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Existence and uniqueness of classical solutions to certain nonlinear integro-differential Fokker-Planck type equations *

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Abstract

A nonlinear Fokker-Planck type ultraparabolic integro-differential equation is studied. It arises from the statistical description of the dynamical behavior of populations of infinitely many (nonlinearly coupled) random oscillators subject to "mean-field" interaction. A regularized parabolic equation with bounded coefficients is first considered, where a small spatial diffusion is incorporated in the model equation and the unbounded coefficients of the original equation are replaced by a special "bounding" function. Estimates, uniform in the regularization parameters, allow passing to the limit, which identifies a classical solution to the original problem. Existence and uniqueness of classical solutions are then established in a special class of functions decaying in the velocity variable.

Introduction

In this paper, we establish the existence and uniqueness of *classical* solutions to a certain nonlinear Fokker-Planck type ultraparabolic integro-differential equation which is encountered in the statistical description of the dynamical behavior of populations of infinitely many (nonlinearly coupled) random oscillators subject to "mean-field" interaction (the space-integral term in the equation accounts for this). Such a model generalizes somehow and improves the results obtained by the celebrated Kuramoto model [10, 11, 17], which describes a variety of phenomena, in particular self-synchronization, in subject areas ranging from biology and medicine to physics and neural networks. Space-degenerate diffusion suggests to consider a regularized equation, where a small spatial diffusion is incorporated into the model equation. The peculiarities of the problem are numerous and include (besides degeneracy) unbounded coefficients, space-periodicity of the sought solution, and a nonlinear space-integral term. Estimates, uniform in regularization parameters, allow passing to the limit, which

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identifies a classical solution to the original problem. Existence and uniqueness of classical solutions are then established in a certain class of functions decaying in the velocity variable. Below, precise estimates, established in [12] for the decay of convolutions of continuous functions with fundamental solutions to linear parabolic equations on unbounded domains, are used repeatedly as an essential tool for general linear parabolic equations in $\mathbb{R}^n$.

Motivation for studying such equations can be provided as follows. Numerous phenomena, pertaining to physics, biology, medicine, and neural networks, are reasonably described in terms of large populations of nonlinearly coupled, often noisy, oscillators. A mathematical model for all these problems is given by a large system of possibly stochastic nonlinearly coupled ordinary differential equations. In the limiting case of infinitely many random oscillators, when the interaction is of the so-called "mean-field" type, a single nonlinear parabolic integro-differential equation, containing an integral term, was derived by Kuramoto [10] (see also [17]). However, an improvement of the finite-dimensional model, accounting for certain observed features, led to the introduction of second-order derivatives on the left-hand side of the above-mentioned system of stochastic differential equations [1, 2, 8, 18, 19]. This suggested that a nonlinear partial integro-differential equation, more general than Kuramoto's equation, could be derived by a similar limiting procedure [2].

Such a new model equation is a Fokker-Planck type equation which, with normalized parameters, takes the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} + \frac{\partial}{\partial \omega}\left[\left(\omega - \Omega - \mathcal{K}_\rho(\theta,t)\right)\rho\right] - \omega\frac{\partial \rho}{\partial \theta},$$

where we set, for short,

$$\mathcal{K}_\rho(\theta,t) := K\int_{-G}^{G}\int_{-\infty}^{+\infty}\int_{0}^{2\pi} g(\Omega')\sin(\theta'-\theta)\rho(\theta',\omega',t,\Omega')\,d\theta' d\omega' d\Omega',$$

with a given "frequency distribution density" function $g(\Omega) \in L^1[-G,G]$, and a "coupling strength" constant $K > 0$. This terminology refers to the physical meaning of $g(\Omega)$ and K, cf. [1, 2]. We look for a *classical* solution, $\rho(\theta,\omega,t,\Omega)$, to this equation, in the unbounded slab $Q_T := \{(\theta,\omega,t,\Omega) \in [0,2\pi]\times\mathbb{R}\times[0,T]\times[-G,G]\}$, which should be 2π-periodic in θ (θ being an angle), nonnegative, and normalized, i.e.,

$$\int_{0}^{2\pi}\int_{-\infty}^{+\infty} \rho(\theta,\omega,t,\Omega)\,d\omega d\theta = 1,$$

for every $t \in [0,T]$ and every $\Omega \in [-G,G]$. These properties are related to the physical meaning referred to above.

In [12, 13], the present authors have proved *existence* of *strong* solutions to such a problem. In this paper, we address the problem of *existence* and *uniqueness* of *classical* solutions in a natural class of functions, under *additional* requirements on the initial data.

Apart from the underlying physical meaning, this problem is interesting from the mathematical point of view for the following sensible reasons, which, occurring all at the same time, make the problem highly nonstandard (even from the point of view of the qualitative theory of *linear* partial differential equations):

(1) The governing equation is of the *second order* with respect to ω, but only of the *first order* with respect to θ and t. Therefore (disregarding the integral term, $\mathcal{K}_\rho$), this equation is neither of the parabolic nor of the hyperbolic type.

(2) The equation is considered in the slab Q_T, which is *unbounded*. The variable ω appears twice in the equation as a coefficient, and is unbounded in Q_T. This fact gives rise to typical *singularity* phenomena.

(3) The coefficient ω, multiplying the time-like derivative ρ_θ, *changes its sign* in Q_T.

(4) The equation contains the integral term $\mathcal{K}_\rho$ extended over an *unbounded* domain.

(5) There is a variable, Ω, the natural frequency of oscillators, with respect to which no derivative appears, but which plays the role of a coefficient of the equation and of an integration variable at the same time.

(6) We are interested *only* in solutions *periodic* in θ, while the governing equation contains only the first (time-like) derivative with respect to θ.

Therefore, the results available in the literature concerning parabolic equations [3, 4, 5, 9, 14], or even integro-parabolic equations [16], cannot be applied to this case. The idea here is to "regularize" the equation, introducing an additional diffusive term with respect to θ (with a small parameter ε in front of it), since such an equation should be considered fully degenerate in θ. We also replace the two *unbounded* coefficients ω with a special "bounding" function, $F_N(\omega)$, in order to face, rather, families of *parabolic* equations with *bounded* coefficients.

