MINIMUM CONDITIONS FOR BOOTSTRAP PERCOLATION ON THE CUBIC GRAPH

by

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HONORS THESIS

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Abstract

Bootstrap percolation is an iterative process on the vertices of a graph. Initially, a proper, non-empty set of vertices is *infected*, and all other vertices are *uninfected*. At each iteration, every uninfected vertex with a certain number of infected neighbors becomes infected, and all infected vertices remain so permanently. At the end of the process, if all vertices are infected, *percolation* occurs. In this case, the initial set of infected vertices *percolates* the graph. Necessary and sufficient conditions for the minimum size of a percolating set and the minimum number of rounds to achieve percolation on a cubic graph are presented.

Acknowledgements

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Finally, I would like to thank my family; my mom, dad, and sisters: Rochelle, David, Kiley, and Erinn Schlortt, and my grandparents: Esther Schlortt, Renee Blaschke, and Kenneth Blaschke, for always pushing me to succeed and to never accept mediocre work from myself. They have always believed in my abilities and have supported me in every endeavor I've undertaken. They have continuously shown their pride in my research, even though they don't really understand it.

Contents

List of Figures

1 Introduction

Bootstrap percolation is a broad term used to describe an iterative process of infection over graphs or 1 networks. This process begins with a certain configuration of infected vertices of a graph or network which infect the other vertices on the graph over a series of rounds, and infected vertices remain so permanently. The infection spread is controlled by a threshold rule, which varies largely over different types of percolation.

Bootstrap percolation was originally introduced in 1979 by Chalupa et al [4] as model of particle interactions in quantum physics. The process implemented in this study was a cellular implementation based on probability models of infection spread over a lattice. This spread, also called the random disease problem, looks at a model where spread from one vertex to another is determined by a certain probability over the edges of the graph [7]. More recently, this process has been used on graphs with a more discrete spread called r-neighbor bootstrap percolation, which defines the spread of infection by a discrete threshold rule that says any vertex with r infected neighbors becomes infected [2]. This version of percolation has been expanded to include (strict) majority bootstrap percolation which has the threshold that a vertex becomes infected if a vertex has a simple majority of its neighbors infected. Mathematically, this threshold rule establishes that a vertex v having $d(v)$ neighbors becomes infected when $\frac{d(v)+1}{2}$ of its neighbors are infected [9]. This threshold rule was further expanded to include strong-majority bootstrap percolation, which is defined by some positive integer k so that a vertex with $d(v)$ neighbors must have at least $\frac{d(v)+k}{2}$ infected neighbors to become infected [10].

There are three main problems that are looked in bootstrap percolation: distributions of vertices in percolating sets, size of percolating sets, and time required to percolate [7]. In this investigation, we consider the latter two problems, as the first has been extensively researched, and many papers have produced results related to the distribution of infected vertices over different graphs.

The results from the different kinds of bootstrap percolation have been used to model ferromagnetism, clustering phenomena, and crack formation in physics, disease spread over a discrete network, influence spread in social networks, market stability in terms of finance, and neural networks [2].

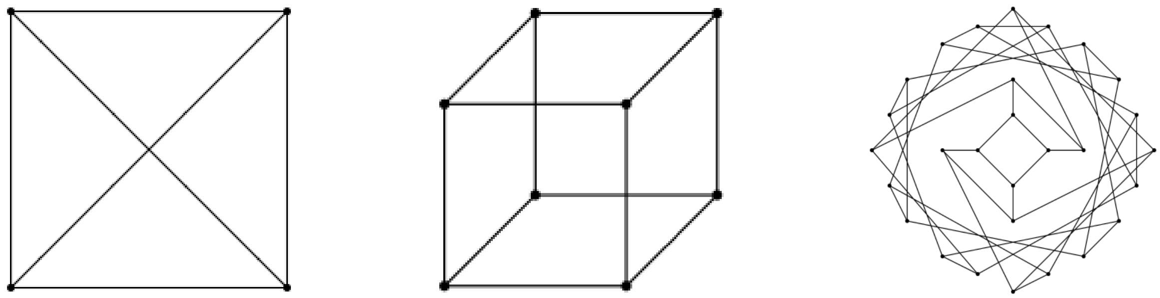
2 Background

The definitions and theorems contained in this section are intended to provide the background needed to understand the structures and processes used in this investigation. Certain known theorems are also presented and explained in order to allow for understanding of different ideas used in the proofs contained in the following sections.

