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Nonlocal Cauchy problems for first-order multivalued differential equations *

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Abstract

We prove the existence of solutions for a nonlocal Cauchy problem for a first-order multivalued differential equation. Our approach is based on the topological transversality theory for set-valued maps.

1 Introduction

In this paper, we investigate the existence of solutions for the nonlocal Cauchy problem

$$\begin{gathered} x'(t) \in F(t, x(t)) \quad t \in (0, T] \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \end{gathered} \tag{1.1}$$

Here $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map, $J = [0, T]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \dots, m$. Nonlocal Cauchy problems for ordinary differential equations (single-valued F) have been investigated by several authors, both for the scalar case and the abstract case (see for instance [3, 7] and the references therein). Also, classical initial value problems for multivalued differential equations have been considered by many authors (see [5, 1, 6] and the references therein). The importance of nonlocal conditions in many applications is discussed in [3, 4] . Also, reference [8] contains examples of problems with nonlocal conditions and references to other works dealing with nonlocal problems.

2 Preliminaries

In this section we introduce notations, definitions and results that will be used in the remainder of this paper.

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Function spaces

Let J be a compact interval in $\mathbb{R}$. $C(J)$ is the Banach space of continuous real-valued functions defined on J, with the norm $\|x\|_0 = \sup\{|x(t)|;\ t \in J\}$ for $x \in C(J)$. $C^k(J)$ is the Banach space of k-times continuously differentiable functions. $L^p(J)$ is the set of measurable functions x such that $\int_J |x(t)|^p dt < +\infty$. Define $\|x\|_{L^p} = (\int_J |x(t)|^p dt)^{1/p}$. The Sobolev spaces $W^{k,p}(J)$ are defined as follows:

$$W^{1,p}(J) := \Big\{x \in L^p(J); \exists\, x' \in L^p(J) \text{ such that } \int_J x\phi' = -\int_J x'\phi \\ \forall \phi \in C^1(J) \text{ with compact support}\Big\}$$

or equivalently,

$$W^{1,p}(J) = \{x : J \to \mathbb{R}; x \text{ absolutely continuous and } x' \in L^p(J),\ 1 \le p \le \infty\}\ .$$

Then we define

$$W^{k,p}(J) = \big\{x \in W^{k-1,p}(J);\ x' \in W^{k-1,p}(J)\big\} \quad k \ge 2.$$

The notation $H^1(J)$ is used for $W^{1,2}(J)$. Let

$$H^1_b(J) := \{u \in H^1(J); u(0) + \sum_{k=1}^m a_k u(t_k) = 0\}.$$

Note that the embeddings $j : W^{k,p}(J) \to C^{k-1}(J)$, $p > 1$, are completely continuous for J compact [2].

Set-valued Maps

Let X and Y be Banach spaces. A set-valued map $G : X \to 2^Y$ is said to be compact if $G(X) = \overline{\cup\{G(x); x \in X\}}$ is compact. G has convex (closed, compact) values if $G(x)$ is convex (closed, compact) for every $x \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subsets B of X. A set-valued map G is upper semicontinuous at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood $\mathcal{M}$ of z_0 such that $G(\mathcal{M}) \subset O$. G is upper semicontinuous on X if it is upper semicontinuous at every point of X. If G is nonempty and compact-valued then G is upper semicontinuous if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $bcc(X)$. A set-valued map $G : J \to bcc(X)$ is measurable if for each $x \in X$, the function $t \mapsto \operatorname{dist}(x, G(t))$ is measurable on J. If $X \subset Y$, G has a fixed point if there exists $x \in X$ such that $x \in Gx$. Also, $|G(x)| = \sup\{|y|;\ y \in G(x)\}$.

Definition A multivalued map $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t, y)$ is measurable for each $y \in \mathbb{R}$;

(ii) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(iii) For each $\sigma > 0$, there exists $h_\sigma \in L^1(J, \mathbb{R}_+)$ such that
$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \le h_\sigma(t)$$
for all $|y| \le \sigma$ and for almost all $t \in J$.

The set of selectors of F that belong to L^1 is denoted by
$$S^1_{F(.,y(.))} = \{v \in L^1(J,) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}$$
By a solution of (1.1) we mean an absolutely continuous function x on J, such that $x' \in L^1$ and
$$\begin{gathered} x'(t) = f(t) \quad \text{a.e. } t \in (0, T] \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \end{gathered} \tag{2.1}$$
where $f \in S^1_{F(.,x(.))}$.

Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ the set $S^1_{F(.,x(.))}$ is not empty (see [9]). For more details on set-valued maps we refer to [5].

Topological Transversality Theory for Set-valued Maps

Let X be a Banach space, C a convex subset of X and U an open subset of C. $K_{\partial U}(\overline{U}, 2^C)$ shall denote the set of all set-valued maps $G : \overline{U} \to 2^C$ which are compact, upper semicontinuous with closed convex values and have no fixed points on ∂U (i.e., $u \notin Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0, 1] \times \overline{U} \to 2^C$ which is compact, upper semicontinuous with closed convex values. If $u \notin H(\lambda, u)$ for every $\lambda \in [0, 1], u \in \partial U$, H is said to be fixed point free on ∂U. Two set-valued maps $F, G \in K_{\partial U}(\overline{U}, 2^C)$ are called homotopic in $K_{\partial U}(\overline{U}, 2^C)$ if there exists a compact homotopy $H : [0, 1] \times \overline{U} \to 2^C$ which is fixed point free on ∂U and such that $H(0, \cdot) = F$ and $H(1, \cdot) = G$. $G \in K_{\partial U}(\overline{U}, 2^C)$ is called essential if every $F \in K_{\partial U}(\overline{U}, 2^C)$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential. For more details we refer the reader to [6].

Theorem 2.1 (Topological transversality theorem) *Let F, G be two homotopic set-valued maps in $K_{\partial U}(\overline{U}, 2^C)$. Then F is essential if and only if G is essential.*

Theorem 2.2 *Let $G : \overline{U} \to 2^C$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential*

Theorem 2.3 (Nonlinear Alternative) *Let U be an open subset of a convex set C, with $0 \in U$. Let $H : [0,1] \times \overline{U} \to 2^C$ be a compact homotopy such that $H_0 \equiv 0$. Then, either*

(i) $H(1,\cdot)$ has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in H(\lambda, u)$.

3 Main results

To prove our main results, we assume the following:

(H0) $a_k \neq 0$ for each $k = 1, 2, \dots, m$ and $\sum_{k=1}^m a_k + 1 \neq 0$.

(H1) $F : J \times \mathbb{R} \to bcc(\mathbb{R}),\ (t,x) \mapsto F(t,x)$ is

- (i) measurable in t, for each $x \in \mathbb{R}$
- (ii) upper semicontinuous with respect to $x \in$ for a.e. $t \in J$

(H2) $|F(t,x)| \leq \psi(|x|)$ for a.e. $t \in J$, all $x \in \mathbb{R}$, where $\psi : [0,+\infty) \to (0,+\infty)$ is continuous nondecreasing and such that $\limsup_{\rho\to\infty} \frac{\psi(\rho)}{\rho} = 0$.

Our first result reads as follows.

Theorem 3.1 *If the assumptions (H0), (H1), and (H2) are satisfied, then the initial-value problem (1.1) has at least one solution.*

Proof This proof will be given in several steps, and uses some ideas from [6].
Step 1. Consider the set-valued operator $\Phi : C(J) \to L^2(J)$ defined as

$$(\Phi x)(t) = F(t, x(t)).$$

Note that Φ is well defined, upper semicontinuous, with convex values and sends bounded subsets of $C(J)$ into bounded subsets of $L^2(J)$. In fact, we have

$$\Phi x := \{u : J \to \mathbb{R} \quad \text{measurable; } u(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.$$

Let $z \in C(J)$. If $u \in \Phi z$ then

$$|u(t)| \leq \psi(|z(t)|) \leq \psi(\|z\|_0).$$

Hence $\|u\|_{L^2} \leq C_0 := \psi(\|z\|_0)$. This shows that Φ is well defined. It is clear that Φ is convex valued.

Now, let B be a bounded subset of $C(J)$. Then, there exists $K > 0$ such that $\|u\|_0 \leq K$ for $u \in B$. So, for $w \in \Phi u$ we have $\|w\|_{L^2} \leq C_1$, where $C_1 = \psi(K)$. Also, we can argue as in [5, p. 16] to show that Φ is upper semicontinuous.

