UPPER SEMICONTINUITY OF ATTRACTORS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS FOR SMALL PERTURBATIONS

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ABSTRACT. We study the problem of upper semicontinuity of compact global attractors of non-autonomous dynamical systems for small perturbations. For the general nonautonomous dynamical systems, we give the conditions of upper semicontinuity of attractors for small parameter. Several applications of these results are given (quasihomogeneous systems, monotone systems, nonautonomously perturbed systems, nonautonomous 2D Navier-Stokes equations and quasilinear functional-differential equations).

1. INTRODUCTION

The problem of upper semicontinuity of global attractors for small perturbations is well studied (see, for example, [15] and references therein) for autonomous and periodical dynamical systems. In the works [1] and [2] this problem was studied for nonautonomous and random dynamical systems.

Our paper is devoted to a systematic study of the problem of upper semicontinuity of compact global attractors and compact pullback attractors of abstract nonautonomous dynamical systems for small perturbations. Several applications of our results are given for different classes of evolutional equations.

The paper is organized as follows. In section 2 we study some general properties of maximal compact invariant sets of dynamical systems. In particular, we prove that the compact global attractor and pullback attractor are maximal compact invariant sets (Theorem 2.6).

Section 3 contains the main results about upper semicontinuity of compact global attractors of abstract non-autonomous dynamical systems for small perturbations (Lemmas 3.3, 3.6 and Theorems 3.10, 3.13, 3.14 and 3.16). In section 4 we give conditions for connectedness and component connectedness of global and pullback attractors (Theorem 4.5). Section 5 is devoted to an application of our general results

obtained in sections 2-4, to the study of different classes of non-autonomous differential equations (quasihomogeneous systems, monotone systems, nonautonomously perturbed systems, nonautonomous 2D Navier-Stokes equations and quasilinear functional-differential equations).

2. Maximal compact invariant sets.

Let W be a complete metric space, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}, \Omega$ a compact metric space, $(\Omega, \mathbb{T}, \sigma)$ a group dynamical system on Ω and $\langle W, \varphi, (\Omega, \mathbb{T}, \sigma) \rangle$ a cocycle with fibre W, i.e. the mapping $\varphi : \mathbb{T}_+ \times W \times \Omega \to W$ is continuous and possesses the following properties: $\varphi(0, x, \omega) = x$ and $\varphi(t + \tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \omega t)$, where $\omega t = \sigma(t, \omega)$.

We denote by $X = W \times \Omega, g = pr_1 : X \mapsto W, (X, \mathbb{T}_+, \pi)$ a semi-group dynamical system on X defined by the equality $\pi = (\varphi, \sigma)$, i.e. $\pi^t x = (\varphi(t, u, \omega), \sigma(t, \omega))$ for every $t \in \mathbb{T}_+$ and $x = (u, \omega) \in X = W \times \Omega$. Let $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \sigma), h \rangle$ be a nonautonomous dynamical system, where $h = pr_2 : X \mapsto \Omega$.

Definition 2.1. A family $\{I_\omega | \omega \in \Omega\}(I_\omega \subset W)$ of nonempty compact subsets of W is called a maximal compact invariant set of cocycle φ, if the following conditions are fulfilled:

(1) $\{I_\omega | \omega \in \Omega\}$ is invariant, i.e.

$$\varphi(t, I_\omega, \omega) = I_{\omega t}$$

for every $\omega \in \Omega$ and $t \in \mathbb{T}_+$;
(2) $I = \bigcup\{I_\omega | \omega \in \Omega\}$ is relatively compact;
(3) $\{I_\omega | \omega \in \Omega\}$ is maximal, i.e. if the family $\{I'_\omega | \omega \in \Omega\}$ is relatively compact and invariant, then $I'_\omega \subseteq I_\omega$ for every $\omega \in \Omega$.

Lemma 2.2. The family $\{I_\omega | \omega \in \Omega\}$ is invariant w.r.t. cocycle φ if and only if the set $J = \bigcup\{J_\omega | \omega \in \Omega\}(J_\omega = I_\omega \times \{\omega\})$ is invariant with respect to the dynamical system $(X, \mathbb{T}_+, \pi)$.

Proof. Let the family $\{I_\omega | \omega \in \Omega\}$ be invariant, $J = \bigcup\{J_\omega | \omega \in \Omega\}$ and $J_\omega = I_\omega \times \{\omega\}$). Then

$$\pi^t J = \bigcup\{\pi^t J_\omega \mid \omega \in \Omega\} = \bigcup\{(\varphi(t, I_\omega, \omega), \omega t) \mid \omega \in \Omega\}$$

$$= \bigcup\{I_{\omega t} \times \{\omega t\} \mid \omega \in \Omega\} = \bigcup\{J_{\omega t} \mid \omega \in \Omega\} = J$$

for all $t \in \mathbb{T}_+$. From the equality (2.1) follows that the family $\{I_\omega | \omega \in \Omega\}$ is invariant w.r.t. cocycle φ if and only if a set J is invariant w.r.t. dynamical system $(X, \mathbb{T}_+, \pi)$. $\square$

Theorem 2.3. Let the family of sets $\{I_\omega | \omega \in \Omega\}$ be maximal, compact and invariant. Then it is closed.

Proof. We note that the set $J = \bigcup\{J_\omega | \omega \in \Omega\}(J_\omega = I_\omega \times \{\omega\})$ is relatively compact and according to Lemma 2.2 it is invariant. Let $K = \overline{J}$, then K is compact. We shall show that K is invariant. If $x \in K$, then there exists $\{x_n\} \subset J$ such that $x = \lim_{n \to +\infty} x_n$. Thus $x_n \in J = \pi^t J$ for all $t \in \mathbb{T}_+$, then for $t \in \mathbb{T}_+$ there exists $\overline{x}_n \in J$ such that $x_n = \pi^t \overline{x}_n$. Since J is relatively compact, the sequence $\{\overline{x}_n\}$ is convergent. We denote by $\overline{x} = \lim_{n \to +\infty} \overline{x}_n$, then $\overline{x} \in \overline{J}, x = \pi^t \overline{x}$ and, consequently, $x \in \pi^t \overline{J}$ for all $t \in \mathbb{T}_+$, i.e. $\overline{J} = \pi^t \overline{J}$. Let $I' = pr_1 K$, then we have $I' = \bigcup\{I'_\omega | \omega \in \Omega\}$,

where $I'_\omega = \{u \in W | (u, \omega) \in K\}$ and $K_\omega = I'_\omega \times \{\omega\}$. Since the set K is invariant, then according to Lemma 2.2 the set I' is also invariant w.r.t. cocycle φ. The set I' is compact, because K is compact and $pr_1 : X \mapsto W$ is continuous. According to the maximality of the family $\{I_\omega | \omega \in \Omega\}$ we have $I'_\omega \subseteq I_\omega$ for every $\omega \in \Omega$ and, consequently, $I' \subseteq I$. On the other hand $I = pr_1 \overline{J} = I'$ and, consequently, $I' = I$. Thus the set I is compact. The theorem is proved. $\square$

Denote by $C(W)$ the family of all compact subsets of W.

Definition 2.4. A family $\{I_\omega \mid \omega \in \Omega\}$ $(I_\omega \subset W)$ of nonempty compact subsets of W is called a compact pullback attractor of the cocycle φ, if the following conditions are fulfilled:

   a. $I = \bigcup \{I_\omega | \omega \in \Omega\}$ is relatively compact ;
   b. I is invariant w.r.t. cocycle φ, i.e. $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{T}_+$ and $\omega \in \Omega$;
   c. for every $\omega \in \Omega$ and $K \in C(W)$

$$\lim_{t \to +\infty} \beta(\varphi(t, K, \omega^{-t}), I_\omega) = 0, \tag{2.2}$$

   where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff and $\omega^{-t} := \sigma(-t, \omega)$.

Definition 2.5. A family $\{I_\omega | \omega \in \Omega\}(I_\omega \subset W)$ of nonempty compact subsets of W is called a compact global attractor, if the following conditions are fulfilled:

   a. a family $\{I_\omega | \omega \in \Omega\}$ is compact and invariant;
   b. for every $K \in C(W)$

$$\lim_{t \to +\infty} \sup_{\omega \in \Omega} \beta(\varphi(t, K, \omega), I) = 0, \tag{2.3}$$

   where $I = \bigcup \{I_\omega \mid \omega \in \Omega\}$.