2.1 Graphs

A *simple graph*, G, is a pair of sets, $G = (V, E)$, where V is a non-empty collection of vertices, $V = \{v_1, v_2, \dots v_n\}$, and E is a collection of unordered pairs of different vertices, $E = \{(u, v) | u \doteq v \text{ and } (u, v) = (v, u)\}$, called *edges*. The *order of* G, n, is defined as $n = |V|$ and the *size of* G, is defined as $|E|$. We say that if $u, v \in V$ and $(u, v) \in E$ then the vertices u and v are *neighbors* or *adjacent*, and these vertices are *incident* to the edge (u, v). The *open neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v, and the *closed neighborhood* of a vertex, denoted $N[v]$ is the set of vertices adjacent to v and v itself. Similarly, the *open neighborhood of a set* of vertices S, $N(S)$ is the set of vertices adjacent to the vertices in S not including any vertex in S, and the *closed neighborhood of* S, $N[S]$ is the union of the open neighborhood of S and the set S. Mathematically, $N(S) = (\bigcup_{v \in S} N(v)) \setminus S$ and $N[S] = N(S) \cup S$. Every vertex $v \in V$ has a non-negative integer, $d(v)$ called the *degree of* v and is defined as the number of neighbors of v, or $|N(v)| = d(v)$. A *regular* graph is defined as a graph G where every vertex v in G has degree r where r is a non-negative integer [5]. For the purposes of this investigation, we will consider *cubic* graphs, which are simply 3-regular graphs, and every graph will be assumed to be cubic unless specifically stated otherwise.

We define a *path* between two vertices u and v on G to be a sequence of k distinct vertices, k a positive integer, $\{a_1, \dots, a_k\}$ with $a_1 = u$ and $a_k = v$ such that for each integer $1 \leq i \leq k-1$, a_i is adjacent to a_{i+1}. The *length* of the path is the number of edges in the path, equal to $k-1$. We then define a *cycle* to be a path where $a_1 = a_k$, and $k > 3$. Since a cycle is a particular path, the length of a cycle is also the number of edges in the cycle, equal to k. We say that a graph G is *connected* if it has exactly one vertex, or if there is a path between every pair of distinct vertices in G. An edge e in a connected graph G is a *bridge*, if the graph obtained by removing e from G, denoted $G - e$, is not connected. A *bridgeless graph* is a graph that does not contain any bridges [5].



(a) The complete graph, called K_4 is the smallest possible cubic graph of order 4 and size 6.

(b) The graph of the geometric cube, for which the cubic graph was named, of order 8 and size 12.

(c) A more complex cubic graph, demonstrating the possible complexity and size, of order 28 and size 42.

Figure 1: Examples of Cubic Graphs

The Handshake Theorem, stated below, is a popular result in graph theory, and it is generally unknown who it can be attributed to. A proof and possible reference to this proof can be found in [5].

Theorem 2.1.1 *The Handshake Theorem Let* $G = (V, E)$ *be a simple graph with* $|E|$ *edges. Then* $2|E| = \sum_{v \in V} d(v)$.

From this theorem, we see that for a cubic graph, $2|E| = \sum_{v \in V} 3 = 3n$. Thus, $|E| = \frac{3n}{2}$. Since the size of a graph must be an integer, then we see that the order of the graph, n must be even for every cubic graph.

Many concepts in graph theory are expressed in terms of colorings of the vertices and/or edges of a graph. For a positive integer k, a *k-coloring* of a graph G is function that assigns to each vertex of G one of k different colors, usually denoted $1, \ldots, k$. Moreover, a *proper k-coloring* of G is a k-coloring of G in which any two adjacent vertices are assigned different colors. The *chromatic number* of a graph G is $c(G)$, defined as the minimum number of colors k in a proper k-coloring of G. One important graph theory concept which can be expressed via colorings is that of vertex independence. A set S of vertices of a graph G is said to be *independent* if no two vertices in S are adjacent in G. As a result, in any proper coloring of a graph G, the set of all vertices assigned to the same color must be an independent set. Moreover, the minimality of the *chromatic number* guarantees that $c(G)$ coincides with the number of sets in a partition of the vertices of G into independent sets [5].

Theorem 2.1.2 *Brook's Theorem [3] The chromatic number of a graph is at most its maximum degree,* Δ *unless the graph is complete, in which case* $\Delta + 1$ *colors are required at most.*

From this theorem, we see that the chromatic number of a cubic graph is three, except for K_4

(see Figure 1(a)), which means that the vertex set of every cubic graph can be partitioned into three independent sets.

Let $G = (V, E)$ be a graph. If there exists a set $M \subseteq E$ such that every vertex $v \in V$ is incident to exactly one edge in M, then M is a *perfect matching* in G. Not every graph has a perfect matching, but the following theorem gives a sufficient condition for a cubic graph to have a perfect matching.

