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Oscillation criteria for a class of nonlinear partial differential equations *

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Abstract

This paper presents sufficient conditions on the function $c(x)$ to ensure that every solution of partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi_p(\frac{\partial u}{\partial x_i}) + B(x,u) = 0, \quad \Phi_p(u) := |u|^{p-1} \operatorname{sgn} u. \quad p > 1$$

is weakly oscillatory, i.e. has zero outside of every ball in $\mathbb{R}^n$. The main tool is modified Riccati technique developed for Schrödinger operator by Noussair and Swanson [11].

1 Introduction

In the oscillation theory of linear second order ordinary differential equation

$$y'' + q(x)y = 0 \tag{1}$$

plays an important role the associated Riccati equation

$$v' + c(x) + v^2 = 0 \tag{2}$$

which can be obtained from (1) by substitution $v(x) = \frac{y'(x)}{y(x)}$, $y(x)$ being a nonzero solution of (1), see e.g. [12]. The use of this substitution, the so-called Riccati technique, has been later developed also for various types of equations, namely discrete, half–linear, Schrödinger and also equations with $p-$Laplacian, see [6, 7, 8, 9, 10, 11, 13].

In this paper we will study the partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi_p\Big(\frac{\partial u}{\partial x_i}\Big) + B(x,u) = 0, \tag{3}$$

where $\Phi_p(u) := |u|^{p-1} \operatorname{sgn} u$, $p > 1$. The nonlinearity $B(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous function odd with respect to the second variable, i.e.

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(i) $B(x,-u) = -B(x,u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

Hence if the function $u(x)$ solves (3), then the function $-u(x)$ is also solution of (3).

Futhermore we suppose that there exist real-valued functions $c(x) \in C(\mathbb{R}^n)$, $\varphi(u) \in C^1(\mathbb{R})$ such that the following conditions hold

(ii) $B(x,u) \geq c(x)\varphi(u)$ for all $u > 0$

(iii) $\varphi(u) > 0$ for $u > 0$,

(iv) there exists $k > 0$ such that $\varphi^{q-2}(u)\varphi'(u) \geq k$ for $u > 0$, where q is the conjugate number to p, i.e., $q = \frac{p}{p-1} > 1$

A significant particular case of (3) we obtain for $B(x,u) = c(x)\Phi_p(u)$. In this case $k = p - 1$ holds in (iv) and (3) has the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big) + c(x)\Phi_p(u) = 0, \tag{4}$$

The study of this equation is motivated by the fact that it is Euler–Lagrange equation for the $p-$degree functional

$$\begin{aligned} \mathcal{F}_p(u;\Omega) &:= \int_\Omega \Big[\sum_{i=1}^{n}\Big|\frac{\partial u}{\partial x_i}\Big|^p - c(x)|u(x)|^p\Big]\,dx \\ &= \int_\Omega \big[\|\nabla u\|_p^p - c(x)|u|^p\big]\;dx. \end{aligned}$$

Equation (4) has been investigated in a series of papers of G. Bognár [1, 2, 3] where the basic properties of the eigenvalue problem have been established. The Picone–type identity and Riccati–type substitution for (4) has been recently introduced by O. Došlý [5].

If $p = 2$ then (4) is linear Schrödinger partial differential equation

$$\Delta u + c(x)u = 0.$$

Oscillation properties of this equation are deeply studied in the literature.

The aim of this paper is to study oscillation properties of equation (3) via modified Riccati technique and derive oscillation criteria for this equation.

The following notation will be used throughout the paper: the p and q-norms in $\mathbb{R}^n$

$$\|x\|_p = \Big(\sum_{i=1}^{n}|x_i|^p\Big)^{1/p}, \quad \|x\|_q = \Big(\sum_{i=1}^{n}|x_i|^q\Big)^{1/q} \quad \text{for } x \in \mathbb{R}^n,$$

and the sets

$$\begin{aligned} \Omega(a,b) &= \{x \in \mathbb{R}^n : a \leq \|x\|_q \leq b\}, \\ \Omega(a) &= \lim_{b\to\infty}\Omega(a,b) = \{x \in \mathbb{R}^n : a \leq \|x\|_q\}, \\ S(a) &= \partial\Omega(a) = \{x \in \mathbb{R}^n : a = \|x\|_q\}. \end{aligned}$$

The norm $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$ and $\omega_{n,q} := \int_{S(1)} \mathrm{d}S$ is the surface of the unit sphere (with respect to the q-norm) in $\mathbb{R}^n$.