Theorem 2.6. *A family $\{I_\omega | \omega \in \Omega\}$ of nonempty compact subsets of W will be maximal compact invariant set w.r.t. cocycle φ, if and only if one of the following two conditions is fulfilled :*

   a. $\{I_\omega \mid \omega \in \Omega\}$ *is a compact pullback attractor w.r.t. cocycle φ;*
   b. $\{I_\omega \mid \omega \in \Omega\}$ *is a compact global attractor w.r.t. cocycle φ.*

Proof. a. Let the family $\{I_\omega \mid \omega \in \Omega\}$ be a compact pullback attractor. If the family $\{I'_\omega \mid \omega \in \Omega\}$ is a compact and invariant set of cocycle φ, then we have

$$\beta(I'_\omega, I_\omega) = \beta(\varphi(t, I'_{\omega^{-t}}, \omega^{-t}), I_\omega) \le \beta(\varphi(t, K, \omega^{-t}), I_\omega) \to 0$$

as $t \to +\infty$, where $K = \overline{\bigcup\{I'_\omega \mid \omega \in \Omega\}}$, and, consequently, $I'_\omega \subseteq I_\omega$ for every $\omega \in \Omega$, i.e. $\{I_\omega \mid \omega \in \Omega\}$ is maximal.

b.) Let the family $\{I_\omega \mid \omega \in \Omega\}$ be a compact global attractor w.r.t. cocycle φ, then according to Theorem 4.1 [6] it is a uniform compact pullback attractor and, consequently, the family $\{I_\omega \mid \omega \in \Omega\}$ is maximal compact invariant set of the cocycle φ. $\square$

Remark 2.7. The family $\{I_\omega \mid \omega \in \Omega\}$ $(I_\omega \subset W)$ is a maximal compact invariant w.r.t. cocycle φ if and only if the set $J = \bigcup\{J_\omega \mid \omega \in \Omega\}$, where $J_\omega = I_\omega \times \{\omega\}$, is a maximal compact invariant in the dynamical system $(X, \mathbb{T}, \pi)$.

Definition 2.8. The cocycle φ is called compact dissipative if there exists a nonempty compact set $K \subseteq W$ such that
$$\lim_{t \to +\infty} \sup\{\beta(U(t,\omega)M, K) \mid \omega \in \Omega\} = 0$$
for all $M \in C(W)$.

Theorem 2.9 ([6])**.** *Let Ω be a compact metric space and the cocycle φ be compact dissipative, then the following assertions are satisfied:*

(1) *the set $I_\omega = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, K, I_\omega)}$ is nonempty, compact and*
$$\lim_{t \to +\infty} \beta(U(t, \omega^{-t})K, I_\omega) = 0$$
for all $\omega \in \Omega$;

(2) *$U(t,\omega)I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}_+$;*

(3)
$$\lim_{t \to +\infty} \sup\{\beta(U(t, \omega^{-t})M, I) \mid \omega \in \Omega\} = 0$$
and
$$\lim_{t \to +\infty} \sup\{\beta(U(t, \omega)M, I) \mid \omega \in \Omega\} = 0$$
for all $M \in C(W)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$.

3. Upper semi-continuity

Lemma 3.1. *Let $\{I_\omega \mid \omega \in \Omega\}$ be a maximal compact invariant set of cocycle φ, then the function $F : \Omega \mapsto C(W)$, defined by equality $F(\omega) = I_\omega$ is upper semi-continuous, i.e. for all $\omega_0 \in \Omega$*
$$\beta(F(\omega_k), F(\omega_0)) \to 0,$$
if $\rho(\omega_k, \omega_0) \to 0$.

Proof. Let $\omega_0 \in \Omega, \omega_k \to \omega_0$ and suppose there exists $\varepsilon_0 > 0$ such that
$$\beta(F(\omega_k), F(\omega_0)) \geq \varepsilon_0.$$
Then there exists $x_k \in I_{\omega_k}$ such that
$$\rho(x_k, I_{\omega_0}) \geq \varepsilon_0. \tag{3.1}$$
As the set I is compact, without loss of generality we can suppose that the sequence $\{x_k\}$ is convergent. Denote by $x = \lim_{k \to +\infty} x_k$, then by virtue of Theorem 2.3 the set $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is compact and hence there exists $\omega_0 \in \Omega$ such that $x \in I_{\omega_0} \subset I$.

On the other hand, according to the inequality (3.1) $x \notin I_{\omega_0}$. This contradiction shows that the function F is upper semi-continuous. $\square$

Remark 3.2. Lemma 3.1 was proved for the pullback attractors of nonautonomous quasi liner differential equations in the work [9, p.13-14].

Lemma 3.3. *Let Λ be a compact metric space and $\varphi : \mathbb{T}_+ \times W \times \Lambda \times \Omega \mapsto W$ verify the following conditions:*

(1) *φ is continuous;*

(2) *for every $\lambda \in \Lambda$ the function $\varphi_\lambda = \varphi(\cdot, \cdot, \lambda, \cdot) : \mathbb{T}_+ \times W \times \Omega \mapsto W$ is a continuous cocycle on Ω with the fibre W;*

(3) *the cocycle φ_λ admits a pullback attractor $\{I_\omega^\lambda \mid \omega \in \Omega\}$ for every $\lambda \in \Lambda$;*

(4) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is precompact, where $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$,*

then the equality

$$\lim_{\lambda \to \lambda_0, \omega \to \omega_0} \beta(I_\omega^\lambda, I_{\omega_0}^{\lambda_0}) = 0 \tag{3.2}$$

holds for every $\lambda_0 \in \Lambda$ and $\omega_0 \in \Omega$ and

$$\lim_{\lambda \to \lambda_0} \beta(I_\lambda, I_{\lambda_0}) = 0 \tag{3.3}$$

for every $\lambda_0 \in \Lambda$.

Proof. Let $Y = \Lambda \times \Omega$ and $\mu : \mathbb{T} \times Y \mapsto Y$ be the mapping defined by the equality $\mu(t, (\lambda, \omega)) = (\lambda, \sigma(t, \omega))$ for every $t \in \mathbb{T}, \lambda \in \Lambda$ and $\omega \in \Omega$. It is clear that the triplet $(Y, \mathbb{T}, \mu)$ is the group dynamical system on Y and $\varphi : \mathbb{T}_+ \times W \times Y \mapsto W$ $(\varphi(t, x, (\lambda, \omega)) = \varphi(t, x, \lambda, \omega))$ is the continuous cocycle on $(Y, \mathbb{T}, \mu)$ with fibre W. Under the conditions of Lemma 3.3 the cocycle φ admits a maximal compact invariant set $\{I_y \mid y \in Y\}$ (where $I_y = I_{(\lambda, \omega)} = I_\omega^\lambda$) because

$$\bigcup \{I_y \mid y \in Y\} = \bigcup \{I_\omega^\lambda \mid \lambda \in \Lambda, \omega \in \Omega\} = \bigcup \{I^\lambda \mid \lambda \in \Lambda\}.$$

According to Lemma 3.1, the function $F : Y \mapsto C(W)$, defined by the equality $F(\lambda, \omega) = I_\omega^\lambda$, is upper semi-continuous and in particular the equality (3.3) holds.

We suppose that the equality (3.3) is not true, then there exist $\varepsilon_0 > 0, \lambda_0 \in \Lambda, \lambda_k \to \lambda_0, \omega_k \in \Omega$ and $x_k \in I_{\omega_k}^{\lambda_k}$ such that

$$\rho(x_k, I_{\lambda_0}) \geq \varepsilon_0. \tag{3.4}$$

Without loss of generality we can suppose that $\omega_k \to \omega_0, x_k \to x_0$ because the sets Ω and $\bigcup\{I_\lambda | \lambda \in \Lambda\}$ are compact. According to the inequality 3.4 we have

$$\rho(x_0, I_{\lambda_0}) \geq \varepsilon_0.$$

On the other hand $x_k \in I_{\omega_k}^{\lambda_k}$ and from the equality 3.3 we have

$$x_0 \in I_{\omega_0}^{\lambda_0} \subset I_{\lambda_0}$$

and, consequently,

$$\varepsilon_0 \leq \rho(x_0, I_{\lambda_0}) \leq \beta(I_{\omega_0}^{\lambda_0}, I_{\lambda_0}) = 0.$$

This contradiction shows that the equality (3.3) holds. $\square$

Corollary 3.4. *Under the conditions of Lemma 3.3 the equality*

$$\lim_{\lambda \to \lambda_0} \beta(I_\omega^\lambda, I_\omega^{\lambda_0}) = 0$$

holds for each $\omega \in \Omega$.

Remark 3.5. The article [2] contains a statement close to Corollary 3.4 in the case when the non-perturbed cocycle φ_{λ_0} is autonomous, i.e. the mapping $\varphi_{\lambda_0} : \mathbb{T}_+ \times W \times \Omega \to W$ does not depend of $\omega \in \Omega$.