Theorem 2.1.3 *Petersen's Theorem [11] Every bridgeless cubic graph has a perfect matching.*

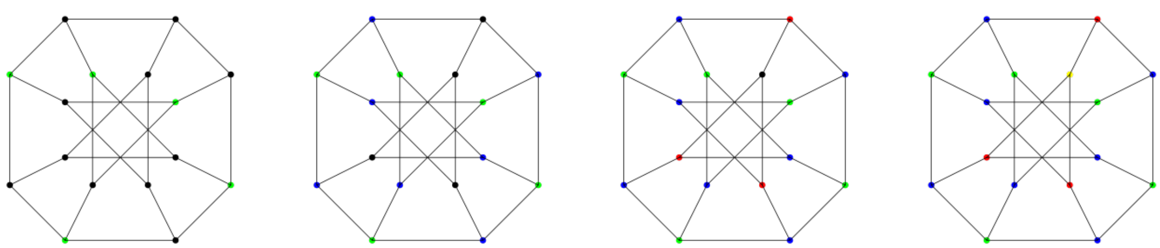
2.2 Bootstrap Percolation

Bootstrap Percolation is an iterative process on a graph $G = (V, E)$ in which a set of *infected* vertices, called I and assumed to have cardinality p, infects *uninfected* vertices, initially $V \setminus I$, according to the threshold rule. For the purpose of this exploration, we use the threshold rule called *Majority Bootstrap Percolation* which states that an uninfected vertex v becomes infected when it has at least $\frac{d(v)+1}{2}$. Once a vertex becomes infected, it remains so permanently. [1] We also note that this threshold rule is equivalent to 2-neighbor bootstrap percolation on cubic graphs. This means that a vertex will become infected if two or more of its neighbors are infected.

We also use the following notation:

- We define a sequence of sets $\{R_i\}_{i=0}^{m}$ where the originally infected set, I is the 0th round, or R_0, and for each i, the set $\bigcup_{j=0}^{i-1} R_j$ infects the vertices in R_i and we say the vertices in $\bigcup_{j=0}^{i-1} Rj$ *percolate* R_i in the ith round.

- If there exists an integer $m \geq 0$ such that $\bigcup_{j=0}^{m} Rj = V$, then we say the originally infected set I *percolates* the graph.



(a) R_0 with $p = 4$ shown in green

(b) R_0 shown in green, R_1 shown in blue

(c) R_0 shown in green, R_1 shown in blue, R_2 shown in red

(d) R_0 shown in green, R_1 shown in blue, R_2 shown in red, R_3 shown in yellow

Figure 2: Bootstrap Percolation Illustrated

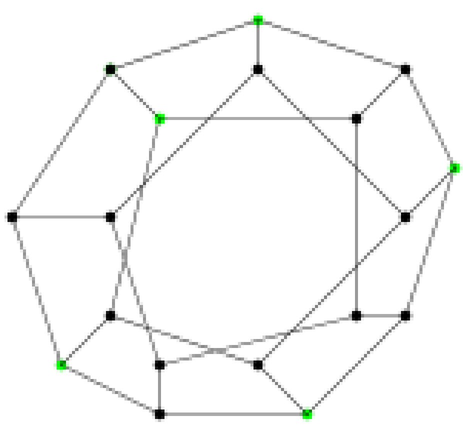
3 Special Families of Graphs

3.1 Generalized Petersen Graphs

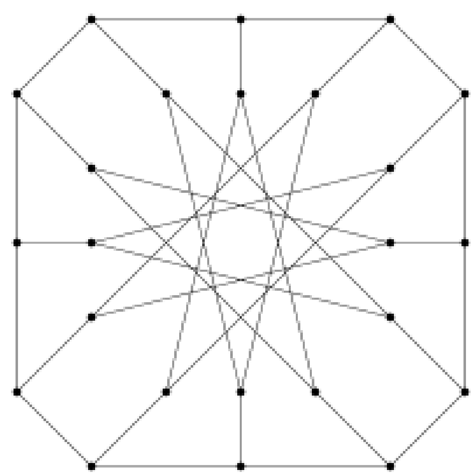
The Petersen graph is an interesting graph to begin this investigation on as it has a unique structure with several unique properties. An overview of some of these properties can be found [6]. In fact, many conjectures made about all cubic graphs can be proved or disproved on the Petersen graph. The generalized Petersen graphs were introduced to extend the unique properties of the Petersen graph to graphs of arbitrarily larger order. Their definition is presented next.

Definition 3.1.1 (Generalized Petersen Graphs, [8]) *Let h and k be positive integers, $h \geq k \geq 1$ The* Generalized Petersen Graph $P(h,k)$ *is defined as a cubic graph with vertex set $A \cup B$ where $A = \{a_0, a_1, \ldots a_{h-1}\}$ and $B = \{b_0, b_1, \ldots b_{h-1}\}$ such that the edge set is defined as follows:*

- *For each integer i $0 \leq i \leq h-1$, a_i is adjacent to b_i*
- *For each integer i $0 \leq i \leq h-1$, a_i is adjacent to $a_{(i-1) \bmod h}$ and $a_{(i+1) \bmod h}$*
- *For each integer i $0 \leq i \leq h-1$, b_i is adjacent to $b_{(i-k) \bmod h}$ and $b_{(i+k) \bmod h}$*