Here is the plan of the paper. In Section 1, we formulate precisely the problem for both the original and the regularized equation. An existence theorem of classical solutions for the *regularized* problem, which has been established earlier by the authors, is then recalled, and a new basic lemma is proved. In Section 2, existence of classical solutions to the original problem is established. Finally, in Section 3, we prove the uniqueness of classical solutions. The article ends with a short summary, which highlights the main results of the paper, i.e., existence and uniqueness of classical solutions in certain classes of *decaying* functions.

1 The statement of the problem and its regularization

The problem considered here is the following. Find a function $\rho(\theta,\omega,t,\Omega)$, satisfying, in the classical sense, the equation

$$\frac{\partial\rho}{\partial t}=\frac{\partial^2\rho}{\partial\omega^2}+\frac{\partial}{\partial\omega}\left[(\omega-\Omega-\mathcal{K}_\rho(\theta,t))\,\rho\right]-\omega\frac{\partial\rho}{\partial\theta}, \tag{1.1}$$

in the unbounded slab Q_T, subject to the boundary and initial data

$$\rho|_{\theta=0}=\rho|_{\theta=2\pi}, \tag{1.2}$$

$$\rho|_{t=0}=\rho_0(\theta,\omega,\Omega). \tag{1.3}$$

The function $\mathcal{K}_\rho(\theta,t)$ is that defined in the previous section.

Definition 1.1 By a "classical solution" to the problem (1.1)–(1.3) in Q_T, we mean a function $\rho(\theta,\omega,t,\Omega)$ which:

(1) is continuous in Q_T and has the continuous partial derivatives ρ_θ, ρ_ω, $\rho_{\omega\omega}$, and ρ_t in $Q_T\cap\{t>0\}$;

(2) is such that the integral $\mathcal{K}_\rho(\theta,t)$ defined on the set $[0,2\pi]\times(0,T]$ converges as a Lebesque integral;

(3) satisfies equation (1.1) in $Q_T\cap\{t>0\}$ as well as the periodicity boundary condition (1.2) and the initial data (1.3) in Q_T, as a function possessing the properties of items (1) and (2) above.

Let $l_0\geq 0$ be an integer and $\alpha_0\in(0,1)$ a real constant. We shall make the following

Assumption 1.2 *The initial profile $\rho_0(\theta,\omega,\Omega)$ is a function: (a_1) belonging to the Hölder space $C^{l_0+\alpha_0}(Q)$, where $Q:=\{(\theta,\omega,\Omega)\in\mathbb{R}\times\mathbb{R}\times[-G,G]\}$; (a_2) 2π-periodic in θ; (a_3) nonnegative, $\rho_0(\theta,\omega,\Omega)\geq 0$ in Q; (a_4) normalized for every $\Omega\in[-G,G]$, i.e.,*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty}\rho_0(\theta,\omega,\Omega)\,d\omega d\theta=1;$$

and (a_5) with exponential decay in ω at infinity (along with some of its partial derivatives), according to the following estimate: For a given integer $l_0\geq 0$, the inequalities

$$\sup_{\theta\in\mathbb{R},\Omega\in[-G,G]}|D^{l_1,l_2,l_3}_{\theta,\omega,\Omega}\rho_0(\theta,\omega,\Omega)|\leq C_0\,\mathrm{e}^{-M_0\omega^2}$$

hold for $\omega\in\mathbb{R}$ and $l_1+l_2+l_3\leq l_0$, with $C_0,M_0>0$ constants and $l_i\geq 0$ $(i=1,2,3)$ integers. Here and in the sequel, D^l_ξ (l and ξ being multi-indices) stands for the differential operator of order l_i with respect to the variable ξ_i, for all i's.

As mentioned above, to study (1.1)–(1.3) we perform a *parabolic regularization* of the governing equation (1.1). Moreover, *to overcome* the problem of facing *unbounded coefficients* in Q_T (cf. ω, appearing twice in (1.1)), we replace ω in (1.1) with an arbitrary fixed bounded ("bounding") function, $F_N(\omega) \in C^{5+\alpha_0}(\mathbb{R})$, with $\alpha_0 \in (0,1)$ (see property (a_1) of ρ_0), such that

$$F_N(\omega) = \begin{cases} \omega & \text{for } |\omega| \le N, \\ \operatorname{sgn}(\omega)(N+1) & \text{for } |\omega| \ge N+1, \end{cases} \qquad \sup_{N>0} \|F_N'(\omega)\|_{C^3(\mathbb{R})} < \infty.$$

Instead of (1.1) we therefore study first its *parabolic regularization*, i.e., the (ε, N)-family of equations

$$\frac{\partial \rho^{\varepsilon,N}}{\partial t} = \frac{\partial^2 \rho^{\varepsilon,N}}{\partial \omega^2} + \varepsilon \frac{\partial^2 \rho^{\varepsilon,N}}{\partial \theta^2} + \frac{\partial}{\partial \omega}(F_N \rho^{\varepsilon,N}) - (\Omega + \mathcal{K}_{\rho^{\varepsilon,N}}) \frac{\partial \rho^{\varepsilon,N}}{\partial \omega} - F_N \frac{\partial \rho^{\varepsilon,N}}{\partial \theta} \quad (1.4)$$

satisfied by $\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$ in $Q_T \cap \{t > 0\}$. We consider as initial data for $\rho^{\varepsilon,N}$, for every $\varepsilon > 0$ and every $N > 0$, the initial profile in (1.3). Having added a term with the second-order derivative with respect to θ, we modify the periodic boundary condition (1.2) as

$$(\rho^{\varepsilon,N}, \rho_\theta^{\varepsilon,N})|_{\theta=0} = (\rho^{\varepsilon,N}, \rho_\theta^{\varepsilon,N})|_{\theta=2\pi} \quad (1.5)$$

for $\omega \in \mathbb{R}$, $t \in (0,T]$, and $\Omega \in [-G,G]$.