Lemma 3.6. *Let the conditions of Lemma 3.3 and the following condition be fulfilled:*

   5. *for certain $\lambda_0 \in \Lambda$ the application $F : \Omega \mapsto C(W)$, defined by equality $F(\omega) = I_\omega^{\lambda_0}$ is continuous, i.e. $\alpha(F(\omega), F(\omega_0)) \to 0$ if $\omega \to \omega_0$ for every $\omega_0 \in \Omega$, where α is the full metric of Hausdorff, i.e. $\alpha(A, B) = \max\{\beta(A, B), \beta(B, A)\}$.*

Then

$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, I_\omega^{\lambda_0}) = 0. \tag{3.5}$$

Proof. Suppose that the equality (3.5) is not correct, then there exist $\varepsilon_0 > 0, \lambda_k \to \lambda_0, \omega_k \in \Omega$ such that

$$\beta(I_{\omega_k}^{\lambda_k}, I_{\omega_k}^{\lambda_0}) \geq \varepsilon_0. \tag{3.6}$$

On the other hand we have

$$\varepsilon_0 \leq \beta(I_{\omega_k}^{\lambda_k}, I_{\omega_k}^{\lambda_0}) \leq \beta(I_{\omega_k}^{\lambda_k}, I_{\omega_0}^{\lambda_0}) + \beta(I_{\omega_0}^{\lambda_0}, I_{\omega_k}^{\lambda_0})$$
$$\leq \beta(I_{\omega_k}^{\lambda_k}, I_{\omega_0}^{\lambda_0}) + \alpha(I_{\omega_k}^{\lambda_0}, I_{\omega_0}^{\lambda_0}). \tag{3.7}$$

According to Lemma 3.3 (see the equality (3.2))

$$\lim_{k \to +\infty} \beta(I_{\omega_k}^{\lambda_k}, I_{\omega_0}^{\lambda_0}) = 0. \tag{3.8}$$

Under the condition 5. of Lemma 3.6 we have

$$\lim_{k \to +\infty} \alpha(I_{\omega_k}^{\lambda_0}, I_{\omega_0}^{\lambda_0}) = 0. \tag{3.9}$$

From (3.7)-(3.9) passing to the limit as $k \to +\infty$, we obtain $\varepsilon_0 \leq 0$. This contradiction shows that the equality (3.5) holds. $\square$

Definition 3.7. The family of cocycle $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is called collectively compact dissipative (uniformly collectively compact dissipative), if there exists a nonempty compact set $K \subseteq W$ such that

$$\lim_{t \to +\infty} \sup\{\beta(U_\lambda(t, \omega)M, K) | \omega \in \Omega\} = 0 \qquad \forall \lambda \in \Lambda \tag{3.10}$$

$$\text{(respectively } \lim_{t \to +\infty} \sup\{\beta(U_\lambda(t, \omega)M, K) \mid \omega \in \Omega, \ \lambda \in \Lambda\} = 0)$$

for all $M \in C(W)$, where $U_\lambda(t, \omega) = \varphi_\lambda(t, \cdot, \omega)$.

Lemma 3.8. *The following conditions are equivalent:*

(1) *the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is collectively compact dissipative;*
(2) (a) *every cocycle φ_λ ($\lambda \in \Lambda$) is compact dissipative;*
    (b) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact.*

Proof. According to the equality (3.10) every cocycle φ_λ ($\lambda \in \Lambda$) is compact dissipative and $\bigcup\{I^\lambda \mid \lambda \in \Lambda\} \subseteq K$.

Suppose that the conditions a. and b. hold. Let $K = \overline{\bigcup\{I^\lambda \mid \lambda \in \Lambda\}}$, then the equality (3.10) holds. $\square$

Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a family of cocycles on $(\Omega, \mathbb{T}, \sigma)$ with fibre W and $\tilde{\Omega} = \Omega \times \Lambda$. On $\tilde{\Omega}$, we define a dynamical system $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ by equality $\tilde{\sigma}(t, (\omega, \lambda)) = (\sigma(t, \omega), \lambda)$ for all $t \in \mathbb{T}, \omega \in \Omega$ and $\lambda \in \Lambda$. By family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is generated a cocycle $\tilde{\varphi}$ on $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ with fibre W, defined in the following way: $\tilde{\varphi}(t, w, (\omega, \lambda)) = \varphi_\lambda(t, w, \omega)$ for all $t \in \mathbb{T}_+, w \in W, \omega \in \Omega$ and $\lambda \in \Lambda$.

Lemma 3.9. *The following conditions are equivalent:*

(1) *the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is uniformly collectively compact dissipative;*
(2) *the cocycle $\tilde{\varphi}$ is compact dissipative.*

Proof. This assertion follows from the fact that

$$\sup\{\beta(\tilde{U}(t, \tilde{\omega})M, K) \mid \tilde{\omega} \in \tilde{\Omega}\} = \sup\{\beta(U_\lambda(t, \omega)M, K) \mid \omega \in \Omega, \ \lambda \in \Lambda\},$$

where $\tilde{U}(t, \tilde{\omega}) = \tilde{\varphi}(t, \cdot, \tilde{\omega})$, and from the corresponding definitions. $\square$

Theorem 3.10. *Let Λ be a compact metric space and $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a family of uniformly collectively compact dissipative cocycles on $(\Omega, \mathbb{T}, \sigma)$ with fibre W, then the following are true:*

(1) *every cocycle φ_λ ($\lambda \in \Lambda$) is compact dissipative;*

(2) *the family of compacts $\{I_\omega^\lambda \mid \omega \in \Omega\} = I^\lambda$ is a Levinson's centre (compact global attractor) of cocycle φ_λ, where $I_\omega^\lambda = I_{(\omega,\lambda)}$ and $I = \{I_{(\omega,\lambda)} \mid (\omega, \lambda) \in \tilde{\Omega}\}$ is a Levinson's centre of cocycle $\tilde{\varphi}$;*

(3) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact.*

Proof. Consider the cocycle $\tilde{\varphi}$ generated by the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$. According to Lemma 3.9 $\tilde{\varphi}$ is compact dissipative and in virtue of the Theorem 4.1 [6] the following assertions take place:

1. $I_{\tilde{\omega}} = \Omega_{\tilde{\omega}}(K) \neq \emptyset$, is compact, $I_{\tilde{\omega}} \subseteq K$ and

$$\lim_{t \to +\infty} \beta(\tilde{U}(t, \tilde{\omega}^{-t})M, I_{\tilde{\omega}}) = 0 \tag{3.11}$$

for every $\tilde{\omega} \in \tilde{\Omega}$, where

$$\Omega_{\tilde{\omega}}(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \tilde{U}(\tau, \tilde{\omega}^{-\tau})K}, \tag{3.12}$$

$\tilde{\omega}^{-\tau} = \tilde{\sigma}(-\tau, \tilde{\omega})$ and K is a nonempty compact appearing in the equality (3.10);

2. $\tilde{U}(t, \tilde{\omega})I_{\tilde{\omega}} = I_{\tilde{\omega}t}$ for all $\tilde{\omega} \in \tilde{\Omega}$ and $t \in \mathbb{T}_+$;

3. the set $I = \bigcup\{I_{\tilde{\omega}} \mid \tilde{\omega} \in \tilde{\Omega}\}$ is compact.

To finish the proof we note that from the collective compact dissipativeness of the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ it follows that every cocycle φ_λ will be compact dissipative. Let $\{I_\omega^\lambda \mid \omega \in \Omega\} = I^\lambda$ be a Levinson's centre of the cocycle φ_λ, then according to Theorem 4.1 [6],

$$I_\omega^\lambda = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} U_\lambda(\tau, \omega^{-\tau})K}. \tag{3.13}$$

From (3.12) and (3.13) it follows that $I_\omega^\lambda = \Omega_{\tilde{\omega}}(K) = I_{\tilde{\omega}}$ and, consequently, $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\} \subseteq \bigcup\{I_\omega^\lambda \mid \omega \in \Omega, \lambda \in \Lambda\} = I$ for all $\lambda \in \Lambda$. Thus $\bigcup\{I^\lambda \mid \lambda \in \Lambda\} \subseteq I$ and, consequently, it is compact. The theorem is proved. $\square$

Definition 3.11. The family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ of autonomous dynamical systems is called collectively (uniformly collectively) asymptotic compact if for every bounded positive invariant set $M \subseteq X$ there exists a nonempty compact K such that

$$\lim_{t \to +\infty} \beta(\pi_\lambda^t M, K) = 0 \qquad \forall \, \lambda \in \Lambda \tag{3.14}$$

$$(\lim_{t \to +\infty} \sup_{\lambda \in \Lambda} \beta(\pi_\lambda^t M, K) = 0).$$

Definition 3.12. The bounded set $K \subset X$ is called absorbing (uniformly absorbing) for the family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ of autonomous dynamical systems if for any bounded subset $B \subset X$ there exists a number $L = L(\lambda, B) > 0$ ($L = L(B) > 0$) such that $\pi_\lambda^t B \subseteq K$ for all $t \geq L(\lambda, B)$ ($t \geq L(B)$) and $\lambda \in \Lambda$.

Theorem 3.13. *Let Λ be a complete metric space. If the family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ of autonomous dynamical systems admits an absorbing bounded set $K \subset X$ and*

is collectively asymptotic compact, then $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ *admits a global compact attractor, i.e. there exists a nonempty compact set* $K \subset X$ *such that*

$$\lim_{t \to +\infty} \beta(\pi_\lambda^t B, K) = 0 \qquad (3.15)$$

for all $\lambda \in \Lambda$ *and bounded* $B \subset X$.

Proof. Let the family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ of autonomous dynamical systems be collectively asymptotic compact and a bounded M be its absorbing set. According to Theorem 3.4 [6] the nonempty set $K = \Omega(M)$ is compact and the equality 3.15 holds. The theorem is proved. $\qquad \square$

Theorem 3.14. *Let* Λ *be a complete compact metric space. If the family* $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ *of autonomous dynamical systems admits a uniformly absorbing bounded set* $K \subset X$ *and it is uniformly collectively asymptotic compact, then* $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ *admits a uniform compact global attractor, i.e. there exists a nonempty compact set* $K \subset X$ *such that*

$$\lim_{t \to +\infty} \sup_{\lambda \in \Lambda} \beta(\pi_\lambda^t B, K) = 0 \qquad (3.16)$$

for all bounded $B \subset X$.