(a) The Generalized Petersen graph P(7, 2) of order 14



(b) The Generalized Petersen graph P(12, 5) of order 24

Figure 3: Examples of Generalized Petersen Graphs

Lemma 3.1.2 *The graph $P(h,k)$, of order $n = 2h$ can percolate with an initial set of vertices of cardinality $p = \left\lceil \frac{n+2}{4} \right\rceil$ or $p = \left\lceil \frac{n+2s}{4} \right\rceil$ where s is the minimum number of disjoint cycles in B.*

Proof. Let $B = \{C_1, \ldots C_t\}$ where each C_i is a cycle in the inner set of vertices. Suppose each $|C_i|$ is even. Then if we place infected vertices in each cycle such that every other vertex in a cycle is infected, then we get $\sum_{i=1}^{t} \frac{|C_i|}{2} = \frac{\sum_{i=0}^{t} |C_i|}{2} = \frac{n}{4}$ infected vertices in B placed so that every vertex in B will percolate. We then place exactly one infected vertex in A, giving $\left\lceil \frac{n+4}{4} \right\rceil$ infected vertices total.

Now suppose we have some integer $1 \leq s \leq t$ such that there are s cycles where $|C_i|$ is odd. Then, for $t - s$ even cycles, we use the same vertex placement as outlined in the first part of this proof. For the s odd cycles, we place $\frac{|C_i|-1}{2}$ vertices in the cycle such that no two infected vertices are adjacent. This will leave two uninfected adjacent vertices in each cycle. We then see that there are $\frac{n}{4} - \frac{s}{2}$ infected vertices in B. Then, for each odd cycle, we add one infected vertex in the outer cycle, such that it is adjacent to one of the two adjacent uninfected inner vertices in the odd cycle of B. This gives us $\frac{n+2s}{4}$ infected vertices with a placement that guarantees all of B will percolate.

Then, every vertex in A has at least one infected neighbor, and at least two vertices in A have two infected neighbors. Thus, we will percolate around A over several rounds, as each newly infected vertex in A will infect at least one other in A. Thus, the whole graph is percolated by a set of cardinality

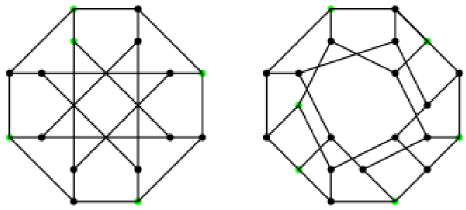
$$p = \frac{h}{2} + 1 = \frac{h}{2} + \frac{4}{4} = \frac{n+4}{4}$$

if there are only even cycles or

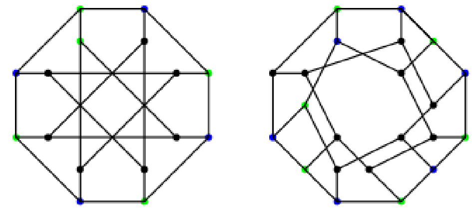
$$p = \frac{n+2s}{4}$$

if there are s odd cycles. Since $\frac{n+4}{4} = \left\lceil \frac{n+2}{4} \right\rceil$ when h is even (the only case where there will only be even cycles), then it is sufficient that $p = \left\lceil \frac{n+2}{4} \right\rceil$ or $p = \frac{n+2s}{4}$. ■

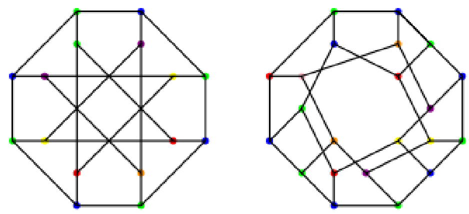
We note that when $gcd(h,k) = 1$, our bound is always $p = \left\lceil \frac{n+2}{4} \right\rceil$, as $s = 1$ in this case. Lemma 3.1.2 is for the sufficiency of this bound. We prove the necessity of this bound in section 4.1.



(a) The Generalized Petersen graphs P(8, 3) and P(11, 2) at Round 0 with percolating set represented in green



(b) Round 0 in green, Round 1 in blue



(c) Round 0 in green, Round 1 in blue, Round 2 in red, Round 3 in yellow, Round 4 in purple, Round 5 in orange, Round 6 in pink

Figure 4: Example of Percolation on Generalized Petersen Graphs

3.2 Cubic Graphs with Special Restrictions

Lemma 3.2.1 *Let G be a cubic graph with a perfect matching M such that there exist cycles A and B with $G - M = A \cup B$ and each edge in M, is incident with a vertex in A and a vertex in B. Then it is sufficient for a percolating set to have cardinality $p = \left\lceil \frac{n+2}{4} \right\rceil$*

The proof of Lemma 3.2.1 is nearly identical to the sufficiency proof presented for Lemma 3.1.2, and follows from this proof.

