol, obviously, belongs to $\mathcal{L}_q^0$.

Proposition 7.6 (directional pseudo-locality)**.** *If supports of κ, κ' are disjoint then $\mathbf{a}_{\kappa\kappa'} \in \mathcal{I}_q^0$.*

Proof. For a cut-off function ψ as above, we have $\psi^2(|z|)\mathbf{a}_{\kappa\kappa'} = [\kappa\psi, \psi\mathbf{a}]\kappa'$, and here we have a pseudodifferential operator of order -1, as in Proposition 7.2. □

Now we consider homogeneous changes of variables. For an operator symbol $\mathbf{a} \in \mathcal{S}_q^\gamma(\mathfrak{K}_0)$ and directional cut-offs κ, κ', the symbol $\mathbf{b} = \kappa(\omega)\mathbf{a}\kappa'(\omega)$ also belongs to $\mathcal{S}_q^\gamma$. Let $\kappa\kappa' = \kappa$ and $\varkappa$ be a diffeomorphisms of a neighbourhood Ω of the support of κ onto another domain Ω' on the sphere S^{k-1}. Then we can define the transformed symbol $\varkappa^*\mathbf{a} = \varkappa \circ \mathbf{a} \circ \varkappa^{-1}$, obtained by the homogeneous change of variables. It is clear that the class $\mathcal{S}_q^\gamma$ is invariant under this transformation. Since such change of variables commutes with multiplication by cut-off functions, such invariance holds also for the classes $\mathcal{L}_q^0$ and $\mathcal{I}_q^0$. At the same time, for any discontinuous scalar symbol b, supported in the cone over Ω we can define a discontinuous scalar symbol $\varkappa^*b$ obtained from β by the usual rule of change of variables $(\varkappa^*b(z',\zeta') = b(\tilde{\varkappa}^{-1}(z'), (D\tilde{\varkappa}')\zeta')$ where $\tilde{\varkappa}(z) = \varkappa(z/|z|))$. For operator symbols obtained by our procedure from discontinuous scalar symbols, the usual rule of change of variables in the leading symbol is preserved.

Proposition 7.7. *Let a be a discontinuous symbol, $\mathbf{a} \in \mathcal{S}_q^0$ be the corresponding operator symbol, $b = \kappa(\omega)a$ and $\mathbf{b} = \mathbf{a}_{\kappa\kappa'}$, where κ, κ' are directional cut-offs, $\kappa\kappa' = \kappa$. Then for a diffeomorphism $\varkappa$,*

$$\varkappa^*\mathbf{b} - OS(\varkappa^* b) \in \mathcal{I}_q^0. \tag{7.10}$$

Proof. Again, since the class $\mathcal{I}_q^0$ is defined by the properties of operators cut-away from the origin, and for such operators (7.8) is just the usual formula of change of variables. □

7.2. **Edge operator symbols.** Now let $\mathbf{M}$ be a compact $k-1$-dimensional manifold, $\mathbf{K}$ be the cone over $\mathbf{M}$ and $\mathfrak{K} = L_2(\mathbf{K})$. Take a covering of $\mathbf{M}$ by co-ordinate neighbourhoods U_j, but instead of usual co-ordinate mappings of U_j to domains in the Euclidean space, we consider such mappings $\varkappa_j : U_j \to \Omega_j$, where Ω_j are domains on the unit $k-1$-dimensional sphere in $\mathbb{R}^k$. For an operator $\mathbf{a}$ in $\mathfrak{K}$ and cut-off functions κ_j, κ_j' with support in U_j, one defines the operator $\varkappa_j^*(\kappa_j \mathbf{a} \kappa_j')$ in $L_2(\mathbb{R}^k)$ obtained by the change of variables.

Definition 7.8. The operator-function $\mathbf{a}(y,\eta), (y,\eta) \in \mathbb{R}^m \times \mathbb{R}^m$ with values being operators in $\mathfrak{K}$, belongs to $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K})$ if each of operator-functions $\varkappa_j^*(\kappa_j \mathbf{a} \kappa_j')$ belongs to $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K}_0)$.

The following theorem follows automatically from Propositions 7.2, 7.4, 7.6, 7.7.

Theorem 7.9. *The class $\mathfrak{S}^0(\mathbb{R}^m, \mathfrak{K})$ is well-defined, i.e. its definition does not depend on the choice made in the construction. This class is an *-algebra.*

We denote by $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^m, \mathfrak{K})$ the algebra obtained by attaching the unit to $\mathfrak{S}^0$. Thus $\mathfrak{S}$ consists of operator symbols of the form $\mathbf{1} + \mathbf{b}$, $\mathbf{b} \in \mathfrak{S}^0$.

