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Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations *

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Abstract

In this paper we prove the existence of infinitely many radially symmetric standing waves in equilibrium with their own electro-magnetic field. The interaction is described by means of the minimal coupling rule; on the other hand the Lagrangian density for the electro-magnetic field is the second order approximation of the Born-Infeld Lagrangian density.

1 Electrically charged fields

Let us consider the nonlinear Klein-Gordon equation

$$\psi_{tt} - \Delta\psi + m_0^2\psi - |\psi|^{p-2}\,\psi = 0, \tag{1.1}$$

where $\psi = \psi\,(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, m_0 is a real constant and $p > 2$. It is known that (1.1) can be used to develop the theory of electrically charged fields (see [8]). Of course we can study the interaction of ψ with its own electro-magnetic field as it has been done in [2]. As usual, the electro-magnetic field is described by the gauge potential $(\phi, \mathbf{A})$

$$\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \quad \mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3.$$

Indeed, from $(\phi, \mathbf{A})$, we obtain the electric field

$$\mathbf{E} = -\nabla\phi - \mathbf{A}_t \tag{1.2}$$

and the magnetic induction field

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.3}$$

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The interaction of ψ with the electro-magnetic field is described by the minimal coupling rule, that is the formal substitution

$$\frac{\partial}{\partial t} \longmapsto \frac{\partial}{\partial t} + ie\phi \tag{1.4}$$

$$\nabla \longmapsto \nabla - ie\mathbf{A} \tag{1.5}$$

where e is the electric charge.

We recall that (1.1) is the Euler-Lagrange equation with respect to the Lagrangian density

$$\mathcal{L}_{\mathrm{NLKG}} = \frac{1}{2}\Big[\big|\frac{\partial\psi}{\partial t}\big|^2 - |\nabla\psi|^2 - m_0^2\,|\psi|^2\Big] + \frac{1}{p}\,|\psi|^p\,. \tag{1.6}$$

Then, by (1.4) and (1.5), the Lagrangian density (1.6) becomes

$$\mathcal{L}_0 = \frac{1}{2}\Big[\big|\frac{\partial\psi}{\partial t} + ie\phi\psi\big|^2 - |\nabla\psi - ie\mathbf{A}\psi|^2 - m_0^2\,|\psi|^2\Big] + \frac{1}{p}\,|\psi|^p\,.$$

The total action of the system is the sum

$$\mathcal{S} = \iint (\mathcal{L}_0 + \mathcal{L}_{\mathrm{e.m.f.}})\,dxdt \tag{1.7}$$

where $\mathcal{L}_{\mathrm{e.m.f.}}$ is the Lagrangian density of the electro-magnetic field. In the classical Maxwell theory, with a suitable choice of constants, we have

$$\mathcal{L}_{\mathrm{e.m.f.}} = \mathcal{L}_{\mathrm{M}} = \frac{1}{8\pi}\Big(|\mathbf{E}|^2 - |\mathbf{B}|^2\Big)\,.$$

The existence of infinitely many solutions for the Euler-Lagrange equations associated to

$$\mathcal{S} = \iint (\mathcal{L}_0 + \mathcal{L}_{\mathrm{M}})\,dxdt$$

has been proved by Benci and Fortunato in [2].

We recall that Maxwell equations coupled with Schrödinger or Dirac equations have been studied respectively in [1], [5], [6] and in [7]. It is well known that the classical theory has two difficulties arising from the divergence of energy (see the first section of [9]). An attempt to avoid this divergence is the Born-Infeld theory, where

$$\mathcal{L}_{\mathrm{BI}} = \frac{b^2}{4\pi}\Big(1 - \sqrt{1 - \frac{1}{b^2}\big(|\mathbf{E}|^2 - |\mathbf{B}|^2\big)}\Big) \tag{1.8}$$

being $b \gg 1$ the so-called Born-Infeld parameter (see [4]).

In [9], the authors consider the second order expansion of (1.8) for

$$\beta = \frac{1}{2b^2} \to 0.$$

They obtain the Lagrangian density

$$\mathcal{L}_1 = \frac{1}{4\pi}\left[\frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) + \frac{\beta}{4}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)^2\right]$$

and they prove some existence results of finite-energy electrostatic solutions.