Proof. Consider the autonomous dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ on $\tilde{X} = X \times \Lambda$ defined by equality $\tilde{\pi}(t, (x, \lambda)) = (\pi_\lambda(t, x), \lambda)$ for all $t \in \mathbb{T}_+, x \in X$ and $\lambda \in \Lambda$. We note that under the conditions of Theorem 3.14 the bounded set $K \times \Lambda$ is absorbing for dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ if the set K is uniformly absorbing for the family $\{(X, \mathbb{T}_+, \pi_\lambda)\}_{\lambda \in \Lambda}$ and $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ is asymptotically compact. According to Theorem 3.4 [5] (see also Theorem 2.2.5 [4]) the dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ admits a compact global attractor $\tilde{K} \subset \tilde{X} = X \times \Lambda$. To finish the proof it is sufficient to note that the set $K = pr_1 \tilde{K} \subset X$ is compact and

$$\sup_{\lambda \in \Lambda} \beta(\pi_\lambda^t B, K) \le \beta(\tilde{\pi}_\lambda^t B, K_0) \to 0$$

as $t \to +\infty$, where $K_0 = K \times \Lambda \supset \tilde{K}$, for all bounded subset $B \subset X$. $\qquad \square$

Let φ be a cocycle on $(\Omega, \mathbb{T}, \sigma)$ with fibre W and $(X, \mathbb{T}_+, \pi)$ be a skew-product dynamical system, where $X = W \times \Omega$ and $\pi(t, (w, \omega)) = (\varphi(t, w, \omega), \omega t)$ for all $t \in \mathbb{T}_+, w \in W$ and $\omega \in \Omega$.

Definition 3.15. The cocycle φ is called asymptotically compact (a family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is called collectively asymptotically compact) if a skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ (a family of skew-product dynamical systems $(X, \mathbb{T}_+, \pi_\lambda)_{\lambda \in \Lambda}$) is asymptotically compact.

Theorem 3.16. *Let* Ω *and* Λ *be compact metric spaces,* W *be a Banach space and* $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ *be a family of cocycles on* $(\Omega, \mathbb{T}, \sigma)$ *with fibre* W. *If there exist* $r > 0$ *and the function* $V_\lambda : W \times \Omega \to \mathbb{R}_+$ *for all* $\lambda \in \Lambda$, *with the following properties:*

(1) *the family of cocycles* $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ *is collectively asymptotically compact;*
(2) *the family of functions* $\{V_\lambda\}_{\lambda \in \Lambda}$ *is collectively bounded on bounded sets and for every* $c \in \mathbb{R}_+$ *the sets* $\{x \in X_r | V_\lambda(x) \le c\}$ *uniformly bounded;*
(3) $V_\lambda'(w, \omega) \le -c(|w|)$ *for all* $w \in W_r = \{w \in W | \ |w| \ge r\}, \omega \in \Omega$ *and* $\lambda \in \Lambda$, *where* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *is positive on* $[r, +\infty)$, $V_\lambda'(w, \omega) = \lim_{t \to 0+} \sup t^{-1} [V_\lambda(\varphi_\lambda(t,$

$w, \omega), \omega t) - V_\lambda(w, \omega)]$ *if* $\mathbb{T} = \mathbb{R}_+$ *and* $V'_\lambda(w, \omega) = V_\lambda(\varphi_\lambda(1, w, \omega), \omega 1) - V_\lambda(w, \omega)$ *if* $\mathbb{T} = \mathbb{Z}_+$.

Then every cocycle φ_λ ($\lambda \in \Lambda$) *admits a uniform compact global attractor* I^λ ($\lambda \in \Lambda$) *and the set* $\bigcup \{I^\lambda \mid \lambda \in \Lambda\}$ *is compact.*

Proof. Let $X = W \times \Omega$ and $(X, \mathbb{T}, \pi_\lambda)$ be a skew-product dynamical system, generated by the cocycle φ_λ, then (X, h, Ω), where $h = pr_2 : X \to \Omega$, is a trivial fibering with fibre W. Under the conditions of Theorem 3.16 and according to Theorem 3.4 [5] the nonautonomous dynamical system $\langle (X, \mathbb{T}_+, \pi_\lambda), (\Omega, \mathbb{T}, \sigma), h \rangle$ admits a compact global attractor J^λ and according to Theorem 4.1 [6] the cocycle φ_λ admits a compact global attractor $I^\lambda = \{I^\lambda_\omega | \omega \in \Omega\}$, where $I^\lambda_\omega = pr_1 J^\lambda_\omega$ and $J^\lambda_\omega = pr_2^{-1}(\omega) \bigcap J^\lambda$.

Let $\tilde{\Omega} = \Omega \times \Lambda$, $(\tilde{\Omega}, \mathbb{T}, \tilde{\sigma})$ be a dynamical system on $\tilde{\Omega}$ defined by the equality $\tilde{\sigma}(t, (\omega, \lambda))$
$= (\sigma(t, \omega), \lambda)$ (for all $t \in \mathbb{T}, \omega \in \Omega$ and $\lambda \in \Lambda$), $\tilde{X} = W \times \tilde{\Omega}$ and $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ be an autonomous dynamical system defined by equality $\tilde{\pi}(t, (w, \tilde{\omega})) = (\pi_\lambda(t, w), (\omega t, \lambda))$ for all $\tilde{\omega} = (\omega, \lambda) \in \tilde{\Omega} = \Omega \times \Lambda$. Note that the triplet $(\tilde{X}, h, \Omega)$, where $h = pr_2 : \tilde{X} \to \tilde{\Omega}$, is a trivial fibering with fibre $W, \langle (\tilde{X}, \mathbb{T}_+, \tilde{\pi}), (\tilde{\Omega}, \mathbb{T}, \tilde{\sigma}), h \rangle$ is a nonautonomous dynamical system. The function $\tilde{V} : \tilde{X}_r = W_r \times \tilde{\Omega} \to \mathbb{R}_+$, defined by the equality $\tilde{V}(\tilde{x}) = V_\lambda(w, \omega)$ for all $\tilde{x} = (w, (\omega, \lambda)) \in \tilde{X}_r$ under the conditions of Theorem 3.16, verifies all the conditions of Theorem 5.3 [7] and, consequently, the dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ admits a compact global attractor. To finish the proof of the theorem it is sufficiently to note that if the dynamical system $(\tilde{X}, \mathbb{T}_+, \tilde{\pi})$ admits a compact global attractor $\tilde{J}$, then the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is uniformly collectively compact dissipative and according to Theorem 3.10 the set $I = \bigcup \{I^\lambda \mid \lambda \in \Lambda\}$ is compact, where $I^\lambda = \{I^\lambda_\omega \mid \omega \in \Omega\}$ is the compact global attractor of cocycle φ_λ. The theorem is proved. $\square$

4. CONNECTEDNESS

Definition 4.1. We will say that the space W possesses the property (S) if for every compact $K \in C(W)$ there exists a compact connected set $V \in C(W)$ such that $K \subseteq V$.

Remark 4.2. 1. It is clear that if the space W possesses the property (S), then it is connected. The inverse statement generally speaking is not true.

2. Every linear vectorial topological space W possesses the property (S), because the set $V(K) = \{\lambda x + (1 - \lambda)y \mid x, y \in K, \lambda \in [0, 1]\}$ is connected, compact and $K \subseteq V(K)$.

If $M \subseteq W$, for each $\omega \in \Omega$, we write

$$\Omega_\omega(M) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M, \omega^{-\tau})}.$$

Lemma 4.3. [6]. *The following all hold:*

(1) *the point* $p \in \Omega_\omega(M)$ *if and only if, when there are* $t_n \to +\infty$ *and* $\{x_n\} \subseteq M$ *such that* $p = \lim_{n \to +\infty} \varphi(t_n, x_n, \omega^{-t_n})$;

(2) $\varphi(t, \Omega_\omega(M), \omega) \subseteq \Omega_{\omega t}(M)$ *for all* $\omega \in \Omega$ *and* $t \in \mathbb{T}_+$;

(3) *If there exists a nonempty compact $K \in C(W)$ such that*

$$\lim_{t \to +\infty} \beta(\varphi(t, M, \omega^{-t}), K) = 0,$$

then $\Omega_\omega(M) \neq \emptyset$, is compact,

$$\lim_{t \to +\infty} \beta(\varphi(t, M, \omega^{-t}), \Omega_\omega(M)) = 0 \tag{4.1}$$

and

$$\varphi(t, \Omega_\omega(M), \omega) = \Omega_{\omega t}(M) \tag{4.2}$$

for all $\omega \in \Omega$ and $t \in \mathbb{T}_+$.

Lemma 4.4. *Suppose that the cocycle φ admits a compact pullback attractor $\{I_\omega \mid \omega \in \Omega\}$, then the following hold:*

 a. *$\emptyset \neq \Omega_\omega(M) \subseteq I_\omega$ for every $M \in C(W)$ and $\omega \in \Omega$;*
 b. *the family $\{\Omega_\omega(M) \mid \omega \in \Omega\}$ is compact and invariant w.r.t. cocycle φ for every $M \in C(W)$;*
 c. *if $I = \bigcup\{I_\omega \mid \omega \in \Omega\} \subseteq M$, then the following inclusion $I_\omega \subseteq \Omega_\omega(M)$ holds for every $\omega \in \Omega$.*

Proof. The first and second assertions follow from the definition of pullback attractor and from the equalities (4.1)-(4.2).