Let us compare the algebra $\mathfrak{S}$ with the algebra of edge operator symbols considered, e.g. in [32, 33, 34, 9]. By usual passing to polar co-ordinates (it was called 'conification of calculus' in [32]) one can see that the main, pseudodifferential part of our symbols is the same as the Mellin part in the usual edge symbol algebra. This latter algebra is constructed in such way that it is the smallest possible *-algebra containing Mellin symbols: thus Green operators arise. On the other hand, our algebra $\mathfrak{S}$ is constructed as the *largest reasonable* algebra containing Mellin symbols. It, surely, contains Green operators since the latter belong to $\mathcal{I}_q^0$.

Now we can apply our index formulas to the operators with symbols in $\mathfrak{S}$, thus generalyzing index theorems from [31, 8]

Theorem 7.10. *Let $\mathbf{a}$ be a symbol in $\mathfrak{S}$, elliptic in the sense of Sect. 3, i.e. $\mathbf{a}(y,\eta)$ is invertible for $|\eta|$ large enough and this inverse is uniformly bounded for such η. Then the pseudodifferential operator $\mathcal{A}$ with symbol $\mathbf{a}$ is Fredholm in $L_2(\mathbb{R}^m; \mathfrak{K})$ and*

$$\operatorname{ind} \mathcal{A} = c_{m,l}\alpha_{m,l}(\mathbf{a}) = c_{m,l}\alpha'_{m,l}(\mathbf{a}), \tag{7.11}$$

where $\alpha_{m,l}(\mathbf{a}), \alpha'_{m,l}(\mathbf{a})$ are given in (5.12),(5.13), and l is any integer such that $2m + 2l - 1 > k$. Moreover, if $\mathbf{r}(y,\eta)$ is a regularizer for $\mathbf{a}$ such that $\mathbf{ar} - \mathbf{1}, \mathbf{ra} - \mathbf{1}$ belong to trace class and have a compact support in η, η then

$$\operatorname{ind} \mathcal{A} = ((2\pi i)^m m!)^{-1} \int_{\mathbb{R}^m \times \mathbb{R}^m} ch(\operatorname{ind} \mathbf{a}), \tag{7.12}$$

where

$$ch(\operatorname{ind} \mathbf{a}) = \operatorname{tr}((d\mathbf{r} d\mathbf{a} + (\mathbf{r} d\mathbf{a})^2)^m). \tag{7.13}$$

Proof. The formula (7.11) is a particular case of Theorem 5.5. As for (7.12), it is obtained by the same way as (6.10), by a homotopy. In fact, both parts in (7.12) are invariant under homotopy of elliptic symbols. The homotopy (6.11), with N large enough, transforms the symbol $\mathbf{a}$ to such symbol to which the formula (7.11) with $l = 0$ can be applied. For the latter symbol, (7.11) gives (7.13) by means of Stokes formula, say, like in [6, 31]. □

7.3. **Ellipticity.** Uniform ellipticity (i.e., invertibility) of the scalar symbol of the operator is, obviously, a necessary condition of ellipticity of the operator symbol $\mathbf{a}$. This condition is, as it is well known, not sufficient. In our abstract setting, one cannot give explicit sufficient conditions of ellipticity of $\mathbf{a}$ without imposing some extra structure on 'Green symbols' in the class $\mathcal{I}_q^0$. We describe here one of possibilities to arrange such structure, still without restricting oneselves to any concrete analytical representation of Green symbols, however, modelling, on the abstract level, the properties of Green symbols in analytical constructions. Consider the one-parametric dilation group $(\mu(t)v)(z) = v(t^{-1}z)$ in $\mathfrak{K} = L_2(\mathbf{K})$. Fix a collection of co-ordinate neighbourhoods on the cone, corresponding diffeomorphisms $\varkappa_j$ and directional cut-offs κ_j, κ_j', as in Proposition 7.7, so that $\{k_j\}$ form a partition of unity. Thus, each $\mathbf{c}_j = \varkappa_j^*(\kappa_j \mathbf{a} \kappa_j')$ is a pseudodifferential operator with discontinuous symbol $c_j(x, \xi)$ plus a symbol from $\mathcal{I}_q^0$. Let $\Phi_j(y, \omega, \eta, \zeta)$ be limit values of the symbol c_j, as in (7.1). Construct the operator symbol acting in $\mathfrak{K}_0$: $\mathbf{a}_j(y, \eta) = OS(\Phi_j(y, z/|z|, \eta, \zeta))$ (so this is a sort of freezing the scalar symbol at the edge). This symbol possesses the skew-homogeneity property: $\mathbf{a}_j(y, t\eta) = \mu(t)^{-1}\mathbf{a}_0(y, \eta)\mu(t),\ t > 0,\ |\eta| \geq \delta$. Consider the Green part of the symbol $\mathbf{a}$: i.e., $\mathbf{g}(y, \eta) = \mathbf{a}(y, \eta) - \sum(\varkappa_j^*)^{-1}OS(c_j(y, z, \eta, \zeta))$. Suppose that there exists a limit in norm operator topology $\lim_{t\to\infty} \mu(t)\mathbf{g}(y, t\eta)\mu(t^{-1}) = \mathbf{g}_0(y, \eta)$. The sum $\mathbf{a}_0 + \mathbf{g}_0$ possesses skew-homogeneity property and plays the same role as the 'indicial family' in [18] or 'the edge symbol' in [33].