The *regularized* problem (1.4), (1.5), (1.3) has been analyzed in [12, 13]. More precisely, the following existence theorem was proved:

Theorem 1.3 *Suppose the data of problem (1.4), (1.5), (1.3) satisfy Assumption 1.2 with* $l_0 = 2$. *Then, for each* $\varepsilon > 0$ *and each* $N > 0$, *there exists a* classical *solution* $\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$ *to the problem (1.4), (1.5), (1.3) in* Q_T. *Such a solution*

(1) *is a continuous function of all variables in* Q_T, *along with its partial derivatives* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}$ *in* Q_T, *for* $l_1+l_2+2l_3+l_4 \le 2$, *and* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}$ *in* $Q_T \cap \{t > 0\}$ *for* $l_1+l_2+2l_3+l_4 \le 4$;

(2) *satisfies, in the classical sense, equation (1.4) in* Q_T, *the boundary data (1.5) in* Q_T, *along with the additional requirement* $\rho_{\theta\theta}^{\varepsilon,N}|_{\theta=0} = \rho_{\theta\theta}^{\varepsilon,N}|_{\theta=2\pi}$, *and the initial data (1.3) in* $Q_T \cap \{t = 0\}$;

(3) *is nonnegative in* Q_T, *and normalized as*

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} \rho^{\varepsilon,N}(\theta,\omega,t,\Omega)\, d\omega d\theta = 1$$

for $t \in [0,T]$ *and* $\Omega \in [-G,G]$;

(4) *has an exponential decay at infinity in* ω*, according to*

$$\sup_{\theta\in[0,2\pi],t\in[0,T],\Omega\in[-G,G]} |D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)| \le C_\varepsilon\,\mathrm{e}^{-M_\varepsilon\omega^2}$$

for $l_1+l_2+2l_3+l_4 \le 2$ *and* $\omega\in\mathbb{R}$*, where the constants* $C_\varepsilon, M_\varepsilon > 0$ *depend on* ε*,* N*,* G*,* T*,* $K\|g\|_{L^1[-G,G]}$*,* C_0*, and* M_0*; moreover,*

$$\sup_{\theta\in[0,2\pi],\Omega\in[-G,G]} |D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)| \le \frac{C_\varepsilon}{\sqrt{t}}\,\mathrm{e}^{-M_\varepsilon\omega^2}$$

for $l_1+l_2+2l_3+l_4 = 3$*,* $\omega\in\mathbb{R}$*, and* $t\in(0,T]$*, with the same constants* $C_\varepsilon, M_\varepsilon>0$ *given above; and*

$$\sup_{\theta\in[0,2\pi],\Omega\in[-G,G]} |D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)| \le \frac{C_\varepsilon}{t}$$

for $l_1+l_2+2l_3+l_4 = 4$*,* $\omega\in\mathbb{R}$*, and* $t\in(0,T]$*, with the same* C_ε *given above;*

(5) *is such that the function* $\mathcal{K}_{\rho^{\varepsilon,N}}(\theta,t)$ *is continuous in* $\Pi := [0,2\pi]\times[0,T]$ *along with the partial derivatives* $D^{k,l}_{t,\theta}\mathcal{K}_{\rho^{\varepsilon,N}}$ *with* $k\le 1$*,* $l\ge 0$*, and the estimate*

$$\|\mathcal{K}_{\rho^{\varepsilon,N}}\|_{C^1(\Pi)} + \sup_{k\le 1,l\ge 0}\|D^{k,l}_{t,\theta}\mathcal{K}_{\rho^{\varepsilon,N}}\|_{C(\Pi)} \le C$$

holds, where the constant C *is independent of* $\varepsilon\in(0,1)$ *and* $N>0$*.*

Remark 1.4 From now on, we use for short the notation D for any derivative $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}$ with $l_1+l_2+2l_3+l_4\le 2$.

We first prove the following basic lemma, using the additional properties of the initial data, that is $l_0=4$ rather than $l_0=2$ in Assumption 1.2, cf. [13]:

Lemma 1.5 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with* $l_0=4$*. Then*

(1) *for arbitrary fixed values of the parameters* $t\in[0,T]$ *and* $\Omega\in[-G,G]$*, the functions* $D\rho^{\varepsilon,N}$ *can be estimated* uniformly *as*

$$\|D\rho^{\varepsilon,N}\|_{W_2^{2,2}} \le \tilde{C}$$

on the set $\{(\theta,\omega)\in[0,2\pi]\times\mathbb{R}\}$*, where the constant* $\tilde{C}$ *is independent of* $\varepsilon\in(0,1)$*,* $N>0$*,* $t\in[0,T]$*, and* $\Omega\in[-G,G]$*;*

(2) *for every fixed* $t\in[0,T]$*, the functions* $D\rho^{\varepsilon,N}$ *satisfy the* uniform *estimates*

$$\|D\rho^{\varepsilon,N}\|_{W_2^{2,2,2}} \le C$$

on the set $\{(\theta,\omega,\Omega)\in[0,2\pi]\times\mathbb{R}\times[-G,G]\}$*, where the constant* C *is independent of* $\varepsilon\in(0,1)$*,* $N>0$*, and* $t\in[0,T]$*;*

(3) *for every fixed* $\Omega \in [-G, G]$, *the functions* $D\rho^{\varepsilon,N}$ *are estimated* uniformly *as*

$$\|D\rho^{\varepsilon,N}\|_{W_2^{2,3,1}} \le C$$

on the set $\{(\theta, \omega, t) \in [0, 2\pi] \times \mathbb{R} \times [0, T]\}$, *where the constant* C *is independent of* $\varepsilon \in (0, 1)$, $N > 0$, *and* $\Omega \in [-G, G]$.