Let I be a subset of M, then

$$I_\omega = \varphi(t, I_{\omega^{-t}}, \omega^{-t}) \subseteq \varphi(t, I, \omega^{-t}) \subseteq \varphi(t, M, \omega^{-t}) \tag{4.3}$$

and according to the equality (4.1) we have $I_\omega \subseteq \Omega_\omega(M)$ for each $\omega \in \Omega$. $\qquad\square$

Theorem 4.5. *Let W possess the property (S) and let the cocycle φ admit a compact pullback attractor $\{I_\omega \mid \omega \in \Omega\}$, then:*

 (1) *the set I_ω is connected for every $\omega \in \Omega$;*
 (2) *if the space Ω is connected, then the set $I = \bigcup\{I_\omega | \omega \in \Omega\}$ also is connected.*

Proof. 1. Since the equality (2.2) holds and the space W possesses the property (S), then there exists a connected compact $V \in C(W)$ such that $I \subseteq V$ and

$$\lim_{t \to +\infty} \beta(\varphi(t, V, \omega^{-t}), I_\omega) = 0, \tag{4.4}$$

for every $\omega \in \Omega$. We shall show that the set I_ω is connected. If we suppose that it is not true, then there are $A_1, A_2 \neq \emptyset$, closes and $A_1 \bigsqcup A_2 = I_\omega$. Let $0 < \varepsilon_0 < d(A_1, A_2)$ and $L = L(\varepsilon_0) > 0$ be such that

$$\beta(\varphi(t, V, \omega^{-t}), I_\omega) < \frac{\varepsilon_0}{3} \tag{4.5}$$

for all $t \geq L(\varepsilon_0)$. We note that the set $\varphi(t, V, \omega^{-t})$ is connected and according to the inclusion (4.3) and the inequality (4.5) the following condition

$$\varphi(t, V, \omega^{-t}) \bigcap (W \setminus [B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})]) \neq \emptyset$$

is fulfilled for every $t \geq L(\varepsilon_0)$ and $\omega \in \Omega$, where $B(A, \varepsilon) = \{u \in W | \rho(u, A) < \varepsilon\}$. Then there exists $t_n \to +\infty$ and $u_n \in W$ such that

$$u_n \in \varphi(t_n, V, \omega^{-t_n}) \bigcap (W \setminus [B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})]). \tag{4.6}$$

According to the equality (4.4) it is possible to suppose that the sequence $\{u_n\}$ is convergent. We denote by $u = \lim_{n \to +\infty} u_n$, then from Lemma 4.3 follows that $u \in$

$\Omega_\omega(V)$. Since $I \subseteq V$, then according to Lemma 4.3 we have $u \in \Omega_\omega(V) \subseteq I_\omega \subseteq I$. On the other hand according to (4.6) we have $u \notin B(A_1, \frac{\varepsilon_0}{3}) \bigsqcup B(A_2, \frac{\varepsilon_0}{3})$. This contradiction shows that the set I_ω is connected.

2. Let the space Ω be connected. According to Lemma 3.3 the function $F : \Omega \mapsto C(W)$, defined by equality $F(\omega) = I_\omega$ is upper semi-continuous and from the corollary 1.8.13 [4] (see also Lemma 3.1 [14]) follows that the set $I = \bigcup \{I_\omega \mid \omega \in \Omega\} = F(\Omega)$ is connected. $\qquad\square$

Corollary 4.6. *Let W be a metric space with the property (S) and let the cocycle φ admit a compact global attractor $\{I_\omega \mid \omega \in \Omega\}$, then:*

(1) *the set I_ω is connected for every $\omega \in \Omega$;*
(2) *if the space Ω is connected, then the set $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ also is connected.*

This affirmation follows from Theorems 2.6, 4.5 and Lemma 3.1.

5. Some applications

Quasihomogeneous systems. Let E and G be two finite dimensional spaces. The function $f \in C(E \times G, E)$ is called [11, 12] homogeneous of order m with respect to variable $u \in E$ if the equality $f(\lambda u, \omega) = \lambda^m f(u, \omega)$ holds for all $\lambda \geq 0, u \in E$ and $\omega \in G$.

Theorem 5.1. *Let $f \in C^1(E), \Phi \in C^1(G), \Omega \subseteq G$ be a compact invariant set of dynamical system*

$$\omega' = \Phi(\omega), \tag{5.1}$$

the function f be homogeneous (of order $m > 1$) and a zero solution of equation

$$u' = f(u) \tag{5.2}$$

be uniformly asymptotically stable. If $F \in C^1(E \times G, E)$ and

$$|F(u, \omega)| \leq c|u|^m$$

for all $|u| \geq r$ and $\omega \in \Omega$, where r and c are certain positive numbers, then there exists a positive number λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following holds:

(1) *a set $I_\omega^\lambda = \{u \in E \mid \sup\{|\varphi_\lambda(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi_\lambda(t, u, \omega)$ is a unique solution of equation $u' = f(u) + \lambda F(u, \omega t)$ satisfying the initial condition $\varphi_\lambda(0, u, \omega) = u$;*
(2) *$\varphi_\lambda(t, I_\omega^\lambda, \omega) = I_{\sigma(t,\omega)}^\lambda$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
(3) *the set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*
(4) *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega_{-t}), I_\omega^\lambda) = 0 \tag{5.3}$$

and

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0$$

take place for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.
(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*
(6) *the equality*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, 0) = 0$$

holds.

Proof. Under the condition of Theorem 5.1 according to Theorem 3.2 [12] by equality

$$V(u) = \int_0^{+\infty} |\pi(t,u)|^k dt,$$

where $\pi(t,u)$ is a solution of equation 5.2 with condition $\pi(0,u) = u$, is defined a continuously differentiable function $V : E \to \mathbb{R}_+$, verifying the following conditions:

a. $V(\mu u) = \mu^{k-m+1} V(u)$ for all $\mu \geq 0$ and $u \in E$;
b. there exist positive numbers α and β such that $\alpha |u|^{k-m+1} \leq V(u) \leq \beta |u|^{k-m+1}$ for all $u \in E$;
c. $V'(u) = DV(u)f(u) = -|u|^k$ for all $u \in E$, where $DV(u)$ is a derivative of Frechet of function V in the point u.

Let us define a function $\mathfrak{V} : X \to \mathbb{R}_+$ ($X = E \times \Omega$) in the following way: $\mathfrak{V}(u,\omega) = V(u)$ for all $(u,\omega) \in X$. Note that

$$\mathfrak{V}'(u,\omega) = \frac{d}{dt} V(\varphi_\lambda(t,u,\omega))|_{t=0} = -|u|^k + DV(u)\lambda F(u,\omega)$$

and there exists $\lambda_0 > 0$ such that the inequality

$$\mathfrak{V}'(u,\omega) \leq -\nu |u|^k$$

holds for all $\omega \in \Omega$ and $|u| \geq r$, where $\nu = 1 - \lambda_0 cL > 0$ (see the proof of Theorem 4.3 [12]).

For finishing the proof of the theorem it is sufficient to refer to Theorem 3.16 and Lemma 3.6. $\qquad \square$

Theorem 5.2. *Let $f \in C^1(E \times G, E), \Phi \in C^1(F), \Omega \subseteq G$ be a compact invariant set of dynamical system (5.1), the function f be homogeneous (of order $m = 1$) w.r.t. variable $u \in E$ and a zero solution of equation*

$$u' = f(u,\omega t) \quad (\omega \in \Omega) \tag{5.4}$$

be uniformly asymptotically stable. If $|F(u,\omega)| \leq c|u|$ for all $|u| \geq r$ and $\omega \in \Omega$, where r and c are certain positive numbers, then there exists a positive number λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following assertions take place:

(1) *a set $I_\omega^\lambda = \{u \in E \mid \sup\{|\varphi_\lambda(t,u,\omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi_\lambda(t,u,\omega)$ is a unique solution of equation*

$$u' = f(u,\omega t) + \lambda F(u,\omega t)$$

verifying the initial condition $\varphi_\lambda(0,u,\omega) = u$;
(2) *$\varphi_\lambda(t, I_\omega^\lambda, \omega) = I_{\sigma(t,\omega)}^\lambda$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
(3) *a set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*
(4) *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t,M,\omega_{-t}), I_\omega^\lambda) = 0$$

and

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t,M,\omega), I^\lambda) = 0 \tag{5.5}$$

take place for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.
(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*
(6) *the following equality holds:*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, 0) = 0 \,.$$

The proof of this assertion is similar to the proof of Theorem 5.1.

Monotone systems. Let $f \in C(E \times \Omega, E)$ be a function satisfying

$$\operatorname{Re}\langle f(u_1, \omega) - f(u_2, \omega), u_1 - u_2 \rangle \leq -k|u_1 - u_2|^\alpha \tag{5.6}$$

for all $\omega \in \Omega$ and $u_1, u_2 \in E$ ($k > 0$ and $\alpha \geq 2$).