In this paper we consider

$$\mathcal{L}_{\text{e.m.f.}} = \mathcal{L}_1.$$

So the total action we study is

$$\mathcal{S} = \iint (\mathcal{L}_0 + \mathcal{L}_1)\, dxdt. \tag{1.9}$$

As in [2], we consider

$$\psi(x,t) = u(x,t)\, e^{iS(x,t)}$$

with $u, S \in \mathbb{R}$; therefore

$$\mathcal{S} = \mathcal{S}(u, S, \phi, \mathbf{A})$$

and the explicit expression of the Lagrangian densities is

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\left\{u_t^2 - |\nabla u|^2 - \left[|\nabla S - e\mathbf{A}|^2 - (S_t + e\phi)^2 + m_0^2\right]u^2\right\} + \frac{1}{p}|u|^p\,, \\
\mathcal{L}_1 &= \frac{1}{4\pi}\left[\frac{1}{2}\left(|\mathbf{A}_t + \nabla\phi|^2 - |\nabla\times\mathbf{A}|^2\right) + \frac{\beta}{4}\left(|\mathbf{A}_t + \nabla\phi|^2 - |\nabla\times\mathbf{A}|^2\right)^2\right].
\end{aligned}$$

The Euler-Lagrange equations associated to (1.9) are

$$d\mathcal{S}[\delta u] = 0 \tag{1.10}$$
$$d\mathcal{S}[\delta S] = 0 \tag{1.11}$$
$$d\mathcal{S}[\delta\phi] = 0 \tag{1.12}$$
$$d\mathcal{S}[\delta\mathbf{A}] = 0. \tag{1.13}$$

By standard calculations, we get

$$\Box u + \left[|\nabla S - e\mathbf{A}|^2 - (S_t + e\phi)^2 + m_0^2\right]u - |u|^{p-2}u = 0 \tag{1.14}$$
$$\frac{\partial}{\partial t}\left[(S_t + e\phi)u^2\right] - \nabla\left[(\nabla S - e\mathbf{A})u^2\right] = 0 \tag{1.15}$$
$$\nabla\cdot\left[\left(1 + \beta|\mathbf{A}_t + \nabla\phi|^2 - \beta|\nabla\times\mathbf{A}|^2\right)(\mathbf{A}_t + \nabla\phi)\right] = 4\pi e(S_t + e\phi)u^2 \tag{1.16}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\left(1 + \beta|\mathbf{A}_t + \nabla\phi|^2 - \beta|\nabla\times\mathbf{A}|^2\right)(\mathbf{A}_t + \nabla\phi)\right] \\
&+\nabla\times\left[\left(1 + \beta|\mathbf{A}_t + \nabla\phi|^2 - \beta|\nabla\times\mathbf{A}|^2\right)(\nabla\times\mathbf{A})\right] = 4\pi e(\nabla S - e\mathbf{A})u^2.
\end{aligned} \tag{1.17}$$

From the physical point of view, it may be interesting to introduce the following notation: we set

$$\rho = -e\left(S_t + e\phi\right)u^2, \tag{1.18}$$
$$\mathbf{J} = e\left(\nabla S - e\mathbf{A}\right)u^2. \tag{1.19}$$

Taking into account (1.2) and (1.3), the equations (1.15), (1.16) and (1.17) become respectively

$$\frac{\partial\rho}{\partial t} + \nabla\mathbf{J} = 0 \tag{1.20}$$
$$\nabla\left[\left(1+\beta\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)\right)\mathbf{E}\right] = 4\pi\rho \tag{1.21}$$
$$\nabla\times\left[\left(1+\beta\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)\right)\mathbf{B}\right] - \frac{\partial}{\partial t}\left[\left(1+\beta\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)\right)\mathbf{E}\right] = 4\pi\mathbf{J}. \tag{1.22}$$

We notice that (1.18) and (1.19) are good definitions respectively of charge density and current density, indeed the continuity equation (1.20) is satisfied.

Equations (1.21) and (1.22) are formally identical to equations (24) and (25) of [9], indeed they replace the second pair of Maxwell equations when we use the second order approximation of the Born-Infeld Lagrangian density.