Step 2. Let x be a possible solution of (1.1). Then there exists a positive constant R^*, not depending on x, such that

$$|x(t)| \le R^* \quad \text{for all } t \text{ in } J.$$

It follows from the definition of solutions of (1.1) that

$$\begin{gathered} x'(t) = f(t) \quad \text{a.e. } t \in (0,T] \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \end{gathered} \tag{3.1}$$

where $f \in S^1_{F(.,x(.))}$. Simple computations give

$$x(t) = \big(1 + \sum_{k=1}^{m} a_k\big)^{-1}\big(- \sum_{k=1}^{m} a_k \int_0^{t_k} f(s)ds\big) + \int_0^t f(s)ds \tag{3.2}$$

Hence

$$|x(t)| \le \big|\big(1 + \sum_{k=1}^{m} a_k\big)^{-1}\big|\big(\sum_{k=1}^{m} |a_k| \int_0^{t_k} |f(s)|ds\big) + \int_0^t |f(s)|ds$$

Assumption (H2) yields

$$|x(t)| \le \big|\big(1 + \sum_{k=1}^{m} a_k\big)^{-1}\big|\big(\sum_{k=1}^{m} |a_k| \int_0^{t_k} \psi(|x(s)|)ds\big) + \int_0^t \psi(|x(s)|)ds$$

Let

$$R_0 = \max\{|x(t)|; t \in J\}.$$

Then

$$R_0 \le \big|\big(1 + \sum_{k=1}^{m} a_k\big)^{-1}\big|\big(\sum_{k=1}^{m} |a_k|\, t_k \psi(R_0)\big) + T\,\psi(R_0)$$

or

$$R_0 \le \Big[\big\{\big|\big(1 + \sum_{k=1}^{m} a_k\big)^{-1}\big| \sum_{k=1}^{m} |a_k|\, t_k\big\} + T\Big]\psi(R_0)$$

The above inequality implies

$$1 \le \Big(T + \big|(1 + \sum_{k=1}^{m} a_k)^{-1}\big| \sum_{k=1}^{m} |a_k|\, t_k\Big)\frac{\psi(R_0)}{R_0}$$

Now, the condition on ψ in (H2) shows that there exists $R^* > 0$ such that for all $R > R^*$,

$$\Big(T + \big|(1 + \sum_{k=1}^{m} a_k)^{-1}\big| \sum_{k=1}^{m} |a_k|\, t_k\Big)\frac{\psi(R)}{R} < 1.$$

Comparing these last two inequalities, we see that $R_0 \leq R^*$. Consequently, we obtain $|x(t)| \leq R^*$ for all $t \in J$.

Step 3. For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$$\begin{gathered} x'(t) \in \lambda F(t, x(t)) \quad t \in J, \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0. \end{gathered} \tag{3.3}$$

It follows from Step 2 that if x is a solution of (3.3) for some $\lambda \in [0, 1]$, then

$$|x(t)| \leq R^* \quad \text{for all } t \in J$$

and R^* does not depend on λ. Define $\Phi_\lambda : C(J) \to L^2(J)$ as

$$(\Phi_\lambda x)(t) = \lambda F(t, x(t)).$$

Step 1 shows that Φ_λ is upper semicontinuous, has convex values and sends bounded subsets of $C(J)$ into bounded subsets of $L^2(J)$. Let $j : H_b^1(J) \to C(J)$ be the completely continuous embedding. The operator $L : H_b^1(J) \to L^2(J)$, defined by $(Lx)(t) = x'(t)$ has a bounded inverse (in fact this follows from the solution of (3.1) which is given by (3.2)), which we denote by L^{-1}. Let $B_{R^*+1} := \{x \in C(J); \|x\|_0 < R^* + 1\}$. Define a set-valued map $H : [0, 1] \times B_{R^*+1} \to C(J)$ by

$$H(\lambda, x) = (j \circ L^{-1} \circ \Phi_\lambda)(x).$$

We can easily show that the fixed points of $H(\lambda, \cdot)$ are solutions of (3.3). Moreover, H is a compact homotopy between $H(0, \cdot) \equiv 0$ and $H(1, \cdot)$. In fact, H is compact since Φ_λ is bounded on bounded subsets and j is completely continuous. Also, H is upper semicontinuous with closed convex values. Since solutions of $(1)_\lambda$ satisfy $\|x\|_0 \leq R^* < R^* + 1$ we see that $H(\lambda, \cdot)$ has no fixed points on ∂B_{R^*+1}.