Theorem 5.3 ([21, 10])**.** *If the function f verifies the condition (5.6), then the following statements are true:*

(1) *the set $I_\omega = \{u \in E \mid \sup_{t \in \mathbb{R}} |\varphi(t, u, \omega)| < +\infty\}$ contains a single point $\{\varphi(\omega)\}$ for all $\omega \in \Omega$, where $\varphi(t, u, \omega)$ is a solution of equation (5.3) with condition $\varphi(0, u, \omega) = u$;*

(2) *the inequalities*

$$|\varphi(t, u, \omega) - \varphi(\omega t)| \leq e^{-kt}|u - \varphi(u)| \quad (\text{if } \alpha = 2),$$

$$|\varphi(t, u, \omega) - \varphi(\omega t)| \leq (|u - \varphi(u)|^{2-\alpha} + (\alpha - 2)t)^{\frac{1}{2-\alpha}} \quad (\text{if } \alpha > 2)$$

hold for all $t \geq 0$, $u \in E$ and $\omega \in \Omega$;

(3) *the function $\gamma : \Omega \to E$, defined by equality $\gamma(\omega) = I_\omega$ is continuous and $\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega)$ for all $t \geq 0$, $u \in E$ and $\omega \in \Omega$.*

Theorem 5.4. *Let $f \in C(E \times \Omega, E)$ be a function verifying the condition (5.6) and $F \in C(E \times \Omega, E)$ be a function with the condition*

$$\operatorname{Re}\langle F(u_1, \omega) - F(u_2, \omega), u_1 - u_2 \rangle \leq L|u_1 - u_2|^\alpha \tag{5.7}$$

for all $u_1, u_2 \in E$ and $\omega \in \Omega$, where L is some positive number.

Then there exists a positive number λ_0 such that for all $|\lambda| \leq \lambda_0$ the following hold:

(1) *the set $I_\omega^\lambda = \{u \in E \mid \sup_{t \in \mathbb{R}} |\varphi_\lambda(t, u, \omega)| < +\infty\}$ contains a single point $\{\varphi_\lambda(\omega)\}$ for each $\omega \in \Omega$, where $\varphi_\lambda(t, u, \omega)$ is a unique solution of the equation*

$$u' = f(u, \omega t) + \lambda F(u, \omega t) \quad (\omega \in \Omega) \tag{5.8}$$

satisfying the initial condition $\varphi_\lambda(0, u, \omega) = u$;

(2) *the function $\varphi_\lambda : \Omega \to E$ defined by equality $\varphi_\lambda(\omega) = I_\omega^\lambda$ is continuous and $\varphi_\lambda(\omega t) = \varphi_\lambda(t, \varphi(\omega), \omega)$ for all $t \geq 0$, $u \in E$ and $\omega \in \Omega$.*

(3)

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} |\varphi_\lambda(\omega) - \varphi(\omega)| = 0.$$

Proof. Let $g = f + \lambda F$, then from (5.6)-(5.9) follows that

$$\operatorname{Re}\langle g(u_1, \omega) - g(u_2, \omega), u_1 - u_2 \rangle \leq (-k + L|\lambda|)|u_1 - u_2|^\alpha \tag{5.9}$$

for all $u_1, u_2 \in E$ and $\omega \in \Omega$. From (5.9) follows that there exists $\lambda_0 > 0$ such that $-k + L|\lambda| \leq -k + L\lambda_0 < 0$ for all $|\lambda| \leq \lambda_0$ and according to Theorem 5.3 the assertions 1. and 2. of the theorem are true.

It is clear that for $|\lambda| \leq \lambda_0$ the cocycle φ_λ generated by the equation (5.8) admits a compact global attractor $I^\lambda = \{\varphi_\lambda(\omega) \mid \omega \in \Omega\}$.

Now we will show that the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda = [-\lambda_0, \lambda_0]\}$ is compact. Let $V(u) = \frac{1}{2}|u|^2$, then

$$
\begin{aligned}
V'(u) =& \frac{d}{dt} V(\varphi_\lambda(t, u, \omega))|_{t=0} \\
=& \operatorname{Re}\langle g(u, \omega), u \rangle \\
=& \operatorname{Re}\langle g(u, \omega) - g(0, \omega), u \rangle + \operatorname{Re}\langle g(0, \omega), u \rangle \\
\leq& (-k + L|\lambda_0|)|u|^2 + C|u| \\
=& |u|^2(-k + L|\lambda_0| + \frac{C}{|u|^{\alpha-1}}),
\end{aligned}
\tag{5.10}
$$

where $C = \max\{|g(0, \omega)| : \omega \in \Omega, \lambda \in \Lambda\}$. From the equality (5.10) follows that there exists $r > 0$ such that for all $|u| \geq r$

$$
V'(u) \leq -\nu |u|^2,
\tag{5.11}
$$

where $\nu = k - L|\lambda_0| - \frac{C}{r^{\alpha-1}} > 0$. Now to finish the proof of Theorem 5.3 it is sufficient to refer to Theorem 3.16. The theorem is proved. $\qquad\square$

Quasilinear systems. Consider a nonautonomous quasilinear system

$$
u' = A(\omega t)u + \lambda f(u, \omega t) \quad (\omega \in \Omega)
$$

on E. Denote by $[E]$ the space of all linear continuous operators acting onto E and equipped with the operational norm.

Theorem 5.5. *Let $A \in C(\Omega, [E]), f \in C(E \times \Omega, E)$ and let the following conditions be fulfilled:*

(1) *there exists a positive constant α_0 such that $\operatorname{Re}\langle A(\omega)u, u \rangle \leq -\alpha_0|u|^2$ for all $u \in E$ and $\omega \in \Omega$;*

(2) *for any $r > 0$ there exists a positive constant $L(r)$ such that*

$$
|f(u_1, \omega) - f(u_2, \omega)| \leq L|u_1 - u_2|
$$

for all $u_1, u_2 \in B[0, r] = \{u \in E \mid |u| \leq r\}$ and $\omega \in \Omega$.

Then there exists a positive constant λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following are true:

(1) *the set $I_\omega^\lambda = \{u \in E \mid \sup\{|\varphi_\lambda(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi_\lambda(t, u, \omega)$ there is a unique solution of equation*

$$
u' = A(\omega t)u + \lambda F(u, \omega t)
$$

satisfying the initial condition $\varphi_\lambda(0, u, \omega) = u$;

(2) *$\varphi_\lambda(t, I_\omega^\lambda, \omega) = I_{\sigma(t, \omega)}^\lambda$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*

(3) *the set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*

(4) *the equalities*

$$
\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega_{-t}), I_\omega^\lambda) = 0,
$$

$$
\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0
$$

hold for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.

(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*

(6) *the following equality is true*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, 0) = 0.$$

Proof. Let λ_0 be a positive number such that $\nu = \alpha_0 - \lambda_0 \alpha > 0$, then the function $F_\lambda(u, \omega) = A(\omega)u + \lambda f(u, \omega)$ verifies the condition

$$\text{Re}\langle F_\lambda(u, \omega), u \rangle \leq -\nu |u|^2 + \lambda_0 \beta \tag{5.12}$$

for all $|\lambda| \leq \lambda_0, \omega \in \Omega$ and $u \in E$.

From the inequality (5.12) follows (see, for example, [9, p.11]) that the inequality

$$|\varphi_\lambda(t, u, \omega)|^2 \leq |u|^2 e^{-2\nu t} + \frac{\lambda_0 \beta}{\nu}(1 - e^{-2\nu t}) \tag{5.13}$$

holds for all $t \in \mathbb{R}_+, \lambda \in \Lambda$ and $(u, \omega) \in E \times \Omega$ and, consequently, the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ admits a bounded absorbing set. Now to finish the proof of the theorem it is sufficient to refer to Theorems 3.10, 3.14 and Lemma 3.6. $\square$

Nonautonomously perturbed systems.

Theorem 5.6. *Suppose that $f \in C(E)$ is uniformly Lipshitzian and an autonomous system (5.1) has a global attractor I. Furthermore suppose that $F \in C(E \times \Omega, E)$ is uniformly Lipshitz in $u \in E$ and it is uniformly bounded on $E \times \Omega$, i.e. $\sup\{|F(u, \omega)| : (u, \omega) \in E \times \Omega\} = K < +\infty$. Then there exists a positive number $\lambda_0 > 0$ such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following are true:*

(1) *the set $I_\omega^\lambda = \{u \in E \mid \sup\{|\varphi_\lambda(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi_\lambda(t, u, \omega)$ there is a unique solution of equation $u' = f(u) + \lambda F(u, \omega t)$ satisfying the initial condition $\varphi_\lambda(0, u, \omega) = u$;*

(2) *$\varphi_\lambda(t, I_\omega^\lambda, \omega) = I_{\sigma(t, \omega)}^\lambda$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*

(3) *the set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*

(4) *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega_{-t}), I_\omega^\lambda) = 0$$

and

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0$$

take place for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.

(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*

(6) *the following equality is true*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, I) = 0 \,.$$

Proof. According to [22, Theorem 22.5] (see also [16, 17]), under its conditions there exists a continuous function $V : E \to \mathbb{R}_+$ with the following properties:

a. V is uniformly Lipshitz in E, i.e. there exists a constant $L > 0$ such that $|V(u_1) - V(u_2)| \leq L|u_1 - u_2|$ for all $u_1, u_2 \in E$;

b. there exist continuous strictly increasing functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = b(0) = 0$ and $0 < a(r) < b(r)$ for all $r > 0$ such that $a(\beta(u, I)) \leq V(u) \leq b(\beta(u, I))$ for all $u \in E$;

c. there exists a constant $c > 0$ such that $V'(u) \leq -cV(u)$ for all $u \in E$, where $V'(u) = \lim_{t \to 0+} \sup t^{-1}[V(\pi(t, u)) - V(u)]$ and $\pi(t, u)$ is a unique solution of equation (5.2) with initial condition $\pi(0, u) = u$.