Proof. Define $Q_T^* := \{(\theta, \omega, t, \Omega) \in \mathbb{R} \times \mathbb{R} \times (0, T) \times [-G, G]\}$, and consider the functions $\rho^{\varepsilon,N}$ in $\overline{Q_T^*}$ as 2π-periodic functions of θ. In view of Theorem 1.3, the so-extended functions $\rho^{\varepsilon,N}$ are bounded classical solutions to the corresponding Cauchy problem (1.4), (1.3) in $\overline{Q_T^*}$, with Ω a fixed parameter in $[-G, G]$. We then consider the corresponding Cauchy problems for the derivatives $D\rho^{\varepsilon,N}$ in $\overline{Q_T^*}$. The equations satisfied by every derivative $D\rho^{\varepsilon,N}$ differ from (1.4) only by low-order terms and right-hand sides. Moreover, these *additional* low-order terms have *uniformly* bounded coefficients, and the right-hand sides have already been *uniformly* estimated in some spaces. Therefore, we obtain for $D\rho^{\varepsilon,N}$ the same estimates as those established in [13] for $\rho^{\varepsilon,N}$. The method is based on differentiating both sides of the equations, multiplying by certain functions, integrating, and using Gronwall's lemma. The proof of Lemma 1.5 is thus similar to that of Theorems 2.2, 2.4, and 3.2 in [13], and hence is omitted here. □

Corollary 1.6 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with* $l_0 = 4$. *Then, the functions* $D\rho^{\varepsilon,N}$ *can be estimated uniformly as*

$$\|D\rho^{\varepsilon,N}\|_{W_2^{2,3,1,2}(Q_T)} \le C,$$

where the constant C *is independent of* $\varepsilon \in (0, 1)$ *and* $N > 0$.

The proof follows from items (2) and (3) of Lemma 1.5, by the definition of the anisotropic Sobolev spaces, see [6].

2 The existence theorem

In this section we prove one of the main results of the paper, that is existence of "decaying" classical solutions. The proof can be given using the additional properties of the initial data, determined by the choice $l_0 = 4$ instead of $l_0 = 2$ in the Assumption 1.2, cf. [13].

We now introduce some auxiliary notation and facts. Let $\mathcal{Q}$ be a domain in $\mathbb{R}^n$ (in particular, $\mathcal{Q}$ may be unbounded), $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a multi-index with $\lambda_i \in (0, 1]$ for $i = 1, 2, \dots, n$, and $[x, y]$ the straight-line segment joining the points $x, y \in \mathbb{R}^n$. Let e_i be the unit vector in $\mathbb{R}^n$ with the ith component equal to 1. For every function $u(x)$ and every parameter value $h \in \mathbb{R}$, set

$$\Delta_i(h)u(x) := \begin{cases} u(x + he_i) - u(x) & \text{if } [x, x + he_i] \subset \mathcal{Q}, \\ 0 & \text{if } [x, x + he_i] \not\subset \mathcal{Q}. \end{cases}$$

Let $C^\lambda(\mathcal{Q})$ denote the set of functions $u(x) \in C(\mathcal{Q})$ with the finite norm

$$\|u\|_{C^\lambda(\mathcal{Q})} := \|u\|_{C(\mathcal{Q})} + \sum_{i=1}^{n} \sup_{x\in\mathcal{Q}, h>0} \frac{|\Delta_i(h)u(x)|}{h^{\lambda_i}}.$$

The following two embedding results are well known, cf. [6, 7, 20]:

Lemma 2.1 *Let $1 < p < \infty$, and let $\mathcal{Q}$ be a bounded domain satisfying the strong l-horn condition* [6] *with the multi-index $l = (l_1, l_2, \dots, l_n)$, where $l_i > 0$, $i = 1, 2, \dots, n$, are integers. If the constant*

$$\Theta := 1 - \frac{1}{p}\Big(\frac{1}{l_1} + \frac{1}{l_2} + \dots + \frac{1}{l_n}\Big) \tag{2.1}$$

is positive, then the anisotropic Sobolev space $W_p^l(\mathcal{Q})$ is embedded in the anisotropic Hölder space $C^\lambda(\mathcal{Q})$ with the multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_i = \Theta l_i$ for $\Theta l_i < 1$ and $\lambda_i < 1$ for $\Theta l_i \ge 1$. Moreover, every function $u(x) \in W_p^l(\mathcal{Q})$ satisfies the inequality

$$\|u\|_{C^\lambda(\mathcal{Q})} \le C\|u\|_{W_p^l(\mathcal{Q})},$$

where the constant C is independent of $u(x)$.

Lemma 2.2 *Let $1 \le p < \infty$ and $\mathcal{Q}$ be a domain satisfying the l-horn condition* [6] (*in particular, $\mathcal{Q}$ may be unbounded*). *If $\Theta > 0$ in (2.1), then the anisotropic Sobolev space $W_p^l(\mathcal{Q})$ is embedded in $C(\mathcal{Q})$. Moreover, every function $u(x) \in W_p^l(\mathcal{Q})$ satisfies the inequality*

$$\|u\|_{C(\mathcal{Q})} \le C\|u\|_{W_p^l(\mathcal{Q})},$$

where the constant C is independent of $u(x)$.

Note that, by the extension theorem in [6] and Corollary 1.6, for $l_1 + l_2 + 2l_3 + l_4 \le 2$ there exist functions $u^{\varepsilon,N}_{l_1,l_2,l_3,l_4}(\theta,\omega,t,\Omega) \in W_2^{2,3,1,2}(\mathbb{R}^4)$ such that

$$u^{\varepsilon,N}_{l_1,l_2,l_3,l_4}(\theta,\omega,t,\Omega) = D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega) \quad \text{in } Q_T,$$
$$\|u^{\varepsilon,N}_{l_1,l_2,l_3,l_4}\|_{W_2^{2,3,1,2}(\mathbb{R}^4)} \le C,$$

where the constant C is independent of $\varepsilon \in (0,1)$ and $N > 0$. The following statement is a special case of [15, Theorem 7.7].

Lemma 2.3 *Suppose that a sequence $\{u_n\}_{n=1}^\infty$ is such that*

$$\|u_n\|_{W_2^{2,3,1,2}(\mathbb{R}^4)} \le C$$

for all $n = 1, 2, \dots$, where the constant C is independent of n. Then there exist a subsequence $\{u_{n_k}\}_{k=1}^\infty$ and a function $u \in W_2^{2,3,1,2}(\mathbb{R}^4)$ such that

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{W_2^{1,2,0,1}(\mathcal{Q})} = 0$$

for every bounded domain $\mathcal{Q} \subset \mathbb{R}^4$.