Proposition 7.11. *Invertibility of* $\mathbf{a}_0 + \mathbf{g}_0$ *for* $|\eta| > \delta$*, together with ellipticity of the scalar symbol, form necessary and sufficient conditions for ellipticity of* $\mathbf{a}$.

Proof. In more concrete situations, such results were established many times, see, e.g., [23, 24, 33, 9, 39] etc. In our case, the reasoning goes quite similarly, so we give just the skeleton of the proof. Take a cut-off function ψ which equals 1 near the vertex of the cone and construct a regularizer to $\mathbf{a}$ in the form

$$\mathbf{r}(y, \eta) = \psi(\mathbf{a}_0 + \mathbf{g}_0)^{-1} + (1 - \psi)\mathbf{r}_0(y, \eta), \tag{7.14}$$

where $\mathbf{r}_0 = OS(a(x, \xi)^{-1})$. If the support of ψ is taken small enough, so that on its support the pseudodifferential symbol a_j are sufficiently close, together with several derivatives, to their limit values as $|z| \to 0$, then the operator symbols $\mathbf{ra} - \mathbf{1}, \mathbf{ar} - \mathbf{1}$ have norm smaller that $1/2$ for η large enough, which guarantees invertibility. Necessity (which we do not need here) is established also in a standard way. □

7.4. **Manifolds with edge, boundary and co-boundary operators.** The symbols considered above act in the Hilbert space $\mathfrak{K}$ of functions defined on a *non-compact* cone $\mathbf{K}$. A somewhat different situation one encounters when considering a *compact* manifold with an edge.

Let $\mathbf{N}$ be a k–dimensional compact manifold with a cone singularity. This means that $\mathbf{N}$ has the structure of a smooth manifold everywhere except the cone vertex z^0.

In other words, $\mathbf{N}$ is the union of a compact manifold $\mathbf{N}_0$ with boundary $\mathbf{M}$, and the finite cone with base $\mathbf{M}$, $\mathbf{K}_0 = (\mathbf{M} \times [0, 1)/(\mathbf{M} \times \{0\})$ (which we consider as a part of the infinite cone $\mathbf{K}$, as above). These two parts are smoothly glued together over $\mathbf{M} \times (\frac{1}{2}, 1)$. The manifold $\mathbf{X}$ with edge is $\mathbb{R}^m \times \mathbf{N}$. We consider operator symbols $\mathbf{a}(y, \eta)$ acting in $\mathfrak{K} = L_2(\mathbf{N})$, glued together from a symbol in $\mathfrak{S}^0(\mathbb{R}^m, L_2(\mathbf{K}))$ supported in $\mathbf{K}_0$ and an operator symbol acting in $L_2(\mathbf{N}_0)$ corresponding to a usual elliptic pseudodifferential operator on $\mathbb{R}^m \times \mathbf{N}_0$, with symbol, stabilyzing in y. Such symbols belong to the class $\mathcal{S}^0_q(\mathbb{R}^m \times \mathbb{R}^m, L_2(\mathbf{N}))$ of Sect. 3, $q > k$, and thus the results of Sect. 4, 5 apply. In particular, the index formulas (7.11), (7.12) hold for elliptic operators in this class. The ellipticity conditions are the same as in Proposition 7.11.

Finally, we show how operator symbols including boundary and co-boundary operators fit into our abstract scheme.