Lemma 3.2.2 *Suppose G is a cubic graph such that there is a cycle that contains every vertex. Then it is sufficient to have a percolating set of cardinality $\frac{n}{2}$.*

Proof. As shown above, a cubic graph must have an even degree. Thus, if every vertex in G belongs to the same cycle, then $\frac{n}{2}$ infected vertices can be placed around the cycle such that every ■

The condition of Lemma 3.2.2 is a well studied property in graphs called Hamiltonicity. Graphs having a cycle containing every vertex are called Hamiltonian, and the study of Hamiltonian graphs is a major field in graph theory. For more information about Hamiltonian graphs, see [5].

4 Minimum Cardinality of Percolating Set

From looking at certain families of graphs, we attempt to expand our study to include all cubic graphs. In order to prove these conditions, we first look at necessary bounds, and then look at sufficiency bounds to determine tightness.

The first minimal aspect that we investigate is the cardinality of the percolating set.

4.1 Necessary Conditions for a Percolating Set

Lemma 4.1.1 *In order for percolation to occur, it is necessary that every cycle in a cubic graph G have at least one infected vertex.*

Proof. Let G be a cubic graph and let C_a be some cycle in G with length $a \geq 3$. Let v be a vertex in C_a. Then v has at least 2 neighbors that also belong to C_a. Now assume that C_a has no infected vertex. Then v will be adjacent to at most one infected vertex. Thus, v will never become infected, and hence C_a will never percolate. Therefore G will never percolate. Hence, there must be at least one infected vertex of C_a. ■

Theorem 4.1.2 *For a cubic graph G of order n to percolate, it is necessary that the set of originally infected vertices have cardinality at least $\frac{n+2}{4}$.*

Proof. We begin with a set, R_0, of p infected vertices. In any cubic graph G, this set can infect at most $\frac{3p}{2}$ other vertices in the graph, ie $R_1 \leq \frac{3p}{2}$. In the next step, as the vertices in R_1 require two neighbors in R_0 in order to have percolated, at most $\frac{3p}{4}$ vertices can become infected, i.e. $R_2 \leq \frac{3p}{4}$. We see by this logic that for each positive integer i, $R_i \leq \frac{3p}{2^i}$. However, since each $R_i < R_{i-1}$, and G is cubic, the process ends at some integer m when $R_{m-1} = 4$ and $R_m = 2$ or $R_{m-1} = 3$ and $R_m = 1$ or $R_{m-1} = 2$ and $R_m = 1$ and there is some vertex left over in another level. We can condense these cases to $R_{m-1} \leq 4$ and $R_m \leq 2$. As this is a geometric series, then we can write

$$n = R_0 + \sum_{i=1}^{m} R_i \leq p + \sum_{i=1}^{m} \frac{3p}{2^i}$$

Since the sum of a geometric series is given by $\sum_{i=1}^{m} ar^i = \frac{ar - ar^{m+1}}{1-r}$, then the sum of the above geometric series, simplified, is $n \leq 4p - 2$, or equivalently, $p \geq \frac{n+2}{4}$. ■

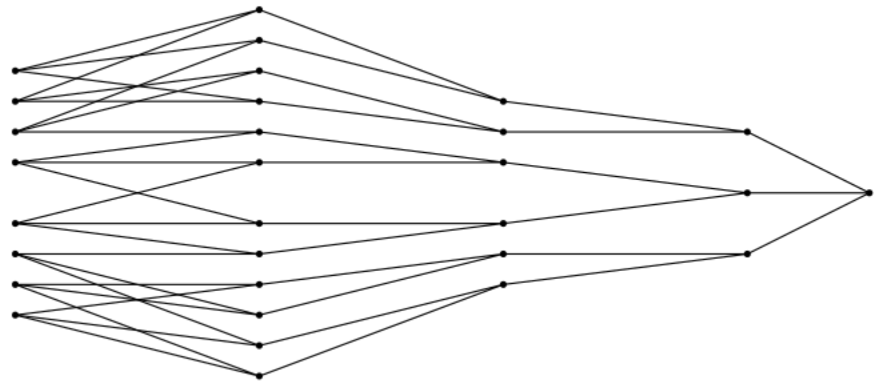


Figure 5: Example of Set Creation with p = 8: Each vertical row of vertices represents a round with the maximum possible connections that would result from percolation in the prior round (i.e. the furthest row to the left contains p vertices, the next row contains $\lfloor \frac{3p}{2} \rfloor$, the next has $\lfloor \frac{3p}{4} \rfloor$, and so on, as described in the proof above).