At this point, a uniform decay property of the derivatives $D\rho^{\varepsilon,N}$ can be established.

Lemma 2.4 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with* $l_0 = 4$, *and let* $a(\omega)$ *be any fixed positive twice-differentiable function, such that*

$$a(\omega) = M|\omega| \quad \text{for} \quad |\omega| \geq 1,$$

$$\|a'\|_{C^1(\mathbb{R})} < \infty, \quad \omega a'(\omega) \geq 0 \quad \text{and} \quad a''(\omega) \geq 0 \quad \text{for} \quad \omega \in \mathbb{R},$$

where $M > 0$ *is a parameter. Then, for arbitrary fixed values of the parameters* $t \in [0,T]$, $\Omega \in [-G,G]$, *and* $M > 0$, *the functions* $D\rho^{\varepsilon,N}$ *can be estimated uniformly as*

$$\|\mathrm{e}^{a(\omega)} D\rho^{\varepsilon,N}\|_{W_2^{2,2}} \leq C(M)$$

on the set $\{(\theta,\omega) \in [0,2\pi] \times \mathbb{R}\}$. *The constant* $C(M)$ *does not depend on* $\varepsilon \in (0,1)$, $N > 0$, $t \in [0,T]$, *nor* $\Omega \in [-G,G]$.

Proof. We only show how to establish the estimate for $\|\mathrm{e}^{a(\omega)}\rho^{\varepsilon,N}\|_{L^2}$ on the set $\{(\theta,\omega) \in [0,2\pi] \times \mathbb{R}\}$. All the other derivatives are estimated in a similar way, cf. [13].

Multiplying both sides of (1.4) by $\mathrm{e}^{2a(\omega)}\rho(\theta,\omega,t,\Omega)$, where we have set $\rho(\theta,\omega,t,\Omega) := \rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$, in order to simplify the notation, and integrating with respect to ω and θ, we conclude after simple transformations that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{2a(\omega)}\rho^2\,d\omega d\theta + \int_0^{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{2a(\omega)}(\rho_\omega^2 + \varepsilon\rho_\theta^2)\,d\omega d\theta \\
&= \int_0^{2\pi}\int_{-\infty}^{+\infty} [a'' + 2a'^2 - \omega a' + (\Omega + \mathcal{K}_\rho)a' + 1/2]\,\mathrm{e}^{2a(\omega)}\rho^2\,d\omega d\theta \\
&\leq \int_0^{2\pi}\int_{-\infty}^{+\infty} \big[\|a''\|_{C(\mathbb{R})} + 2\|a'\|_{C(\mathbb{R})}^2 \\
&\quad + (G + K\|g\|_{L^1[-G,G]})\|a'\|_{C(\mathbb{R})} + 1\big]\,\mathrm{e}^{2a(\omega)}\rho^2\,d\omega d\theta.
\end{aligned}$$

By Gronwall's lemma we obtain

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{2a(\omega)}\rho^2\,d\omega d\theta + 2\int_0^t\int_0^{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{2a(\omega)}(\rho_\omega^2 + \varepsilon\rho_\theta^2)\,d\omega d\theta dt \leq C,$$

where the constant C is independent of $\varepsilon > 0$, $N > 0$, $t \in [0,T]$, and $\Omega \in [-G,G]$. The lemma is thus proved. □

We are now ready to establish the following existence result:

Theorem 2.5 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with* $l_0 = 4$*. Then, there exists a* classical *solution,* $\rho(\theta,\omega,t,\Omega)$*, to the problem* (1.1)–(1.3) *in* Q_T*, such that:*

(1) $\rho(\theta,\omega,t,\Omega)$ *is a continuous bounded function in* Q_T *along with its partial derivatives* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho(\theta,\omega,t,\Omega)$ *for* $l_1+l_2+2l_3+l_4 \le 2$*; moreover, the derivatives* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho$ *with* $l_1+l_2+2l_3+l_4 \le 2$ *belong to the anisotropic Sobolev space* $W_2^{2,3,1,2}(Q_T)$ *and to the Hölder spaces* $C^{\lambda,\lambda,\frac{1}{12},\frac{1}{2}}(\mathcal{Q}_R)$ *for all* $\lambda \in (0,1)$ *and* $R > 0$*, where* $\mathcal{Q}_R := Q_T \cap \{\omega \in [-R,R]\}$*;*

(2) $\rho(\theta,\omega,t,\Omega)$ *satisfies equation (1.1) in the classical sense in* Q_T*, and satisfies the boundary data in (1.2) and the initial data in (1.3) as a continuous function in* Q_T*; moreover,* $(\rho_\theta,\rho_{\theta\theta})|_{\theta=0} = (\rho_\theta,\rho_{\theta\theta})|_{\theta=2\pi}$ *in* Q_T*;*

(3) $\rho(\theta,\omega,t,\Omega) \ge 0$ *in* Q_T *and is normalized as*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} \rho(\theta,\omega,t,\Omega)\,d\omega d\theta = 1$$

for all $t \in [0,T]$ *and* $\Omega \in [-G,G]$*;*

(4) for any value of the parameter $M > 0$*, there exists a constant* $C = C(M) > 0$ *such that the estimate*

$$|D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho(\theta,\omega,t,\Omega)| \le C\mathrm{e}^{-M|\omega|}$$

holds in Q_T *for* $l_1+l_2+2l_3+l_4 \le 2$*.*

A classical solution to the problem (1.1)–(1.3) *in* Q_T*, satisfying the item* (4)*, is unique.*