2 Statement of the main result

In this paper we look for solutions of (1.14)-(1.17) such that

$$u = u\left(x\right),$$
$$S = \omega t,$$
$$\phi = \phi\left(x\right), \tag{2.1}$$
$$\mathbf{A} = \mathbf{0}. \tag{2.2}$$

We recall that solutions

$$\psi\left(x,t\right) = u\left(x\right)e^{i\omega t}$$

are called *standing waves*. On the other hand, (2.1) and (2.2) characterize a purely *electrostatic field*.

With the above *ansatz*, equations (1.15) and (1.17) are identically satisfied; (1.14) and (1.16) take the form

$$-\Delta u + \left[m_0^2 - \left(\omega+\phi\right)^2\right]u - |u|^{p-2}u = 0 \tag{2.3}$$
$$\Delta\phi + \beta\Delta_4\phi = 4\pi\left(\omega+\phi\right)u^2, \tag{2.4}$$

where we have taken $e = 1$. Now we can state our main result.

Theorem 1 *If $|\omega| < |m_0|$ and $4 < p < 6$, the system of equations (2.3) and (2.4) has infinitely many solutions (u, ϕ) such that*

$$\int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx < +\infty \tag{2.5}$$

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \int_{\mathbb{R}^3} |\nabla \phi|^4 \, dx < +\infty. \tag{2.6}$$

Moreover the fields u and ϕ are radially symmetric.

The fact that $u \in H^1\left(\mathbb{R}^3\right)$ is radially symmetric implies that u decays to 0 at infinity (see [12]), then $\psi(x,t) = u(x)\, e^{i\omega t}$ can be called *solitary wave* . Moreover, from (2.6) we deduce that the electrostatic field $\mathbf{E} = -\nabla \phi$ has finite energy (in the sense of [9]).

3 Variational setting

Let $H^1(\mathbb{R}^3)$ denote the usual Sobolev space with norm

$$\|u\|_{H^1} = \Big(\int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \Big)^{1/2}$$

and D denote the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm

$$\|\phi\|_D = \|\nabla \phi\|_{L^2} + \|\nabla \phi\|_{L^4} .$$

If $D^{1,2}\left(\mathbb{R}^3\right)$ denotes the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm

$$\|\phi\|_{D^{1,2}} = \|\nabla \phi\|_{L^2} ,$$

it is obvious that D is continuously embedded in $D^{1,2}\left(\mathbb{R}^3\right)$. On the other hand, by well known Sobolev inequality, $D^{1,2}\left(\mathbb{R}^3\right)$ is continuously embedded in $L^6\left(\mathbb{R}^3\right)$.

Moreover it can be easily proved that D is continuously embedded in $L^\infty\left(\mathbb{R}^3\right)$ (see Proposition 8 of [9]).

Consider the functional

$$\begin{aligned} &F(u,\phi) \\ &= \int \left[\frac{1}{2} |\nabla u|^2 - \frac{1}{8\pi} |\nabla \phi|^2 + \frac{1}{2} \left(m_0^2 - (\omega + \phi)^2 \right) u^2 - \frac{\beta}{16\pi} |\nabla \phi|^4 - \frac{1}{p} |u|^p \right] dx. \end{aligned}$$

For the rest of this article, the integration domain is $\mathbb{R}^3$. By the above remarks, for every $(u, \phi) \in H^1\left(\mathbb{R}^3\right) \times D$

$$F(u, \phi) \in \mathbb{R}.$$

Proposition 2 *The functional F is C^1 on $H^1\left(\mathbb{R}^3\right) \times D$ and its critical points are solutions of (2.3) and (2.4) and satisfy (2.5) and (2.6).*

Proof. Let $F'_u(u,\phi)$ and $F'_\phi(u,\phi)$ denote the partial derivatives of F at $(u,\phi) \in H^1(\mathbb{R}^3) \times D$. For every $v \in H^1(\mathbb{R}^3)$ and $w \in D$,

$$F'_u(u,\phi)[v] = \int \left\{ (\nabla u \mid \nabla v) + \left[m_0^2 - (\omega+\phi)^2 \right] uv - |u|^{p-2} uv \right\} dx \qquad (3.1)$$

$$F'_\phi(u,\phi)[w] = -\int \left\{ \frac{1}{4\pi} \left(\left(1 + \beta |\nabla\phi|^2 \right) \nabla\phi \mid \nabla w \right) + (\omega+\phi) u^2 w \right\} dx. \qquad (3.2)$$