In the above situation, let $\mathbf{P1}(y, \eta)$ be an operator acting from $\mathbb{C}^p$ to $L_2(\mathbf{N})$, $\mathbf{s}(y, \eta)$ be an operator acting from $\mathbb{C}^s$ to $L_2(\mathbf{N})$, and $\delta(y, \eta)$ be a $p \times s$ matrix, $(y, \eta) \in (\mathbb{R}^m \times \mathbb{R}^m)$. We suppose that the estimates of the form (3.1) holds for $\mathbf{P1}, \mathbf{q}, \mathbf{d}$, with $\gamma = 0$; due to finite rank of operators, the estimates of the form (3.2) hold automatically, with any given q.

We construct the composite symbol $\tilde{\mathbf{a}}(y, \eta) = \begin{pmatrix} \mathbf{a}(y, \eta) & \mathbf{P1}(y, \eta) \\ \mathbf{s}(y, \eta) & \mathbf{d}(y, \eta). \end{pmatrix}$ acting from $L_2(\mathbf{N}) \oplus \mathbb{C}^p$ to $L_2(\mathbf{N}) \oplus \mathbb{C}^s$. In order to apply the results of Sect. 3-5 to this symbol, fix an operator $\mathbf{u}$ which establishes an isometry of Hilbert spaces $L_2(\mathbf{N}) \oplus \mathbb{C}^s$ and $L_2(\mathbf{N}) \oplus \mathbb{C}^p$. For the operator symbol $\mathbf{b}(y, \eta) = \mathbf{u}\tilde{\mathbf{a}}(y, \eta)$, the start and target Hilbert spaces are now the same. Provided the symbol $\mathbf{b}$ is elliptic, the index formulas of the form (7.11), (7.12) hold for the corresponding pseudodifferential operator. Therefore, such formulas hold for the pseudodifferential operators with elliptic symbols of the form $\tilde{\mathbf{a}}$.

References

[1] B. Blackadar, *K-theory for Operator Algebras*, Springer, NY, 1986.

[2] A. Connes, *Noncommutative Differential geometry*, Publ. Math. I.H.E.S, Vol. 62, 1985, pp. 257-360.

[3] A. Connes, *Noncommutative Geometry*, Academic Press, NY, 1994.

[4] C. Dorschfeldt, *Algebras of pseudodifferential operators near edge and corner singularities*, Whiley-VCH Verlag, Berlin, 1998.

[5] C.Dorschfeldt, B.-W. Schulze, *Pseudo-differential operators with operator-valued symbols in the Mellin-edge-approach*, Ann. Global Anal. Geom., Vol. 12, 1994, pp. 135–171.

[6] B. Fedosov, *Analytic formulas for the index of elliptic operators*, Trans. Moscow Math. Soc., Vol. 30, 1974, pp. 159–240.

[7] B. Fedosov, *Analytic formulas for the index of an elliptic boundary value problem*, Math. USSR Sbornik, Vol. 22, pp. 61-90, 1974. Vol. 24, pp. 511-535, 1974. Vol 30, pp. 341-359, 1976.

[8] B. Fedosov, B.-W. Schulze, *On the index of elliptic operators on a cone*, Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras, pp. 348–372, Akademie Verlag, Berlin, 1996.

[9] B.Fedosov, B.-W. Schulze, N. Tarkhanov, *On the index of elliptic operators on a wedge*, J. Funct. Anal., Vol. 157, 1998, pp. 164–209.

[10] V. Guillemin, *Gauged Lagrangian distributions*, Adv. Math., Vol. 102, 1993, pp. 184–201.

[11] L. Hörmander, *On the index of pseudodifferential operators* Elliptische Differentialgleichungen, Band II, Akademie-Verlag, pp. 127-146, 1969.

[12] B. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., Vol. 127, 1972.

[13] J.J. Kohn, L. Nirenberg, *An algebra of pseudodifferential operators*, Commun. Pure Appl. Math., Vol. 18, 1965, pp. 269–305.