Proof. Consider the subdomain $\mathcal{Q}_{R,T_0,\Omega_0} := Q_T \cap \{\omega \in [-R,R]\} \cap \{t = T_0\} \cap \{\Omega = \Omega_0\}$. In view of item (1) of Lemma 1.5 and Lemma 2.1, for arbitrary fixed values of the parameters $T_0 \in [0,T]$ and $\Omega_0 \in [-G,G]$, the functions $D\rho^{\varepsilon,N}(\theta,\omega,T_0,\Omega_0)$ can be estimated *uniformly* as

$$\|D\rho^{\varepsilon,N}\|_{C^{\lambda_1,\lambda_2}(\mathcal{Q}_{R,T_0,\Omega_0})} \le C,$$

for any fixed $\lambda_1,\lambda_2 \in (0,1)$, where the constant C is independent of $\varepsilon \in (0,1)$, $N > 0$, T_0, and Ω_0. The constant C depends only on λ_1, λ_2, R, and the constant $\tilde{C}$ of Lemma 1.5. In particular, this implies that

$$\|D\rho^{\varepsilon,N}\|_{C(\mathcal{Q}_R)} \le C, \quad \frac{|D\rho^{\varepsilon,N}(\theta_1,\omega,t,\Omega) - D\rho^{\varepsilon,N}(\theta_2,\omega,t,\Omega)|}{|\theta_1-\theta_2|^{\lambda_1}} \le C, \tag{2.2}$$

$$\frac{|D\rho^{\varepsilon,N}(\theta,\omega_1,t,\Omega) - D\rho^{\varepsilon,N}(\theta,\omega_2,t,\Omega)|}{|\omega_1-\omega_2|^{\lambda_2}} \le C \tag{2.3}$$

for all $(\theta_i,\omega,t,\Omega),(\theta,\omega_i,t,\Omega)\in\mathcal{Q}_R$, $i=1,2$, where $\mathcal{Q}_R=Q_T\cap\{\omega\in[-R,R]\}$ and the constant C is independent of $\varepsilon\in(0,1)$ and $N>0$.

In view of item (2) of Lemma 1.5 and Lemma 2.1, we obtain similarly

$$\frac{|D\rho^{\varepsilon,N}(\theta,\omega,t,\Omega_1)-D\rho^{\varepsilon,N}(\theta,\omega,t,\Omega_2)|}{|\Omega_1-\Omega_2|^{1/2}}\le C \tag{2.4}$$

for all $(\theta,\omega,t,\Omega_i)\in\mathcal{Q}_R$, $i=1,2$, for some constant C independent of $\varepsilon\in(0,1)$ and $N>0$. Then, item (3) of Lemma 1.5 and Lemma 2.1 imply that

$$\frac{|D\rho^{\varepsilon,N}(\theta,\omega,t_1,\Omega)-D\rho^{\varepsilon,N}(\theta,\omega,t_2,\Omega)|}{|t_1-t_2|^{1/12}}\le C \tag{2.5}$$

for all $(\theta,\omega,t_i,\Omega)\in\mathcal{Q}_R$, $i=1,2$, with a constant C independent of $\varepsilon\in(0,1)$ and $N>0$.

Summing up the estimates in (2.2)–(2.5), we conclude that

$$\|D\rho^{\varepsilon,N}\|_{C^{\lambda,\lambda,\frac{1}{12},\frac{1}{2}}(\mathcal{Q}_R)}\le C \tag{2.6}$$

for any fixed $\lambda\in(0,1)$ and $R>0$, where the constant C is independent of $\varepsilon\in(0,1)$ and $N>0$, but it depends, in general, on λ and R.

Lemmas 2.2 and 2.4 mean that, for any value of the parameter $M>0$, there exists a constant $C=C(M)>0$ such that the estimate

$$|D\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)|\le C\mathrm{e}^{-M|\omega|} \tag{2.7}$$

holds in Q_T, for all $\varepsilon\in(0,1)$ and $N>0$.

Therefore, there exist a sequence $\rho_n(\theta,\omega,t,\Omega):=\rho^{\varepsilon_n,N_n}(\theta,\omega,t,\Omega)$ with $\lim_{n\to\infty}\varepsilon_n=0$ and $\lim_{n\to\infty}N_n=+\infty$, and a function $\rho(\theta,\omega,t,\Omega)\in C^{2,2,1,2}_{\theta,\omega,t,\Omega}(Q_T)$, such that

$$\lim_{n\to\infty}\|D\rho_n-D\rho\|_{C(\mathcal{Q}_R)}=0,\quad \lim_{n\to\infty}\|D\rho_n-D\rho\|_{W_2^{1,2,0,1}(\mathcal{Q}_R)}=0,$$
$$D\rho\in W_2^{2,3,1,2}(Q_T),$$

for every $R>0$ (see (2.6) and Lemma 2.3). Taking the limit for $n\to\infty$ in equation (1.4) and in the initial-boundary conditions (1.5) and (1.3), at any fixed point of the slab Q_T, we infer that the limiting function $\rho(\theta,\omega,t,\Omega):=\lim_{n\to\infty}\rho_n(\theta,\omega,t,\Omega)$ is a classical solution to problem (1.1)–(1.3) in Q_T. Passage to the limit in $\mathcal{K}_{\rho_n}$ is permissible since there exists a summable majorant as shown in (2.7).

We omit here the proof of uniqueness of the solution, as a stronger result will be established in the next section. The theorem is thus proved. □

3 Uniqueness of solutions

In this section we establish the second main result of the paper, namely a uniqueness theorem. The proof is based on a version of the maximum principle, properly adapted to the case under investigation (see [14], e.g.). We first identify a certain class of functions, to be denoted by $f(\omega)$.

Assumption 3.1 The function $f(\omega)$ belongs to $C^2(\mathbb{R})\cap L^1(\mathbb{R})$, is positive, and the estimate

$$\mathcal{F}(A) := \sup_{\omega\in\mathbb{R},\alpha\in[-A,A]} \frac{f''(\omega)+(\omega+\alpha)f'(\omega)}{f(\omega)} < +\infty$$

holds for every $A > 0$.

Definition 3.2 Correspondingly to a given function $f(\omega)$ satisfying Assumption 3.1, denote by $\mathcal{D}_f(Q_T)$ the set of functions $\rho(\theta,\omega,t,\Omega)$ defined in Q_T, such that

$$\sup_{\theta\in[0,2\pi],t\in[0,T],\Omega\in[-G,G]} |\rho(\theta,\omega,t,\Omega)| = o(f(\omega)) \quad \text{as} \quad \omega\to\pm\infty.$$

We can then prove the following uniqueness result:

Theorem 3.3 *Suppose that*

(1) $\rho_1(\theta,\omega,t,\Omega)$ and $\rho_2(\theta,\omega,t,\Omega)$ are any two classical solutions to problem (1.1)–(1.3) in Q_T, belonging to the class $\mathcal{D}_f(Q_T)$;

(2) the function $\rho_2(\theta,\omega,t,\Omega)$, in addition, is such that $\|f^{-1}\frac{\partial\rho_2}{\partial\omega}\|_{C(Q_T)} < \infty$.