Using standard computations we show that F'_u and F'_ϕ are continuous; therefore F is C^1 on $H^1(\mathbb{R}^3) \times D$. Moreover, from (3.1) and (3.2) we get obviously that the critical points of F are solutions of (2.3) and (2.4). $\square$

The functional F is strongly indefinite, i.e. it is unbounded from above and from below, even modulo compact perturbations. Following [2] we are going to study a functional of the only variable u whose critical points give rise to critical points of F.

Lemma 3 *For every $u \in H^1(\mathbb{R}^3)$ there exists a unique $\phi = \Phi[u] \in D$ solution of (2.4).*

Proof. For every fixed $u \in H^1(\mathbb{R}^3)$, the solutions of (2.4) are critical points of the functional

$$I(\phi) = \int \left\{ \frac{1}{8\pi} |\nabla\phi|^2 + \frac{\beta}{16\pi} |\nabla\phi|^4 + \omega u^2 \phi + \frac{1}{2}\phi^2 u^2 \right\} dx \qquad (3.3)$$

defined on D. This functional is coercive; indeed, by the continuous embedding of D in $L^\infty(\mathbb{R}^3)$,

$$I(\phi) \geq \frac{1}{8\pi} \|\nabla\phi\|_{L^2}^2 + \frac{\beta}{16\pi} \|\nabla\phi\|_{L^4}^4 - c \left\| u^2 \right\|_{L^1} \left(\|\nabla\phi\|_{L^2} + \|\nabla\phi\|_{L^4} \right).$$

Furthermore I is weakly lower semicontinuous since each term in (3.3) is continuous and convex. Therefore I admits a global minimum. The solution of (2.4) is unique because the operator

$$\mathcal{A} = -\Delta - \beta\Delta_4 + 4\pi u^2$$

is strictly monotone. $\square$

By Lemma 3 we can consider the map

$$\Phi : H^1(\mathbb{R}^3) \to D$$

which is implicitly defined by $F'_\phi(u,\phi) = 0$.

From standard arguments we deduce that the map Φ is C^1. Fixed $u \in H^1(\mathbb{R}^3)$, since $\Phi[u]$ is the solution of (2.4), we have

$$\left\langle \frac{1}{4\pi}\Delta\Phi[u] + \frac{\beta}{4\pi}\Delta_4\Phi[u] - \Phi[u]\,u^2, \Phi[u] \right\rangle = \left\langle \omega u^2, \Phi[u] \right\rangle,$$

i.e.

$$-\frac{1}{4\pi}\int|\nabla\Phi[u]|^2\,dx-\frac{\beta}{4\pi}\int|\nabla\Phi[u]|^4\,dx-\int(\Phi[u])^2\,u^2dx=\omega\int u^2\Phi[u]\,dx. \tag{3.4}$$

Now we consider the functional

$$J:H^1\left(\mathbb{R}^3\right)\to\mathbb{R},\quad J(u)=F(u,\Phi[u]).$$

Using (3.4) we obtain

$$\begin{aligned}J(u)=&\frac{1}{2}\int|\nabla u|^2\,dx+\frac{1}{2}\left(m_0^2-\omega^2\right)\int u^2dx+\frac{1}{2}\int(\Phi[u])^2\,u^2dx+\\&+\frac{1}{8\pi}\int|\nabla\Phi[u]|^2\,dx+\frac{3}{16\pi}\beta\int|\nabla\Phi[u]|^4\,dx-\frac{1}{p}\int|u|^p\,dx.\end{aligned}$$

Remark 4 Since F and Φ are C^1, also J is C^1. Moreover, if $u\in H^1\left(\mathbb{R}^3\right)$ is a critical point of J, then $(u,\Phi[u])$ is a critical point of F (see Proposition 7 of [2]).