Motivated by the terminology in [11], we define an oscillation of (3) as follows

Definition 1 (Weak oscillation). A function $f : \Omega \to \mathbb{R}$ is called *(weakly) oscillatory*, if and only if $f(x)$ has zero in $\Omega \cap \Omega(a)$ for every $a > 0$. Equation (3) is called *(weekly) oscillatory* in Ω whenever every solution u of (3) is oscillatory in Ω.

Since we will not deal with another definition of oscillation, we will refer weak oscillation simply as *oscillation*.

The paper is organized as follows. The next section contains the presentation of the main results. In Section 3 we prove some auxiliary results used in the proofs, which are contained in Section 4.

2 Main results

Theorem 1. *Let $a_0 \in \mathbb{R}^+$, $\alpha \in C^1((a_0,\infty),\mathbb{R}^+)$ and $l > 1$. If*

$$\lim_{r\to\infty} \int_{\Omega(a_0,r)} \Big[\alpha(\|x\|_q)c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1} \alpha^{1-p}(\|x\|_q)|\alpha'(\|x\|_q)|^p\Big]\,dx = +\infty \quad (5)$$

and

$$\lim_{r\to\infty} \int_{a_0}^{r} \frac{1}{\big(r^{n-1}\alpha(r)\big)^{\frac{1}{p-1}}} = +\infty, \quad (6)$$

then (3) is oscillatory in $\mathbb{R}^n$.

Remark 1. Remark that Theorem 1 does not deal with the existence of solution. In other words it states that if there exists a solution, then this solution is oscillatory function (in the sense of Definition 1).

A suitable choice of the function α in Theorem 1 leads to effective oscillation criteria for equations (3) and (4). This is the content of the following corollaries. The first one is a Leighton–type oscillation criterion (see [12, Th. 2.24, p. 70]).

Corollary 2. *Suppose that $p \ge n$ and*

$$\lim_{r\to\infty} \int_{\Omega(1,r)} c(x)\,dx = +\infty. \quad (7)$$

Then (3) is oscillatory in $\mathbb{R}^n$.

We remark that the condition $p \ge n$ cannot be removed, which is known already from the study of Schrödinger equation (for $p = 2$).

Another choice of the function α improves this criterion criterion, if $p > 2$.

Corollary 3. *Let $p \geq n$, $p > 2$ and*

$$\lim_{r\to\infty}\int_{\Omega(1,r)} \ln(\|x\|_q)c(x)\,dx = +\infty. \tag{8}$$

Then (3) *is oscillatory in* $\mathbb{R}^n$.

The following theorem covers also the case when $p < n$.

Corollary 4. *Let*

$$\liminf_{r\to\infty}\frac{1}{\ln r}\int_{\Omega(1,r)} \|x\|_q^{p-n}c(x)\,dx > \omega_{n,q}\frac{|p-n|^p}{p(kq)^{p-1}}. \tag{9}$$

Then (3) *is oscillatory in* $\mathbb{R}^n$.

Corollary 5. *Let*

$$\liminf_{r\to\infty}\frac{1}{\ln r}\int_{\Omega(1,r)} \|x\|_q^{p-n}c(x)\,dx > \omega_{n,q}\Big|\frac{p-n}{p}\Big|^p. \tag{10}$$

Then (4) *is oscillatory in* $\mathbb{R}^n$.

Remark 2. The constant $\omega_{n,q}\big|\frac{p-n}{p}\big|^p$ in (10) is optimal and cannot be decreased. This follows from the example of equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big) + \Big|\frac{p-n}{p}\Big|^p \|x\|_q^{-p}\Phi_p(u) = 0.$$

This equation is not oscillatory, since it has nonoscillatory solution $u(x) = \|x\|_q^{\frac{p-n}{p}}$ and the function $c(x) = \big|\frac{p-n}{p}\big|^p\|x\|_q^{-p}$ produces equality in condition (10).

Remark 3. We have already mention that the function $\Phi_p(u) := |u|^{p-1}\operatorname{sgn} u$ satisfies hypothesis (iii) and (iv) with $k = p-1$. On the other hand in most real applications we claim $B(x,0) = 0$ for all x and consequently $\varphi(0) = 0$. In this case integration of (iv) implies $\varphi(u) \geq \left(\frac{k}{p-1}\right)^{p-1} u^{p-1}$ and the function $\varphi(u)$ must satisfy this growth condition.