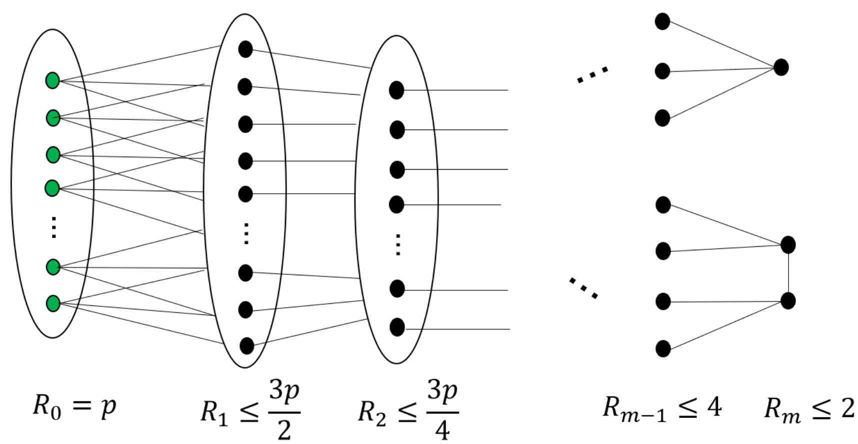


Figure 6: A general representation of the creation of the sets sorted by rounds with maximal set cardinality

4.2 Sufficient Conditions for a Percolating Set

In the previous examples of specific graphs, we see that graphs of order n can percolate from sets of cardinality greater than or equal to $\frac{n+2}{4}$. While it is logical to try and extend these results to all cubic graphs, we find that this conditions fails on the cubic graph of order 10 presented in Figure 7. Using the above bound, we would need to be able to percolate the graph using only 3 infected vertices. However, there is no placement of three vertices that will infect the graph, as each component on either side of the bridge requires two infected vertices so that every cycle has at least one infected vertex.

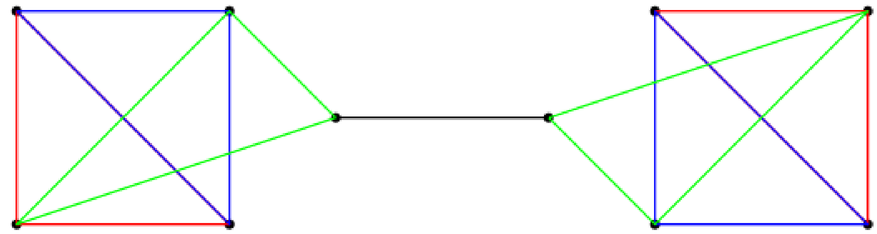


Figure 7: Each cycle is highlighted in a different color. There is no placement of three infected vertices so that every cycle has at least one infected vertex.

Lemma 4.2.1 *A cubic graph G of order n, can percolate with a set of cardinality at most $\frac{2n}{3}$.*

Proof. By Brook's Theorem, every cubic graph has $\frac{n}{3}$ independent vertices i.e. there is a set In of cardinality $\frac{n}{3}$ such that no two vertices in the set are adjacent. As such, if we let I be the set $V \setminus In$, then every vertex in In must percolate, and thus G percolates. ■

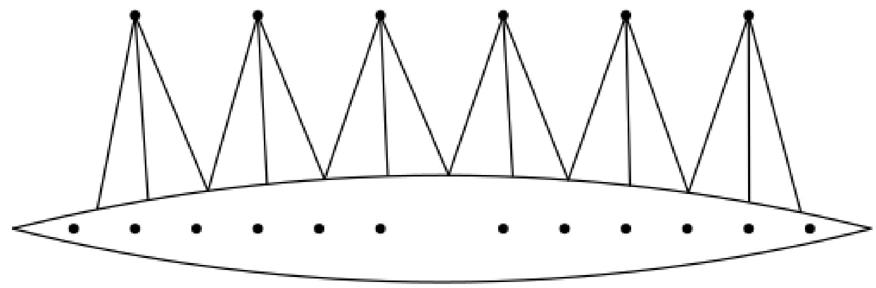


Figure 8: By Brook's theorem, one third of the vertices must be independent. Infecting the rest of the vertices, shown in the oval, will guarantee percolation.

Lemma 4.2.2 *Let G be cubic graph of order n. If G contains a perfect matching M then*

a) $G - M$ is a cycle or a collection of two or more pairwise disjoint cycles.

b) G can percolate with a set of $\frac{n+s}{2}$ where s is the minimum number of odd cycles in $G - M$.

Proof. Let G be a cubic graph with at least one perfect matching. Since G has at least one perfect matching, then we can consider the set $\{M : \text{M is a perfect matching of } G\}$ which is not empty. Then we let $s = \min\{\text{number of odd cycles in } G - M : M \text{ is a perfect matching of } G\}$ to minimize the number of odd cycles, and take the matching M which correlates with our s.

We now consider $G - M$. Since G is cubic, then $G - M$ will be a collection of k disjoint cycles. Since $G - M$ is G with fewer edges, then if we can percolate $G - M$, we can percolate G.