Now, $H(0, \cdot)$ is essential by Theorem 2. Hence H_1 is essential. This implies that $j \circ L^{-1} \circ \Phi$ has a fixed point. Therefore problem (1.1) has a solution . This completes the proof of Theorem 3.1. □

Our next result is based on an application of the nonlinear alternative. We shall replace condition (H2) by

(H2') $|F(t, x)| \leq p(t)\psi(|x|)$ for a.e. $t \in J$, all $x \in \mathbb{R}$, where $p \in L^1(J, \mathbb{R}_+)$, $\psi : [0, +\infty) \to (0, +\infty)$ is continuous nondecreasing and such that

$$\sup_{\delta \in (0,\infty)} \frac{\delta}{[\{|(1 + \sum_{k=1}^{m} a_k)^{-1}| \sum_{k=1}^{m} |a_k|\} + T] \, \|p\|_{L^1} \psi(\delta)} > 1$$

Now, we state our second result.

Theorem 3.2 *If assumptions (H0), (H1), and (H2') are satisfied, then the initial value problem (1.1) has at least one solution.*

Proof This proof is similar to the proof of Theorem 3.1. Let $M_0 > 0$ be defined by

$$\frac{M_0}{[\{|(1+\sum_{k=1}^m a_k)^{-1}|\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)ds\} + \|p\|_{L^1}]\psi(M_0)} > 1.$$

Let $U := \{x \in C(J); \|x\|_0 < M_0\}$. Then consider the compact homotopy (see Step 3 above) $H : [0,1] \times U \to C(J)$ defined by

$$H(\lambda, x) = (j \circ L^{-1} \circ \Phi_\lambda)(x).$$

Suppose that alternative (ii) in Theorem 2.3 holds. This means that there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in H(\lambda, u)$, or equivalently

$$u'(t) \in \lambda F(t, u(t)) \quad t \in J,$$
$$u(0) + \sum_{k=1}^m a_k u(t_k) = 0$$

Now, as in Step2 above, assumption (H2') yields

$$|u(t)| \leq \big|\big(1 + \sum_{k=1}^m a_k\big)^{-1}\big|\big(\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)\psi(|u(s)|)ds\big) + \int_0^t p(s)\psi(|u(s)|)ds$$

Since ψ is increasing,

$$|u(t)| \leq \big|\big(1 + \sum_{k=1}^m a_k\big)^{-1}\big|\big(\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)\psi(\|u\|_0)ds\big) + \int_0^t p(s)\psi(\|u\|_0)ds\,.$$

Since for $u \in \partial U$ we have $\|u\|_0 = M_0$ this last inequality implies that

$$M_0 \leq \big|\big(1 + \sum_{k=1}^m a_k\big)^{-1}\big|\big(\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)\psi(M_0)ds\big) + \int_0^t p(s)\psi(M_0)ds$$

which, in turn gives

$$M_0 \leq \Big[\Big\{\big|\big(1 + \sum_{k=1}^m a_k\big)^{-1}\big|\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)ds\Big\} + \int_0^t p(s)ds\Big]\psi(M_0)$$

Hence,

$$M_0 \leq \Big[\Big\{\big|\big(1 + \sum_{k=1}^m a_k\big)^{-1}\big|\sum_{k=1}^m |a_k| \int_0^{t_k} p(s)ds\Big\} + \|p\|_{L^1}\Big]\psi(M_0)$$

This, clearly, contradicts the definition of M_0. Therefore, condition (ii) of Theorem 2.3 does not hold. Consequently, $H(1,.)$ has a fixed point, which is a solution of problem (1.1).

Remark For nonlocal initial values of the form $x(0)+\sum_{k=1}^m a_k x(t_k)=x_0$, where x_0 is a given nonzero real number, we let $y(t)=x(t)-x_0(1+\sum_{k=1}^m a_k)^{-1}$. Then y is a solution to the problem

$$
\begin{gathered}
y'(t)\in F(t,y(t)+x_0(1+\sum_{k=1}^m a_k)^{-1})\\
y(0)+\sum_{k=1}^m a_k y(t_k)=0
\end{gathered}
$$

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