Thus, to get solutions of our problem, we look for critical points of J. For every $a\in\mathbb{R}^3$ and $v:\mathbb{R}^3\to\mathbb{R}$, we set

$$v_a(x)=v(x+a). \tag{3.5}$$

Lemma 3 implies that for every $u\in H^1\left(\mathbb{R}^3\right)$,

$$\Phi[u_a]=(\Phi[u])_a\,;$$

therefore $J(u_a)=J(u)$.

Since J is invariant under translations (3.5), there is still lack of compactness. For this reason we restrict J to the subspace

$$H_r^1\left(\mathbb{R}^3\right)=\left\{u\in H^1\left(\mathbb{R}^3\right)\big|\,u=u(|x|)\right\}$$

which is a natural constraint for J in the sense of following lemma.

Lemma 5 *If $u\in H_r^1\left(\mathbb{R}^3\right)$ is a critical point of $J\big|_{H_r^1(\mathbb{R}^3)}$, then u is a critical point of J.*

Proof. For every field v defined almost everywhere in $\mathbb{R}^3$ and for every $g\in O(3)$ we set

$$(T_gv)(x)=v(gx). \tag{3.6}$$

Of course (3.6) defines an action of $O(3)$ on $H^1\left(\mathbb{R}^3\right)$.

Since $H_r^1\left(\mathbb{R}^3\right)$ is the set of fixed points for this action, by the well known Principle of Symmetric Criticality (see [10]), it is enough to prove that J is T_g-invariant, i.e. for every $u\in H^1\left(\mathbb{R}^3\right)$ and $g\in O(3)$

$$J(T_gu)=J(u). \tag{3.7}$$

The main point is to prove that, for every $u \in H^1\left(\mathbb{R}^3\right)$

$$\Phi\left[T_g u\right] = T_g \Phi\left[u\right]. \tag{3.8}$$

It is well known that

$$\Delta T_g \Phi\left[u\right] = T_g\left(\Delta \Phi\left[u\right]\right).$$

Analogously one can show that

$$\Delta_4 T_g \Phi\left[u\right] = T_g\left(\Delta_4 \Phi\left[u\right]\right).$$

Then it is simple to verify that $\Phi\left[T_g u\right]$ and $T_g \Phi\left[u\right]$ solve the same equation

$$\Delta\phi + \beta\Delta_4\phi = 4\pi\left(\omega + \phi\right)\left(T_g u\right)^2.$$

Therefore we obtain (3.8). So, by the T_g-invariance of the norms in $H^1\left(\mathbb{R}^3\right)$, D and $L^p\left(\mathbb{R}^3\right)$ we get (3.7). □

4 Proof of Theorem 1

By Proposition 2, Remark 4 and Lemma 5, the thesis of Theorem 1 will follow if we show that the functional J has infinitely many critical points on $H_r^1\left(\mathbb{R}^3\right)$.

First of all we prove that on $H_r^1\left(\mathbb{R}^3\right)$ there is no lack of compactness. Indeed the following lemma holds true.

Lemma 6 *If $|\omega| < |m_0|$ and $4 < p < 6$, the functional $J|_{H_r^1(\mathbb{R}^3)}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset H_r^1\left(\mathbb{R}^3\right)$ be a Palais-Smale sequence, i.e. $J\left(u_n\right) = M_n$ is bounded and

$$J'|_{H_r^1(\mathbb{R}^3)}\left(u_n\right) = \varepsilon_n \to 0$$

in $\left(H_r^1\left(\mathbb{R}^3\right)\right)'$ (the dual space of $H_r^1\left(\mathbb{R}^3\right)$).