Since $G - M$ is a collection of k disjoint cycles, $\{C_1, \ldots, C_k\}$, we attempt to percolate each cycle individually. Since there are s odd cycles, there must be $k - s$ even cycles. Similar to the proof of Lemma 3.1.2, the even cycles can each be percolated with $\frac{|C_i|}{2}$ infected vertices placed so that each uninfected vertex has two infected neighbors. Similarly, the odd cycles can be percolated with $\frac{|C_i|+1}{2}$ infected vertices such that every uninfected vertex has two infected neighbors. Taking the sum of the infected vertices in these cycles gives $\frac{n+s}{2}$ infected vertices, placed so that every vertex will become infected, and the graph will percolate. ■

We note that this result verifies our result of Lemma 3.2.2, as if there is a cycle that contains every vertex, then there is a perfect matching that leaves only this cycle. Since n must be even, then $G - M$ is simply one even cycle. By Lemma 4.2.2, then G can percolate with $\frac{n+0}{2} = \frac{n}{2}$ vertices, producing the same bound as introduced in Lemma 3.2.2.

We also see that since by Petersen's theorem, every cubic graph contains a perfect matching, the bound from Lemma 4.2.2 applies to every bridgeless cubic graph.

5 Minimum Number of Rounds

We know consider the problem of determining a lower bound for the minimum number of rounds required for percolation from a given set of infected vertices. Before creating an algorithm for this, we need the following lemma.

Lemma 5.0.1 *For each* $1 \leq i \leq m-1$*,* $|R_i| \geq |R_{i+1}|$.

Proof. For each new round, the only change is the addition of $|R_i|$ vertices to the infected set. Suppose a vertex is part of the $i+1$ round. Then is is not a part of the ith rounds, and before the ith round, it can have at most one infected neighbor. Thus, the vertices infected in the $i+1$ round are entirely dependent on the vertices in the ith round. Furthermore, since the vertices in the ith round have become infected, they must have two already infected neighbors, and thus can infect at most one other vertex. Thus, at most, the $i+1$ round can have cardinality $|R_i|$, hence concluding the proof. ■

5.1 Algorithm to find minimum number of rounds

Like above, the first step to determining the minimum number of rounds required is to discover an algorithm that works to find the minimum number of rounds given certain parameters, then finding a numerical representation of this algorithm.

Given a cubic graph G of order n with a percolating set of cardinality p, $2 \leq p \leq 2n$ and $n \leq 2p-1$, E, a set containing extra vertices, and count, a Boolean variable containing either 0 or 1, the algorithm to determine the minimum number of rounds to percolate a graph G of order n is given below.

Set count to false, or 0. Place the p infected vertices in the set R_0 such that every infected vertex has exactly three uninfected neighbors in the set R_1 of uninfected vertices, and every vertex in R_1 is adjacent to exactly two infected vertices. This forces R_1 to have $\left\lfloor \frac{3p}{2} \right\rfloor$ vertices. If p is odd, then one infected vertex should have only two adjacent edges, and every other infected vertex should have exactly three adjacent edges, and we change count to true, or 1, and place the vertex with two adjacent neighbors in E. Otherwise, every infected vertex should have exactly three adjacent edges, and every uninfected vertex should have exactly two adjacent edges.

For every two vertices in R_1, we create a vertex placed in the set R_2 such that it is adjacent to both vertices in R_1. If there is an odd number of vertices in R_1, we designate one vertex to E. If $count = 0$ then we change count to 1. If $count = 1$, then we create a new vertex in R_2 that is adjacent to both vertices in E. We then set $E = \varnothing$ and *count* = 0.

This process continues such that for each k a positive integer, for every two vertices in R_k, we create a vertex placed in the set R_{k+1} such that it is adjacent to both vertices in R_k. If there is an odd number of vertices in R_k, we designate one vertex to E. If *count* = 0 then we change count to 1. If *count* = 1, then we create a new vertex in R_{k+1} that is adjacent to both vertices in E. We then set $E = \varnothing$ and *count* = 0.

The algorithm terminates at some positive integer m when either $|R_m| = 2$ and *count* = 0, or when $|R_m| = 1$ and *count* = 1, or $\sum_{i=0}^{m} |R_i| = n$. Then m is the minimum number of rounds that a cubic graph of size n with percolating set of cardinality p requires to percolate.

Below, we illustrate this algorithm, which we find to be very similar to the previous algorithm used to determine the minimum cardinality.