Define a function $\mathfrak{V} : X \to \mathbb{R}_+$ ($X = E \times \Omega$) in the following way: $\mathfrak{V}(x) := V(u)$ for all $x = (u, \omega) \in X$. Note that

$$\mathfrak{V}'(u, \omega) = \lim_{t \to 0+} \sup V(\varphi_\lambda(t, u, \omega))|_{t=0} \leq LKr - c\mathfrak{V}(u, \omega)$$

(see [17, p.11]) for all $u \in E$. Then there exist $\lambda_0 > 0$ and $r_0 > 0$ such that

$$\mathfrak{V}'(u, \omega) \leq -LK\lambda_0$$

for all $|u| \geq r_0$ and $\omega \in \Omega$.

To finish the proof of the theorem it is sufficient to refer to Theorem 3.16 and Lemma 3.6.   □

Remark 5.7. Similar theorem was proved in [17, Th.4.1] for the pullback attractors of nonautonomously perturbed systems.

Non-autonomous 2D Navier Stokes equation. Let $G \subset \mathbb{R}^2$ be a bounded domain with C^2 smooth boundary,

$$V = \{u \in (\dot{W}_2^1(G))^2, \ \text{div}u(x) = 0\}, \ H = \overline{V}^{(L_2(G))^2},$$

V' be the dual space of V, $(\dot{W}_2^1(G))^2$ denotes the Sobolev space of functions having two components, and let π be the orthogonal projector from $(L_2(G))^2$ onto H. The operator $F(u, v) = \pi(u, \nabla)v$ has values in V'.

Let Ω be a compact complete metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω, $\mathcal{F} \in C(V \times \Omega, V)$ and and satisfy the the following conditions:

  (i) $|\mathcal{F}(u_1, \omega) - \mathcal{F}(u_2, \omega)| \leq L|u_1 - u_2|$ for all $u_1, u_2 \in V$ and $\omega \in \Omega$;
  (ii) $\text{Re}\langle \mathcal{F}(u, \omega), u \rangle \leq M|u|^2 + N$ for all $u \in V$ and $\omega \in \Omega$, where L, M and N are some positive constants.

Consider the perturbed $2D$ Navier Stokes equation

$$u' + \nu Au + B(u, u) = \mathcal{F}(u, \omega t) \quad (\omega \in \Omega) \tag{5.14}$$

on H, where $B : V \times V \to V'$ is a bilinear continuous form and A is the extension of $-\pi\nabla$ with zero boundary conditions on V and $\nu > 0$. In particular, there exists $\lambda_1 > 0$ such that

$$\langle Au, u \rangle \ \geq |u|_V^2 \geq \lambda_1 |u|_H^2$$

for any $u \in V$. According to [20],[19] by equation (5.14) is generated a cocycle $\varphi(t, u, \omega)$ on $(\Omega, \mathbb{R}, \sigma)$ with fibre H, where $\varphi(t, u, \omega)$ is a unique solution of equation (5.14) with the condition $\varphi(0, u, \omega) = u$.

Lemma 5.8. *Under the conditions (i) and (ii) the following holds:*

  (1) *for any $T > 0$, $\nu > 0$, $\omega \in \Omega$ and any $u \in H$ the equation (5.11) has a unique solution $\varphi(t, u, \omega)$ with path in $C([0, T], H)$;*
  (2) *the energy inequality holds*

$$\frac{d}{dt}|\varphi(t, u, \omega)|_H^2 + \nu\lambda_1\varphi(t, u, \omega)|_H^2 \leq \frac{|\mathcal{F}(0, \omega t)|_H^2}{\nu\lambda_1} + 2L|\varphi(t, u, \omega)|_H^2 \tag{5.15}$$

  for all $t \in [0, T], u \in H$ and $\omega \in \Omega$;
  (3) *the mapping $\varphi : \mathbb{R}_+ \times H \times \Omega \to H$ is continuous.*

Proof. The assertions 1. and 2. follow from [13] (see also Lemma 3.1 [19]).

Now we will prove that the mapping $\varphi : \mathbb{R}_+ \times H \times \Omega \to H$ is continuous. Let $t_0 \in \mathbb{R}_+, u_0 \in H$ and $\omega_0 \in \Omega$, then we have

$$|\varphi(t, u, \omega) - \varphi(t_0, u_0, \omega_0)|_H$$
$$\leq |\varphi(t, u, \omega) - \varphi(t, u_0, \omega_0)|_H + |\varphi(t, u_0, \omega_0) - \varphi(t_0, u_0, \omega_0)|_H. \quad (5.16)$$

Denote by $w(t) = \varphi(t, u, \omega) - \varphi(t, u_0, \omega_0)$ and $f(t) = \mathcal{F}(\varphi(t, u, \omega), \omega t) - \mathcal{F}(\varphi(t, u_0, \omega_0), \omega_0 t)$, then the function $w(t)$ verifies the following equation

$$\frac{dw}{dt} + \nu A w + B(w, w) + B(w, u_1) + B(u_1, w) = f(t), \quad (5.17)$$

where $u_1 = \varphi(t, u_0, \omega_0)$. Using the well-known identity $\langle B(u, v), v \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the scalar product in H, we obtain

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 = \langle \dot{w}, w \rangle = \langle -\nu A w - B(w, w) - B(w, u_1) - B(u_1, w) + f(t), w \rangle$$
$$= -\nu\langle A w, w \rangle - \langle B(w, w), w \rangle - \langle B(w, u_1), w \rangle - \langle B(u_1, w), w \rangle + \langle f(t), w \rangle$$
$$= -\nu\langle A w, w \rangle - \langle B(w, u_1), w \rangle + \langle f(t), w \rangle.$$

Bearing in mind the inequality (see [18]) $|u|_{L_4}^2 \leq |w|_H |w|_V$ we obtain

$$|\langle B(w, u_1), w \rangle| \leq |w|_{L_4}^2 |u_1|_V \leq |w|_H |w|_V |u_1|_V \leq \frac{\nu}{2}|w|_V^2 + \frac{1}{2\nu}|w|_H^2 |u_1|_V^2.$$

Taking into account that $|(f, w)| \leq |f|_H |w|_H$, we get from (5.18)

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 \leq -\nu\lambda_1 |w|_V^2 + \frac{\nu\lambda_1}{2}|w|_V^2 + \frac{1}{2\nu\lambda_1}|w|_H^2 |u_1|_V^2 + |f|_H |w|_H. \quad (5.19)$$

We remark that

$$|f(t)|_H = |\mathcal{F}(\varphi(t, u, \omega), \omega t) - \mathcal{F}(\varphi(t, u_0, \omega_0), \omega_0 t)|_H$$
$$\leq L|\varphi(t, u_0, \omega_0) - \varphi(t, u_0, \omega_0)| + |\mathcal{F}(\varphi(t, u_0, \omega_0), \omega t) - \mathcal{F}(\varphi(t, u_0, \omega_0), \omega_0 t)| \quad (5.20)$$

and, consequently, from (5.18)-(5.20), we obtain

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 \leq (\frac{1}{2\nu}|u_1|_V^2 + L + \frac{1}{2})|w|_H^2 + \frac{|f|^2}{2}. \quad (5.21)$$

From this differential inequality we deduce that

$$|w(t)|_H^2 \leq \exp\Big(\int_0^t (\frac{1}{2\nu}|\varphi(\tau, u_0, \omega_0)|_V^2 + L + \frac{1}{2})d\tau\Big)|u - u_0|_H^2$$
$$+ \int_0^t \exp\Big(-\int_0^\tau (\frac{1}{2\nu}|\varphi(s, u_0, \omega_0)|_V^2 + L + \frac{1}{2})ds\Big) \quad (5.22)$$
$$\times \frac{1}{2}\big|\mathcal{F}(\varphi(\tau, u_0, \omega_0), \omega\tau) - \mathcal{F}(\varphi(\tau, u_0, \omega_0), \omega_0\tau)\big|d\tau.$$

Since $\mathcal{F} \in C(H \times \Omega, H)$, then

$$\max_{0 \leq t \leq T} |\mathcal{F}(\varphi(t, u_0, \omega_0), \omega t) - \mathcal{F}(\varphi(t, u_0, \omega_0), \omega_0 t)| \to 0$$

as $\omega \to \omega_0$ and, consequently, from (5.22) we obtain

$$\max_{0 \leq t \leq T} |w(t)| \to 0. \quad (5.23)$$

From (5.16) and (5.23) we obtain the continuity of mapping φ. The lemma is proved. $\qquad\square$

Corollary 5.9. *Under the conditions (i) and (ii) there exists a positive number $L_0 < \frac{\nu\lambda_1}{2}$ such that if $L < L_0$, then the following inequality*

$$|\varphi(t,u,\omega)|^2 \le e^{(-\nu\lambda_1 + 2L_0)t}|u|^2 + \frac{|f|^2}{\nu\lambda_1(-2L_0 + \nu\lambda_1)}$$

holds for all $t \ge 0$, $u \in H$ and $\omega \in \Omega$, where $|f| = \max_{\omega \in \Omega} |\mathcal{F}(0,\omega)|$.