We want to prove that $\{u_n\}$ contains a convergent subsequence. From the definition of J we have that

$$J'\left(u_n\right) = F_u'\left(u_n, \Phi\left[u_n\right]\right) + F_\phi'\left(u_n, \Phi\left[u_n\right]\right)\Phi'\left[u_n\right] = F_u'\left(u_n, \Phi\left[u_n\right]\right) = \varepsilon_n$$

since $F_\phi'\left(u_n, \Phi\left[u_n\right]\right) = 0$. Then

$$\begin{aligned}
J(u_n) - \frac{1}{p}\left\langle J'\left(u_n\right), u_n\right\rangle &= J\left(u_n\right) - \frac{1}{p}\left\langle F_u'\left(u_n, \Phi\left[u_n\right]\right), u_n\right\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\int \left|\nabla u_n\right|^2 dx + \left(\frac{1}{2} - \frac{1}{p}\right)\left(m_0^2 - \omega^2\right)\int u_n^2 dx + \\
&+ \left(\frac{1}{2} + \frac{1}{p}\right)\int \left(\Phi\left[u_n\right]\right)^2 u_n^2 dx + \frac{2}{p}\omega\int \Phi\left[u_n\right] u_n^2 dx + \\
&+ \frac{1}{8\pi}\int \left|\nabla\Phi\left[u_n\right]\right|^2 dx + \frac{3}{16\pi}\beta\int \left|\nabla\Phi\left[u_n\right]\right|^4 dx.
\end{aligned}$$

On the other hand

$$J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle = M_n - \frac{1}{p}\langle \varepsilon_n, u_n\rangle .$$

Hence we can write

$$c_1 \int |\nabla u_n|^2 \, dx + c_2 \int u_n^2 dx + A_n = M_n - \frac{1}{p}\langle \varepsilon_n, u_n\rangle \tag{4.1}$$

where

$$\begin{aligned} A_n = & \frac{1}{8\pi}\int |\nabla\Phi[u_n]|^2 \, dx + \frac{3}{16\pi}\beta \int |\nabla\Phi[u_n]|^4 \, dx + \\ & + \left(\frac{1}{2} + \frac{1}{p}\right)\int (\Phi[u_n])^2 u_n^2 dx + \frac{2}{p}\omega \int \Phi[u_n] u_n^2 dx \end{aligned} \tag{4.2}$$

and c_1, c_2 are positive constants.

Equation (3.4) holds also for the pair $(u_n, \Phi[u_n])$ and so

$$\begin{aligned} A_n = & \frac{1}{4\pi}\left(\frac{1}{2} - \frac{2}{p}\right)\int |\nabla\Phi[u_n]|^2 \, dx + \frac{\beta}{4\pi}\left(\frac{3}{4} - \frac{2}{p}\right)\int |\nabla\Phi[u_n]|^4 \, dx + \\ & + \left(\frac{1}{2} - \frac{1}{p}\right)\int (\Phi[u_n])^2 u_n^2 dx. \end{aligned} \tag{4.3}$$

Then, since $4 < p < 6$, there exists $c_3 > 0$ such that

$$A_n \geq c_3 \left(\int |\nabla\Phi[u_n]|^2 \, dx + \int |\nabla\Phi[u_n]|^4 \, dx + \int (\Phi[u_n])^2 u_n^2 dx\right). \tag{4.4}$$

Thus, being $A_n \geq 0$, from (4.1) we obtain that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Moreover, the equation (4.1) implies that $\{A_n\}$ is bounded and so, from (4.4), also $\{\Phi[u_n]\}$ is bounded in D. Hence, up to subsequence,

$$\begin{gathered} u_n \rightharpoonup u \text{ in } H_r^1(\mathbb{R}^3) \\ \Phi[u_n] \rightharpoonup \bar{\Phi} \text{ in } D. \end{gathered}$$

Now we prove that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. We know that

$$-\Delta u_n + \left[m_0^2 - (\omega + \Phi[u_n])^2\right] u_n - |u_n|^{p-2} u_n = \varepsilon_n,$$

i.e.

$$-\Delta u_n + \left(m_0^2 - \omega^2\right) u_n = 2\omega\Phi[u_n] u_n + (\Phi[u_n])^2 u_n + |u_n|^{p-2} u_n + \varepsilon_n.$$

Let $L : H_r^1(\mathbb{R}^3) \to \left(H_r^1(\mathbb{R}^3)\right)'$ be the isomorphism

$$Lu = -\Delta u + \left(m_0^2 - \omega^2\right) u.$$

Then

$$u_n = 2\omega L^{-1}\left(\Phi\left[u_n\right]u_n\right) + L^{-1}\left(\left(\Phi\left[u_n\right]\right)^2 u_n\right) + L^{-1}\left(\left|u_n\right|^{p-2}u_n\right) + L^{-1}\left(\varepsilon_n\right). \tag{4.5}$$