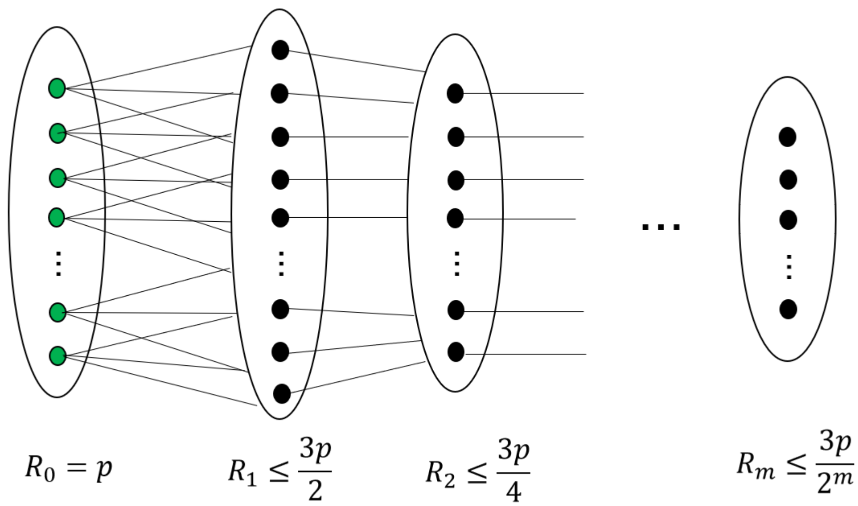


Figure 9: The algorithm used to find the minimum number of rounds is similar to that used to find the minimum percolating set cardinality, but finishes when the sum of the vertices in each rounds reaches the order of the graph in the mth round.

Theorem 5.1.1 *The algorithm described in section 5.1 effectively obtains the minimum number of rounds needed for a graph G to percolate from a given set I.*

Proof. In order to show that this algorithm gives a lower bound for the number of rounds needed to percolate, it is sufficient to prove that the structure outlined above is the ideal case for percolation. By Lemma 5.0.1, each round of percolation must infect a number of vertices less than or equal to the number of vertices in the previous round. Thus, to minimize the number of rounds required, we must maximize the number of vertices infected in the first round and in subsequent rounds. The structure described above maximizes the number of vertices infected in the first round by ensuring that every infected vertex is adjacent to the maximum possible number of uninfected vertices in order to still percolate. In each subsequent round, the percolation is maximized as every subsequent round uses every possible vertex with an open edge to spread the infection. As the structure forms a binary tree after the first round, and the first round will always percolate, the structure will always percolate. Since the tree will always percolate and each round maximizes the number of vertices percolated in that round, the structure provides the best case possible for percolation. Thus, the number rounds found by this algorithm gives the minimum bound for a cubic graph of percolating set size p and graph order of n to percolate. ■

5.2 Lower Bound for Minimum Number of Rounds

With an established algorithm that finds the minimum number of rounds given a specific p and n, we then strive to use this algorithm to find a numerical lower bound for the number of bounds required to achieve percolation.

Theorem 5.2.1 *Let G be a cubic graph of order n. A percolating set of cardinality p needs at least $\log_2\left(\frac{3p}{4p-n}\right)$ rounds to percolate G.*

Proof. Let G be a graph with order n and percolating set $p \geq \frac{n+2}{4}$. By the proof of Theorem 4.1.2, we know that each round i can add at most $\frac{3p}{2^i}$ infected vertices. Thus to find the minimum number of rounds to percolate, we maximize the number of vertices percolated in each round. Then, let m be our minimum number of rounds.

$$n = R_0 + \sum_{i=1}^{m} R_i \leq p + \sum_{i=1}^{m} \frac{3p}{2^i}$$

Using the sum of a geometric series, given in the proof of Theorem 4.1.2 and simplifying, we obtain $\frac{3p}{2^m} \leq 4p - n$, and equivalently $\log_2\left(\frac{3p}{4p-n}\right) \leq m$. ■

6 Applications and Future Work

While we have found a lower bound for the cardinality of the percolating set and have shown this bound is tight through the example of the generalized Petersen graph, we believe that this bound can be tight for every bridgeless cubic graph. Thus, we propose the following conjecture, and still attempt to prove the sufficiency condition.

Conjecture 6.0.1 *Let G be a bridgeless cubic graph. It is both necessary and sufficient that the percolating set have cardinality* $\lceil \frac{n+2}{4} \rceil$.

Further research includes proving this conjecture as well as finding the minimum product of the number of rounds and the cardinality of the percolating set in an attempt to further minimize the resources required for percolation. In other words, if the number of rounds decreases significantly with the addition of several vertices in the percolating set or vice versa, it would be beneficial to minimize this product to minimize necessary resources.

Furthermore, research could be expanded to include a different percolation rule where there is the possibility of vertices reverting to a healthy state. This rule would model real life information and idea spreads more accurately as people have the ability to change their mind and ideas more than once. In this case, it would be interesting to see which percolating set configurations and cardinalities would lead to either a domination, steady state, or infinitely variable state of different graphs.

The applications of this work relate more closely to information and idea spreads, as spread is based on the number of neighbors instead of a probabilistic spread over the edges. Research shows that the more often an idea is encountered, the more likely it is to spread, whereas most infection spreads are based on a probability of infection with each interaction.

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