Then, $\rho_1(\theta,\omega,t,\Omega)\equiv\rho_2(\theta,\omega,t,\Omega)$ in Q_T.

Proof. Consider the quantity

$$\tilde\rho(\theta,\omega,t,\Omega) := \frac{\rho_1(\theta,\omega,t,\Omega)-\rho_2(\theta,\omega,t,\Omega)}{f(\omega)}\,\mathrm{e}^{-\lambda t},$$

with

$$\lambda := 2+2\pi K\|g\|_{L^1[-G,G]}\|f\|_{L^1(\mathbb{R})}\left\|\frac{1}{f(\omega)}\frac{\partial\rho_2}{\partial\omega}\right\|_{C(Q_T)} + \mathcal{F}(A), \qquad (3.1)$$

where $A := G+\|\mathcal{K}_{\rho_1}\|_{C(Q_T)}$. The parameters A and λ have finite values by the assumptions of the theorem.

The function $\tilde\rho(\theta,\omega,t,\Omega)$ solves, in the classical sense (in Q_T), the problem

$$\begin{gathered}\frac{\partial\tilde\rho}{\partial t} = \frac{\partial^2\tilde\rho}{\partial\omega^2} + \left[2\frac{f'}{f}+\omega-\Omega-\mathcal{K}_{\rho_1}\right]\frac{\partial\tilde\rho}{\partial\omega} - \omega\frac{\partial\tilde\rho}{\partial\theta} \\ + \left[\frac{f''}{f}+(\omega-\Omega-\mathcal{K}_{\rho_1})\frac{f'}{f}+1-\lambda\right]\tilde\rho - \frac{\mathcal{K}_{(\tilde\rho f)}}{f}\frac{\partial\rho_2}{\partial\omega}, \\ \tilde\rho|_{\theta=0} = \tilde\rho|_{\theta=2\pi}, \quad \tilde\rho|_{t=0}\equiv 0.\end{gathered}$$

Note that Definition 1.1 and equation (1.1) imply the relation $\omega\rho_\theta|_{\theta=0} = \omega\rho_\theta|_{\theta=2\pi}$ in $Q_T \cap \{t > 0\}$, for any classical solution $\rho(\theta,\omega,t,\Omega)$ to problem (1.1)–(1.3). In view of the continuity of ρ_θ, the equality $\rho_\theta|_{\theta=0} = \rho_\theta|_{\theta=2\pi}$ in $Q_T \cap \{t > 0\}$ follows, and thus the same additional property for $\tilde\rho(\theta,\omega,t,\Omega)$,

$$\tilde\rho_\theta|_{\theta=0} = \tilde\rho_\theta|_{\theta=2\pi}, \tag{3.2}$$

holds in $Q_T \cap \{t > 0\}$.

According to Definition 3.2 and assumption (1) of the theorem, the function $\tilde\rho(\theta,\omega,t,\Omega)$ possesses a decay as $|\omega| \to \infty$, that is there exists a function $\varphi(\delta)$, defined for $\delta \geq 0$, such that

$$|\tilde\rho(\theta,\omega,t,\Omega)| \leq \varphi(|\omega|) \quad \text{in } Q_T, \quad \lim_{\delta\to\infty} \varphi(\delta) = 0.$$

Therefore, there exists a point $M = (\theta^*,\omega^*,t^*,\Omega^*)$, with $t^* > 0$, such that

$$|\tilde\rho(M)| = \|\tilde\rho\|_{C(Q_T)}. \tag{3.3}$$

Consider now two possible occurrences.
Case 1: M is the point of the nonnegative maximum of the function $\tilde\rho(\theta,\omega,t,\Omega)$, i.e.,

$$\tilde\rho(M) = \|\tilde\rho\|_{C(Q_T)}. \tag{3.4}$$

In this case, the relations

$$\tilde\rho_t(M) \geq 0, \quad \tilde\rho_\omega(M) = 0, \quad \tilde\rho_{\omega\omega}(M) \leq 0 \tag{3.5}$$

hold. If $\theta^* \in (0, 2\pi)$, then, obviously, $\tilde\rho_\theta(M) = 0$. If $\theta^* = 0$ or $\theta^* = 2\pi$, then, using (3.2), we get $\tilde\rho_\theta(M) = 0$. Therefore, we obtain in any case

$$\tilde\rho_\theta(M) = 0. \tag{3.6}$$

Relations (3.5), (3.6), and the equation for $\tilde\rho(\theta,\omega,t,\Omega)$ imply that

$$0 \leq \left[\frac{f''}{f} + (\omega - \Omega - \mathcal{K}_{\rho_1})\frac{f'}{f} + 1 - \lambda\right]\tilde\rho - \frac{\mathcal{K}_{(\tilde\rho f)}}{f}\frac{\partial\rho_2}{\partial\omega} \tag{3.7}$$

at the point M. In view of assumption (1) of the theorem, the estimate

$$\|\mathcal{K}_{(\tilde\rho f)}\|_{C(Q_T)} \leq 2\pi K \|g\|_{L^1[-G,G]} \|f\|_{L^1(\mathbb{R})} \|\tilde\rho\|_{C(Q_T)} < \infty \tag{3.8}$$

holds. Using (3.1), (3.4), (3.7), and (3.8), we conclude that $\|\tilde\rho\|_{C(Q_T)} \leq 0$.
Case 2: M is the point of the nonpositive minimum of the function $\tilde\rho(\theta,\omega,t,\Omega)$, i.e., $-\tilde\rho(M) = \|\tilde\rho\|_{C(Q_T)}$. Then we obtain, similarly, that $\|\tilde\rho\|_{C(Q_T)} \leq 0$, using the relation

$$0 \geq \left[\frac{f''}{f} + (\omega - \Omega - \mathcal{K}_{\rho_1})\frac{f'}{f} + 1 - \lambda\right]\tilde\rho - \frac{\mathcal{K}_{(\tilde\rho f)}}{f}\frac{\partial\rho_2}{\partial\omega}$$

at the point M. It follows that $\|\tilde\rho\|_{C(Q_T)} = 0$ (see (3.3) and Cases 1 and 2). The theorem is thus proved. □