Example 1. Let us consider perturbed equation (4)

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big) + c(x)\Phi_p(u) + \sum_{i=1}^m q_i(x)\psi_i(u) = 0, \quad p \in (1,2] \tag{11}$$

where $c(x), q_i(x) \in C(\mathbb{R}^n)$, $\psi_i(u) \in C^1(\mathbb{R})$, $\psi_i(-u) = -\psi_i(u)$ for all $i = 1..m$ and all $u \in \mathbb{R}$, and $\psi_i(u)$ are positive and nondecreasing functions for $u > 0$ and all $i = 1..m$. Define

$$q(x) = \min\{c(x), q_1(x), q_2(x), \dots, q_m(x)\}$$

and

$$\varphi(u) = \Phi_p(u) + \sum_{i=1}^{m} \psi_i(u).$$

Then

$$c(x)\Phi_p(u) + \sum_{i=1}^{m} q_i(x)\psi_i(u) \geq q(x)\varphi(u) \quad \varphi'(u)\varphi^{q-2}(u) \geq p-1$$

and hence Theorem 1 can be applied. Remark that we suppose no sign restrictions for the functions q_i and so (11) needs not to be majorant for (4) in the sense of Sturmian theory.

3 Auxiliary results

A modification of Riccati substitution from [5] is presented in the following lemma.

Lemma 1. *Let $a_0 \in \mathbb{R}^+$, $\alpha \in C^1((a_0,\infty),\mathbb{R}^+)$. If $u \in C^2(\mathbb{R}^n,\mathbb{R})$ is a solution of (3) on $\Omega(a_0)$ such that $u(x) \neq 0$ for $x \in \Omega(a_0)$, then the vector function $\vec{w}(x)$ is well-defined on $\Omega(a_0)$ by*

$$\vec{w}(x) = \big(w_i(x)\big)_{i=1}^{n}, \quad w_i(x) = -\frac{\alpha(\|x\|_q)}{\varphi(u(x))}\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big) \tag{12}$$

and satisfies the inequality

$$\operatorname{div}\vec{w} \geq \alpha(\|x\|_q)c(x) + k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}\sum_{i=1}^{n} w_i\nu_i, \tag{13}$$

where $\nu_i = \Phi_q\Big(\frac{x_i}{\|x\|_q}\Big)$.

Proof. In view of (i), without loss of generality let us consider that $u(x) > 0$ on $\Omega(a_0)$. It holds

$$\begin{aligned}\frac{\partial w_i}{\partial x_i} =& -\frac{\alpha(\|x\|_q)}{\varphi(u)}\frac{\partial}{\partial x_i}\Big(\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big)\Big) - \Phi_p\Big(\frac{\partial u}{\partial x_i}\Big)\frac{\alpha'(\|x\|_q)}{\varphi(u)}\frac{\partial\|x\|_q}{\partial x_i} \\ &+ \alpha(\|x\|_q)\Big|\frac{\partial u}{\partial x_i}\Big|^p\frac{\varphi'(u)}{\varphi^2(u)}.\end{aligned}$$

Since $\frac{\partial\|x\|_q}{\partial x_i} = \Phi_q\Big(\frac{x_i}{\|x\|_q}\Big) = \nu_i$, we get

$$\begin{aligned}&\frac{\partial w_i}{\partial x_i} \\ &= -\frac{\alpha(\|x\|_q)}{\varphi(u)}\frac{\partial}{\partial x_i}\Big(\Phi_p\Big(\frac{\partial u}{\partial x_i}\Big)\Big) + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}w_i\nu_i + \varphi'(u)\varphi^{q-2}(u)\alpha^{1-q}(\|x\|_q)|w_i|^q.\end{aligned}$$

From this equation and from (3) it follows

$$\begin{aligned}&\operatorname{div}\vec{w}\\&=\alpha(\|x\|_q)\frac{B(x,u)}{\varphi(u)}+\varphi'(u)\varphi^{q-2}(u)\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q+\frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}\sum_{i=1}^n w_i\nu_i.\end{aligned}$$

Taking into account conditions (ii), (iii) and (iv) we obtain inequality (13). □

Lemma 2. *It holds*

$$\frac{\|x\|_p^p}{p}+\sum_{i=1}^n x_iy_i+\frac{\|y\|_q^q}{q}\geq 0$$

for every $x,y\in\mathbb{R}^n$, $x=(x_i)_{i=1}^n$, $y=(y_i)_{i=1}^n$.

For the proof of this lemma, see [5].