This assertion follows from the second assertion of Lemma 5.8.

Theorem 5.10. *There exists a positive number $L_0 > 0$ such that the cocycle φ generated by (5.14) admits a compact global attractor, if $L \le L_0$.*

Proof. According to Lemma 3.1 [19] there exists $L_0 > 0$ (for example $L_0 < \frac{\nu\lambda_1}{2}$) such that the cocycle φ admits a bounded absorbing set if $L < L_0$. On the other hand the cocycle φ is compact, i.e. the mapping $\varphi(t,\cdot,\cdot) : V \times \Omega \to V$ is completely continuous for all $t > 0$. To finish the proof of the theorem it is sufficient to refer to Theorem 1.3 [6]. $\qquad\square$

Theorem 5.11. *Under the conditions (i) and (ii) there exists a positive number λ_0 such that the following is true:*

(1) *the set $I_\omega^\lambda = \{u \in H \mid \sup_{t \in \mathbb{R}} |\varphi_\lambda(t,u,\omega)| < +\infty\}$ is not empty, compact and connected for all $\omega \in \Omega$ and $\lambda \in \Lambda = [-\lambda_0, -\lambda_0]$, where $h \in H$ and $\varphi_\lambda(t,u,\omega)$ is a unique solution of the equation*

$$u' + \nu Au + F(u,u) + h = \lambda\mathcal{F}(u,\omega t) \quad (\omega \in \Omega) \tag{5.24}$$

satisfying the initial condition $\varphi_\lambda(0,u,\omega) = u$;

(2) *$\varphi_\lambda(t,I_\omega^\lambda,\omega) = I_{\sigma(t,\omega)}^\lambda$ for all $\lambda \in \Lambda$, $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*

(3) *the set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*

(4) *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t,M,\omega_{-t}), I_\omega^\lambda) = 0$$

and

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t,M,\omega), I^\lambda) = 0$$

take place for all $\lambda \in \Lambda$, $\omega \in \Omega$ and bounded subset $M \subseteq E$.

(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact and connected;*

(6) *the equality*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, I) = 0$$

holds, where I is a Levinson's centre for the equation

$$u' + \nu Au + F(u,u) + h = 0.$$

Proof. Let $\tilde{F}(u,\omega t) = -h + \lambda\mathcal{F}(u,\omega t)$ and $\lambda_0 < \frac{\nu\lambda_1}{2L}$, then for the equation

$$u' + \nu Au + F(u,u) = \lambda\tilde{F}(u,\omega t) \quad (\omega \in \Omega)$$

the conditions of Theorem 5.10 are fulfilled. Let φ_λ be a cocycle generated by equation (5.20), then according to Corollary 5.9 the family of cocycle $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ admits a collectively absorbing bounded set. Since the imbedding V into H is

compact, to finish the proof of the theorem it is sufficient to refer to Theorem 3.16 and Lemma 3.6. The theorem is proved. $\square$

Quasilinear Functional-Differential Equations. Functional-differential equations are a very important class of systems with infinite-dimensional phase space [15]. Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of continuous functions $\nu : [a,b] \to \mathbb{R}^n$ with the sup-norm . If $[a,b] = [-r,0]$, then suppose $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $\sigma \in \mathbb{R}, A \geq 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. For any $t \in [\sigma, \sigma + A]$ define $u_t \in \mathcal{C}$ by the equality $u_t(\theta) = u(t + \theta), -r \leq \theta \leq 0$. Let us define by $C(\Omega \times C, \mathbb{R}^n)$ the space of all continuous functions $f : \Omega \times \mathcal{C} \to \mathbb{R}^n$, with compact-open topology and let $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on the compact metric space Ω. Consider the equation

$$u' = f(\omega t, u_t) \qquad (\omega \in \Omega), \tag{5.25}$$

where $f \in C(\Omega \times C, \mathbb{R}^n)$. We will suppose that the function f is regular, that is for any $\omega \in \Omega$ and $u \in \mathcal{C}$ the equation (5.25) has a unique solution $\varphi(t, u, \omega)$ which is defined on $\mathbb{R}_+ = [0, +\infty)$. Let $X := \mathcal{C} \times \Omega$, and $\pi : X \times \mathbb{R}_+ \to X$ be a dynamical system on X defined by the following rule: $\pi((u, \omega), \tau) = (\varphi_\tau(u, \omega), \omega \tau)$, then the triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle (h = pr_2 : X \to \Omega)$ is a nonautonomous dynamical system, where $\varphi_\tau(u, \omega)(\theta) = \varphi(\tau + \theta, u, \omega)$. From the general properties of solutions of (5.25) (see, for example [15]), we have the following statement.

Theorem 5.12. *The following statements are true:*

(1) *The nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ generated by equation (5.22) is conditionally completely continuous;*

(2) *Let Ω be compact and the function $f : \Omega \times \mathcal{C} \to \mathbb{R}^n$ be bounded on $\Omega \times B$ for any bounded set $B \subset \mathcal{C}$, then the nonautonomous dynamical system generated by the equation (5.25) is conditionally completely continuous (in particular, it is asymptotically compact).*

Denote by $[\mathcal{C}]$ the space of all linear continuous operators acting onto $\mathcal{C}$ and equipped with the operational norm.

Theorem 5.13 ([3])**.** *Let $A \in C(\Omega, [\mathcal{C}]), f \in C(C \times \Omega, E)$ and let the following conditions be fulfilled:*

(1) *a zero solution of equation*

$$u' = A(\omega t)u_t \tag{5.26}$$

is uniformly asymptotically stable, i.e. there exist positive numbers N and ν such that $|\varphi_0(t, u, \omega)| \leq Ne^{-\nu t}|u|$ for all $t \geq 0$ and $\omega \in \Omega$, where $\varphi_0(t, u, \omega)$ is a solution of equation (5.26) with condition that $\varphi_0(0, u, \omega) = u$;

(2) *there exists a positive constant L such that*

$$|f(u_1, \omega) - f(u_2, \omega)| \leq L|u_1 - u_2|$$

for all $u_1, u_2 \in \mathcal{C}$ and $\omega \in \Omega$.

Then there exists a positive constant $\varepsilon_0 (\varepsilon_0 < \frac{\nu}{N})$ such that

$$|\varphi(t, u, \omega)| \leq \frac{NM}{\nu - NL} + (N|u| - \frac{NM}{\nu - NL})e^{-(\nu - NL)t}$$

for all $t \geq 0$, $u \in \mathcal{C}$ and $\omega \in \Omega$, where $\varphi(t, u, \omega)$ is a unique solution of the equation

$$u' = A(\omega t)u_t + f(u_t, \omega t)$$

with the condition $\varphi(0, u, \omega) = u$ and $M = \max\{|f(0, \omega)| : \omega \in \Omega\}$.

Consider a nonautonomous quasilinear system

$$u' = A(\omega t)u_t + \lambda f(u_t, \omega t) \quad (\omega \in \Omega) \tag{5.27}$$

on $\mathcal{C}$.

Theorem 5.14. *Let $f \in C(\mathcal{C} \times \Omega, E)$ and let the inequality*

$$|f(u_1, \omega) - f(u_2, \omega)| \le L|u_1 - u_2|$$

take place for all $u_1, u_2 \in \mathcal{C}$ and $\omega \in \Omega$, where L is some positive number.

Then there exists a positive number λ_0 such that for all $\lambda \in \Lambda = [-\lambda_0, \lambda_0]$ the following statements are true:

(1) *the set $I_\omega^\lambda = \{u \in \mathcal{C} | \sup\{|\varphi_\lambda(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$, where $\varphi_\lambda(t, u, \omega)$ there is a unique solution of equation (5.27) satisfying the initial condition $\varphi_\lambda(0, u, \omega) = u$;*
(2) *$\varphi_\lambda(t, I_\omega^\lambda, \omega) = I_{\sigma(t, \omega)}^\lambda$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
(3) *the set $I^\lambda = \bigcup\{I_\omega^\lambda \mid \omega \in \Omega\}$ is compact and connected;*
(4) *the equalities*

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega_{-t}), I_\omega^\lambda) = 0,$$

$$\lim_{t \to +\infty} \beta(\varphi_\lambda(t, M, \omega), I^\lambda) = 0$$

hold for all $\lambda \in \Lambda, \omega \in \Omega$ and bounded subset $M \subseteq E$.
(5) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is compact;*
(6) *the following equality holds*

$$\lim_{\lambda \to 0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, 0) = 0.$$

Proof. Let λ_0 be a positive number such that $\nu = \lambda_0 L < \nu/N$, then the function $F_\lambda(u, \omega) = A(\omega)u + \lambda f(u, \omega)$ satisfies the condition

$$|F_\lambda(u, \omega)| \le \nu|u| + M \tag{5.28}$$

(with $M = \max_{\omega \in \Omega} |f(0, \omega)|$) for all $|\lambda| \le \lambda_0$, $\omega \in \Omega$ and $u \in \mathcal{C}$. From the inequality (5.28) and Theorem 5.12 follows that the family of cocycles $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ admits a bounded absorbing set. Now to finish the proof of theorem it is sufficient to refer to Theorems 3.10, 3.14, 5.11 and Lemma 3.6. $\square$

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