4 Summary of results

The solutions to the *nonlinear partial integro-differential* equation (1.1) with the *periodic* boundary condition (1.2) and the initial data (1.3) have been investigated. Equation (1.1) possesses a number of peculiarities. In particular, it could be treated as a *parabolic* equation *fully degenerate* in one of the space variables (as the Fokker-Planck equation in transport theory is). Moreover, it is considered *over an unbounded domain* and has *unbounded coefficients.* A regularized integroparabolic equation for which existence and regularity of solutions were studied in [12, 13] has been first considered. In this paper, the passage to the limit on the regularization parameters has been first justified, and thus existence of *decaying classical* solutions to the original problem (1.1)–(1.3) has been established. Uniqueness of classical solutions in a special class of functions has also been proved. The high points of the paper can be summarized in the following

Theorem 4.1 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with* $l_0 = 4$*. Then, there exists a* unique classical *solution,* $\rho(\theta, \omega, t, \Omega)$*, to the problem* (1.1)–(1.3) *in* Q_T*, belonging to the set* $\mathcal{D}_f(Q_T)$ *introduced in the Definition 3.2, with* $f(\omega)$ *such that* $\mathrm{e}^{-M|\omega|} = o(f(\omega))$ *as* $\omega \to \pm\infty$*, where* $M > 0$ *is a fixed constant.*

Proof. By Theorem 2.5, there exists a classical solution $\rho(\theta, \omega, t, \Omega)$ to problem (1.1)–(1.3) in Q_T, which belongs to $\mathcal{D}_f(Q_T)$ (see item (4) of Theorem 2.5 and the assumptions of the theorem). This solution also satisfies the estimate $\|f^{-1}\rho_\omega\|_{C(Q_T)} < \infty$. Uniqueness of the solution $\rho(\theta, \omega, t, \Omega)$ in the class $\mathcal{D}_f(Q_T)$ therefore follows from Theorem 3.3, with $\rho_2(\theta, \omega, t, \Omega) := \rho(\theta, \omega, t, \Omega)$. This completes the proof. $\square$

Assumption 3.1 on the function $f(\omega)$ is an important condition for the uniqueness result above. This assumption is used defining the class $\mathcal{D}_f(Q_T)$ and is necessary to establish a “decay” property of the function f. In fact, the limits of $f(\omega)$ as $\omega \to \pm\infty$ may not exist, in general, but $f \in L^1(\mathbb{R})$, and this property somehow characterizes the behavior of $f(\omega)$ at infinity. Integrability of the function f, along with the inequality in Assumption 3.1, make it possible to prove the uniqueness theorem in Section 3, namely Theorem 3.3.

Here we give some examples of functions f, satisfying Assumption 3.1. Consider all positive functions $f(\omega) \in C^2(\mathbb{R})$ such that:

$$f(\omega) = |\omega|^{-\beta} \quad \text{for} \quad |\omega| \geq M,$$

or

$$f(\omega) = |\omega|^{-1}(\log|\omega|)^{-\beta} \quad \text{for} \quad |\omega| \geq M,$$

or

$$f(\omega) = (|\omega| \log|\omega|)^{-1}(\log\log|\omega|)^{-\beta} \quad \text{for} \quad |\omega| \geq M,$$

and so on, where $\beta \in \mathbb{R}$ and $M > \mathrm{e}^2$ are fixed constants. Then, if $\beta > 1$, the function $f(\omega)$ satisfies Assumption 3.1; if $\beta \leq 1$, $f(\omega)$ does not satisfy Assumption 3.1, as $f \notin L^1(\mathbb{R})$.

Thus, Theorem 4.1 could be reformulated in the following weaker form:

Corollary 4.2 *Suppose the data of problem* (1.1)–(1.3) *satisfy Assumption 1.2 with $l_0 = 4$, and $\beta > 1$ is a fixed constant. Then, there exists a* unique classical *solution, $\rho(\theta, \omega, t, \Omega)$, to the problem* (1.1)–(1.3) *in Q_T, belonging to the set of functions satisfying the condition*

$$\sup_{\theta\in[0,2\pi],t\in[0,T],\Omega\in[-G,G]} |\rho(\theta, \omega, t, \Omega)| = O\Big(\frac{1}{|\omega|^\beta}\Big) \quad as \quad \omega \to \pm\infty.$$

On the other hand, if $f(\omega)$ is merely a *rapidly* decaying function, then, in general, a classical solution $\rho(\theta, \omega, t, \Omega)$ belonging to the class $\mathcal{D}_f(Q_T)$ does *not* exist. The condition $e^{-M|\omega|} = o(f(\omega))$ as $\omega \to \pm\infty$ guarantees that, in the class $\mathcal{D}_f(Q_T)$, there exists at least one classical solution to the problem (1.1)–(1.3) in Q_T (Theorem 2.5). Assumption 1.2 (with $l_0 = 4$) guarantees the fulfilment of the condition in item (2) of Theorem 3.3 and thus uniqueness in the classes described above.

In closing, here are some remarks concerning Assumption 1.2.

(1) Item (a_4) of Assumption 1.2, which has the physical meaning of being the probability integral over all space of the distribution function ρ, is not necessary for all mathematical constructions. Existence and uniqueness results are true regardless to this condition.

(2) Items (a_1)–(a_3) of Assumption 1.2, in contrast, are essential and cannot be generalized in the framework of the technique used here.

(3) The exponential function in item (a_5) of Assumption 1.2 has been considered here because this function is natural in transport theory and, moreover, it respects the decay properties enjoyed by the fundamental solutions to linear parabolic partial differential equations. Such assumption, however, is not optimal and can be relaxed.

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