4 Proofs of the main results

Proof of Theorem 1. Suppose, by contradiction, that u is a solution of (3) which is positive on $\Omega(a_0)$ for some $a_0>0$. Then $\vec{w}$ is defined on $\Omega(a_0)$. From inequality (13), using integration over the domain $\Omega(a_0,r)$ and the Gauss–Ostrogradski divergence theorem, follows

$$\begin{aligned}&\int_{S(r)}\vec{w}\vec{n}\,\mathrm{d}S-\int_{S(a_0)}\vec{w}\vec{n}\,\mathrm{d}S\geq\\&\geq\int_{\Omega(a_0,r)}\Big(\alpha(\|x\|_q)c(x)+k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q+\frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}\sum_{i=1}^n w_i\nu_i\Big)\,\mathrm{d}x, \qquad (14)\end{aligned}$$

where $\vec{n}$ is the outward normal unit vector to $\Omega(a_0,r)$ i.e. $\vec{n}=\pm\frac{\vec{\nu}}{\|\vec{\nu}\|}$, $\vec{\nu}=(\nu_i)_{i=1}^n$ and ν_i is defined in Lemma 1. Observe that $\|\vec{\nu}\|_p=1$.

Now, let $l^*=\frac{l}{l-1}>1$ be the conjugate number to the number l. Then

$$\begin{aligned}&k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q+\frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}\sum_{i=1}^n w_i\nu_i=\\&=\frac{kq}{l}\alpha^{1-q}(\|x\|_q)\Big(\frac{\|\vec{w}\|_q^q}{q}+\frac{l\alpha'(\|x\|_q)\alpha^{q-2}(\|x\|_q)}{qk}\sum_{i=1}^n w_i\nu_i\Big)+\\&\qquad+\frac{k}{l^*}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q.\end{aligned}$$

Using Lemma 2 we obtain

$$
\begin{aligned}
&k\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q + \frac{\alpha'(\|x\|_q)}{\alpha(\|x\|_q)}\sum_{i=1}^n w_i\nu_i \\
&\geq -\frac{qk}{lp}\alpha^{1-q}(\|x\|_q)\Big\|\frac{l\alpha'(\|x\|_q)\alpha^{q-2}(\|x\|_q)}{qk}\vec{\nu}\Big\|_p^p + \frac{k}{l^*}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q \\
&= -\frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\alpha^{1-p}(\|x\|_q)|\alpha'(\|x\|_q)|^p + \frac{k}{l^*}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q.
\end{aligned}
$$

This inequality together with (14) yields

$$
\begin{aligned}
&\int_{S(r)}\vec{w}\vec{n}\,\mathrm{d}S - \int_{S(a_0)}\vec{w}\vec{n}\,\mathrm{d}S \\
&\geq \int_{\Omega(a_0,r)}\Big[\alpha(\|x\|_q)c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\alpha^{1-p}(\|x\|_q)|\alpha'(\|x\|_q)|^p\Big]\,\mathrm{d}x \\
&\quad + \frac{k}{l^*}\int_{\Omega(a_0,r)}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q\,\mathrm{d}x. \quad (15)
\end{aligned}
$$

In view of (5), there exists $r_0 > a_0$ such that

$$
\begin{aligned}
&\int_{\Omega(a_0,r)}\Big[\alpha(\|x\|_q)c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1}\alpha^{1-p}(\|x\|_q)|\alpha'(\|x\|_q)|^p\Big]\,\mathrm{d}x+ \\
&\qquad + \int_{S(a_0)}\vec{w}\vec{n}\,\mathrm{d}S \geq 0
\end{aligned}
$$

and now (15) implies

$$
\int_{S(r)}\vec{w}\vec{n}\,\mathrm{d}S \geq \frac{k}{l^*}\int_{\Omega(a_0,r)}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q\,\mathrm{d}x \quad (16)
$$

for $r > r_0$. Application of the Hölder inequality in $\mathbb{R}^n$ yields

$$
\int_{S(r)}\vec{w}\vec{n}\,\mathrm{d}S \leq \int_{S(r)}\|\vec{w}\|_q\|\vec{n}\|_p\,\mathrm{d}S.
$$

Since $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent norms in $\mathbb{R}^n$, there exists $K > 0$ such that $\|\vec{n}\|_p \leq K\|\vec{n}\| = K$. This fact and another application of Hölder inequality gives

$$
\int_{S(r)}\vec{w}\vec{n}\,\mathrm{d}S \leq K\Big(\omega_{n,q}r^{n-1}\Big)^{1/p}\Big(\int_{S(r)}\|\vec{w}\|_q^q\,\mathrm{d}S\Big)^{1/q} \quad (17)
$$