[14] M. Karoubi, *Homologie cyclique et K-théorie*, Astérisque, No 149, 1987.
[15] G. Luke, *Pseudodifferential operators on Hilbert bundles*, Journal Diff. Equat., Vol. 12, pp. 566-589, 1972.
[16] R. Melrose, *The eta invariant and families of pseudodifferential operators*, Math. Res. Lett., Vol. 2, 1995, pp. 541–561.
[17] R. Melrose, *Fibrations, compactifications and algebras of pseudodifferential operators*, Partial Diff. Equations and Math. Physics. The Danish-Swedish Analysis Seminar, Birkhäuser, 1996, pp. 246-261.
[18] R. Melrose, P. Piazza, *Analytic K-theory for manifolds with corners*, Advances Math., Vol. 92, 1992, pp. 1–27.
[19] B. Plamenevsky, *Algebras of pseudodifferential operators* Mathematics and its Applications (Soviet Series), 43, Kluwer Academic Publishers Group, Dordrecht, 1989.
[20] B. Plamenevsky, G. Rozenblum, *On the index of pseudodifferential operators with isolated singularities in the symbols* (Russian), Algebra i Analiz, Vol. 2, 1990, No. 5, pp. 165–188. Translation in Leningrad Math. J., Vol. 2, 1991, 1085–1110.
[21] B Plamenevsky, G. Rozenblum, *Topological characteristics of the algebra of singular integral operators on the circle with discontinuous symbols* (Russian), Algebra i Analiz, Vol. 3 (5), 1991, pp. 155–167. Translation in St. Petersburg Math. J., Vol. 3, 1992.
[22] B. Plamenevsky, G. Rozenblum, *Pseudodifferential operators with discontinuous symbols: K-theory and index formulas* (Russian), Funktsional. Anal. i Prilozhen., Vol. 26, 1992 No. 4, pp. 45–56. Translation in Functional Anal. Appl., Vol. 26, 1993, pp. 266–275.
[23] B. Plamenevsky, V. Senichkin, *Representations of an algebra of pseudodifferential operators with multidimensional discontinuities in the symbols* (Russian), Izv. Akad. Nauk SSSR Ser. Mat., Vol. 51, 1987, pp. 833–859. Translation in Math. USSR-Izv. Vol. 31, 1988, pp. 143–169.
[24] B.Plamenevsky, V. Senichkin, *Solvable operator algebras*, St. Petersburg Math. J., Vol. 6, 1995, pp. 895-968.
[25] B. Plamenevsky, G. Tashchiyan, *A convolution operator in weighted spaces* (Russian), Nonlinear equations and variational inequalities. Linear operators and spectral theory, pp. 208–237, 1990. Probl. Mat. Anal., 11, Leningrad. Univ. Leningrad.
[26] G. Rozenblum, *On a class of pseudodifferential operators with operator-valued symbols*, Preprint Linköping University, 1992.
[27] G. Rozenblum, *Index formulae for pseudodifferential operators with discontinuous symbols*, Ann. Global Anal. Geom., Vol. 15, 1997, pp. 71–100.
[28] G. Rozenblum, *The index of cone Mellin operators*, Geometric aspects of partial differential equations (Roskilde, 1998), 1999 pp. 43–49, Contemp. Math., 242, Amer. Math. Soc., Providence, RI.
[29] E. Schrohe, *Fréchet Algebras of Pseudodifferential Operators and Boundary Problems*, 2000, Birkhäuser, Basel.
[30] E. Schrohe, *A short introduction to Boutet de Monvel's calculus*, Preprint 2000/03, Univ. of Potsdam, 2000.
[31] E. Schrohe, J. Seiler, *An analytic index formula for pseudodifferential operators on wedges*, Preprint, Max-Planck-Institut für Mathematik, 96-172, 1996.
[32] B.-W. Schulze, *Pseudodifferential operators on manifolds with singularities*, 1991, North Holland, Amsterdam.
[33] B.-W. Schulze, *Pseudo-differential Boundary Value Problems, Conical Singularities, and Asymptotics,* 1994, Academie-Verlag, Berlin.
[34] B.-W. Schulze, *Boundary Value Problems and Singular Pseudodifferential Operators*, J. Whiley, Chichester, 1998.
[35] B.-W. Schulze, *Operator algebras with symbol hierarchies on manifolds with singularities*, Preprint 99/30, Universitet Potsdam, 1999.
[36] B.-W. Schulze, B. Sternin, V. Shatalov, *Differential Equations on Singular Manifolds: Semiclassical Theory and Operator Algebra*, 1999, Wiley, Weinheim.
[37] B.-W. Schulze, B. Sternin, V. Shatalov, *On the index of differential operators on manifolds with conical singularities*, Ann. Global Anal. Geom., Vol. 16, 1998, pp. 141–172.
[38] B.-W Schulze, N. Tarkhanov, *Pseudodifferential operators on manifolds with corners*, preprint 2000/13, Univ. of Potsdam, 2000.
[39] V. Senichkin, *Pseudodifferential operators on manifolds with edges*, J. Math. Sci., Vol. 73, 1995, pp. 711–747.

Grigori Rozenblum
Dept. of Mathematics, Chalmers University of Technology,, 412 96 Göteborg, Sweden
E-mail address: grigori@math.chalmers.se