To prove the strong convergence of $\{u_n\}$, it is sufficient to prove the strong convergence of each term in the right hand side of (4.5). Obviously the sequence $\left\{L^{-1}\left(\varepsilon_n\right)\right\}$ converges strongly. Since $H_r^1\left(\mathbb{R}^3\right)$ is compactly embedded into $L^q\left(\mathbb{R}^3\right)$ for $2 < q < 6$ (see [12], [3]), we have that

$$L^{p'}\left(\mathbb{R}^3\right) \text{ is compactly embedded into } \left(H_r^1\left(\mathbb{R}^3\right)\right)', \tag{4.6}$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p. Thus, if we set

$$v_n = \left|u_n\right|^{p-2}u_n$$

an easy calculation shows that

$$\|v_n\|_{L^{p'}} = \|u_n\|_{L^p}^{p-1}. \tag{4.7}$$

Since $\{u_n\}$ is bounded in $H_r^1\left(\mathbb{R}^3\right)$, from Sobolev embedding theorem, it is bounded in $L^p\left(\mathbb{R}^3\right)$ and so, by (4.7), we have that $\{v_n\}$ is bounded in $L^{p'}\left(\mathbb{R}^3\right)$.

Then, up to subsequence, $\{v_n\}$ converges weakly in $L^{p'}\left(\mathbb{R}^3\right)$ and, by (4.6), converges strongly in $\left(H_r^1\left(\mathbb{R}^3\right)\right)'$. So we conclude that $\left\{L^{-1}(|u_n|^{p-2}u_n)\right\}$ converges strongly in $H_r^1\left(\mathbb{R}^3\right)$.

Finally, it remains to show the strong convergence of $\left\{L^{-1}\left(\Phi\left[u_n\right]u_n\right)\right\}$ and $\left\{L^{-1}\left(\left(\Phi\left[u_n\right]\right)^2 u_n\right)\right\}$. Since $H_r^1\left(\mathbb{R}^3\right)$ is compactly embedded into $L^3\left(\mathbb{R}^3\right)$, we have that $L^{3/2}\left(\mathbb{R}^3\right)$ is compactly embedded into $\left(H_r^1\left(\mathbb{R}^3\right)\right)'$and then it is sufficient to prove that $\{\Phi\left[u_n\right]u_n\}$ and $\left\{\left(\Phi\left[u_n\right]\right)^2 u_n\right\}$ are bounded in $L^{3/2}\left(\mathbb{R}^3\right)$. By Hölder inequality

$$\begin{aligned}\left\|\left(\Phi\left[u_n\right]\right)^2 u_n\right\|_{L^{3/2}} &\leq \|\Phi\left[u_n\right]\|_{L^6}^3 \|u_n\|_{L^3}^{3/2}\\ \|\Phi\left[u_n\right]u_n\|_{L^{3/2}} &\leq \|\Phi\left[u_n\right]\|_{L^6}\|u_n\|_{L^2}.\end{aligned}$$

Hence, being $\{u_n\}$ and $\{\Phi\left[u_n\right]\}$ bounded respectively in $H_r^1\left(\mathbb{R}^3\right)$ and D and from well known Sobolev inequalities, we conclude that $\{\Phi\left[u_n\right]u_n\}$ and $\{(\Phi[u_n])^2u_n\}$ are bounded in $L^{3/2}(\mathbb{R}^3)$. □

Since J is *even* and $J|_{H_r^1(\mathbb{R}^3)}$ satisfies the Palais-Smale condition, the existence of infinitely many critical points follows from the equivariant version of the mountain pass theorem (Theorem 9.12 of [11]). Let $J|_{H_r^1(\mathbb{R}^3)}$ satisfy the geometrical assumptions:

(G$_1$) There exists α, $\rho > 0$ such that for every $u \in H_r^1\left(\mathbb{R}^3\right)$ with $\|u\|_{H_r^1} = \rho$

$$J\left(u\right) \geq \alpha$$

(G_2) For every finite dimensional subspace V of $H_r^1(\mathbb{R}^3)$ there exists $R>0$ such that for every $u \in H_r^1(\mathbb{R}^3)$ with $\|u\|_{H_r^1} \geq R$

$$J(u) \leq 0.$$

Then we deduce that $J|_{H_r^1(\mathbb{R}^3)}$ has infinitely many critical points. Therefore we are left to prove the geometrical assumptions (G_1) and (G_2).