Denote

$$
g(r) = \int_{\Omega(a_0,r)}\alpha^{1-q}(\|x\|_q)\|\vec{w}\|_q^q\,\mathrm{d}x.
$$

Then it holds

$$
g'(r) = \alpha^{1-q}(r)\int_{S(r)}\|\vec{w}\|_q^q\,\mathrm{d}S.
$$

and (17) gives

$$\int_{S(r)} \vec{w}\vec{n}\,\mathrm{d}S \le K\omega_{n,q}^{1/p} r^{\frac{n-1}{p}} \Big(\alpha^{q-1}(r)g'(r)\Big)^{\frac{1}{q}}. \tag{18}$$

Combining (16) and (18) we obtain the inequality

$$\frac{k}{l^*} g(r) \le K\omega_{n,q}^{1/p} r^{\frac{n-1}{p}} \Big(\alpha^{q-1}(r)g'(r)\Big)^{1/q}$$

for $r > r_0$. Hence

$$\Big(\frac{1}{r^{n-1}\alpha(r)}\Big)^{\frac{1}{p-1}} \le \frac{l^* \omega_{n,q}^{\frac{q}{p}}}{kK^q} \frac{g'(r)}{g^q(r)}.$$

Integration of this inequality over $[r_0, \infty]$ gives the divergent integral on the left hand side, according to the assumption (6), and the convergent integral on the right hand side. This contradiction completes the proof. ☐

The Proof of Corollary 2 follows immediately from Theorem 1 for $\alpha(r) \equiv 1$.

Proof of Corollary 3. Let $a_0 > e$ be arbitrary and $\alpha(r) = \ln(r)$ on $[a_0, \infty)$. Since

$$\lim_{r\to\infty} \frac{\alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}}}{\frac{1}{r\ln r}} = \lim_{r\to\infty} r^{\frac{p-n}{p-1}} \ln^{\frac{p-2}{p-1}} r \ge 1,$$

the integral (6) diverges by ratio-convergence test. Further, since

$$\begin{aligned}\int_{\Omega(a_0,r)} |\alpha'(\|x\|_q)|^p \alpha^{1-p}(\|x\|_q)\,\mathrm{d}x &= \omega_{n,q} \int_e^r \xi^{n-1-p} \ln^{1-p}\xi\,\mathrm{d}\xi \\ &\le \omega_{n,q} \int_{a_0}^r \xi^{-1} \ln^{1-p}\xi\,\mathrm{d}\xi = \omega_{n,q} \frac{1}{p-2}[1 - \ln^{2-p} r],\end{aligned}$$

the limit $\lim_{r\to\infty} \int_{\Omega(a_0,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|)\,\mathrm{d}x$ converges and (8) is equivalent to the condition (5) of Theorem 1. All conditions of Theorem 1 are satisfied and the proof is complete. ☐

Proof of Corollary 4. Let $\alpha(r) = r^{p-n}$. Then (6) holds and it is sufficient to prove that also (5) holds, i.e. that there exists $l > 1$ such that

$$\lim_{r\to\infty} \int_{\Omega(1,r)} \Big[\|x\|_q^{p-n} c(x) - \frac{1}{p}\Big(\frac{l}{kq}\Big)^{p-1} |p-n|^p \|x\|_q^{-n}\Big]\,\mathrm{d}x = +\infty. \tag{19}$$

According to (9) there exists $m > 1$, $\varepsilon > 0$ and $r_0 > 1$ such that

$$\int_{\Omega(1,r)} \|x\|_q^{p-n} c(x)\,\mathrm{d}x > (m+\varepsilon)\omega_{n,q} \frac{|p-n|^p}{p(kq)^{p-1}} \ln r \tag{20}$$

for $r > r_0$. Since

$$\int_{\Omega(1,r)} \|x\|_q^{-n}\,\mathrm{d}x = \omega_{n,q} \int_1^r \frac{1}{s}\,\mathrm{d}s = \omega_{n,q} \ln r,$$

can be (20) written in the form

$$\int_{\Omega(1,r)} \Big[\|x\|_q^{p-n} c(x) - m \frac{|p-n|^p}{p(kq)^{p-1}} \|x\|_q^{-n} \Big] \,\mathrm{d}x > \varepsilon \omega_{n,q} \frac{|p-n|^p}{p(kq)^{p-1}} \ln r \qquad (21)$$

which implies (19). The proof is complete. □

The Proof of Corollary 5 follows immediately from Corollary 4.

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