In the sequel c_i denotes a positive constant. From the definition of J, for every $u \in H_r^1(\mathbb{R}^3)$

$$J(u) \geq c_1 \|u\|_{H^1}^2 - c_2 \|u\|_{L^p}^p .$$

Since $4 < p < 6$, by well known Sobolev inequalities, we obtain (G_1).

As to (G_2), let V be a finite dimensional subspace of $H_r^1(\mathbb{R}^3)$. For every $u \in V$

$$\begin{aligned} J(u) \leq & c_3 \|u\|_{H^1}^2 + \frac{1}{8\pi} \|\nabla\phi[u]\|_{L^2}^2 + \frac{3}{16\pi}\beta \|\nabla\phi[u]\|_{L^4}^4 + \\ & + \frac{1}{2} \|\phi[u]u\|_{L^2}^2 - \frac{1}{p}\|u\|_{L^p}^p . \end{aligned} \tag{4.8}$$

We know that

$$\frac{1}{4\pi}\|\nabla\phi[u]\|_{L^2}^2 + \frac{\beta}{4\pi}\|\nabla\phi[u]\|_{L^4}^4 + \|\phi[u]u\|_{L^2}^2 = -\omega \int u^2\phi[u]\,dx.$$

Moreover, by Hölder and Sobolev inequalities

$$\left|\omega \int u^2\phi[u]\,dx\right| \leq c_4 \|u\|_{L^{12/5}}^2 \|\phi[u]\|_{L^6} \leq c_5 \|u\|_{L^{12/5}}^2 \|\nabla\phi[u]\|_{L^2}$$

and so

$$\begin{aligned} \frac{1}{4\pi}\|\nabla\phi[u]\|_{L^2}^2 + \frac{\beta}{4\pi}\|\nabla\phi[u]\|_{L^4}^4 + \|\phi[u]u\|_{L^2}^2 \leq \\ \leq c_5 \|u\|_{L^{12/5}}^2 \|\nabla\phi[u]\|_{L^2} . \end{aligned} \tag{4.9}$$

Then

$$\frac{1}{4\pi}\|\nabla\phi[u]\|_{L^2} \leq c_5 \|u\|_{L^{12/5}}^2 \tag{4.10}$$

being

$$\frac{1}{4\pi}\|\nabla\phi[u]\|_{L^2}^2 \leq c_5 \|u\|_{L^{12/5}}^2 \|\nabla\phi[u]\|_{L^2} . \tag{4.11}$$

Thus, from (4.9), (4.10) and (4.11)

$$\begin{aligned} \frac{1}{4\pi}\|\nabla\phi[u]\|_{L^2}^2 &\leq c_5^2 \|u\|_{L^{12/5}}^4 \\ \frac{\beta}{4\pi}\|\nabla\phi[u]\|_{L^4}^4 &\leq c_5^2 \|u\|_{L^{12/5}}^4 \\ \|\phi[u]u\|_{L^2}^2 &\leq c_5^2 \|u\|_{L^{12/5}}^4 . \end{aligned}$$

Hence, from (4.8),

$$\begin{aligned}J(u) \leq & c_3 \|u\|_{H^1}^2 + \frac{c_5^2}{2} \|u\|_{L^{12/5}}^4 + \frac{c_5^2}{2} \|u\|_{L^{12/5}}^4 + \frac{3}{4} c_5^2 \|u\|_{L^{12/5}}^4 + \\ & - \frac{1}{p} \|u\|_{L^p}^p = c_3 \|u\|_{H^1}^2 + \frac{7}{4} c_5^2 \|u\|_{L^{12/5}}^4 - \frac{1}{p} \|u\|_{L^p}^p\end{aligned}$$

and, since V is finite dimensional and $p > 4$, we obtain (G_2).

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