

High regularity of the solution of a nonlinear parabolic boundary-value problem *

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Abstract

The aim of this paper is to report some results concerning high regularity of the solution of a nonlinear parabolic problem with a linear parabolic differential equation in one spatial dimension and nonlinear boundary conditions. We show that any regularity can be reached provided that appropriate smoothness of the data and compatibility assumptions are required.

1 Introduction

We consider the parabolic boundary value problem (BVP):

$$y_t - y_{xx} + gy = f(x, t), \quad \text{in } D_T, \quad (1.1)$$

$$\begin{aligned} y_x(0, t) &\in \beta_1(y(0, t)), \\ -y_x(1, t) &\in \beta_2(y(1, t)), \quad 0 < t < T, \end{aligned} \quad (1.2)$$

$$y(x, 0) = y_0(x), \quad 0 < x < 1, \quad (1.3)$$

where $D_T := \{(x, t) \in \mathbb{R}^2; 0 < x < 1, 0 < t < T\}$ for a fixed $T \in (0, \infty)$, where $g \geq 0$ is a given constant, $f : \overline{D_T} \rightarrow \mathbb{R}$ and $y_0 : [0, 1] \rightarrow \mathbb{R}$ are given functions, and $\beta_i : D(\beta_i) \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, are nonlinear mappings that might possibly be multivalued. So, this BVP is a nonlinear problem.

Notice that the BVP is a model for heat conduction and diffusion processes. Also, it can be viewed as a reduced model for a singular perturbation problem associated with an electrical circuit, in which the specific inductance is negligible and is set equal to zero [2]. In [2] some nonhomogeneous boundary conditions appear, but they can easily be homogenized by a simple change of the unknown function:

$$\tilde{y}(x, t) = y(x, t) + x(1 - x)[a(t)x + b(t)]. \quad (1.4)$$

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In the case of integrated circuits with negligible specific inductances, we can set them equal to zero and thus arrive at a similar BVP, but with n equations instead of (1.1) [8, 11].

An existence theory for the BVP in the case in which β_1, β_2 are monotone mappings can be found in [12, Chapter 3, §3], even in a more general framework. In the present paper we are concerned with higher order regularity of the solution. As we shall see, this requires higher regularity of f, β_1, β_2, y_0 and higher order compatibility conditions. The main difficulty in this problem is due to the nonlinearity of (1.2) whereas the linear case is well-known from [9, Chapter VII, §10]. There the compatibility conditions, corresponding to our particular BVP with linear conditions, consist in the fact that the time derivatives $(\partial^k y / \partial t^k)(x, 0)$ ($k \geq 0$), which can be calculated from (1.1) and (1.3), for $x = 0$ and $x = 1$ must satisfy the boundary conditions and the relations obtained from their differentiation with respect to t . Of course, the case of the nonlinear condition (1.2) is more delicate.

The regularity question for the BVP is also important as an intermediate step in developing an asymptotic analysis of the telegraph system with small specific inductance (see [2, 3, 4]).

2 Preliminaries

Let us first recall the following result (see [7, Appendix]), which will be used later:

Theorem 2.1 *Let X be a reflexive real Banach space and let $u \in L^p(a, b; X)$ with $-\infty < a < b < \infty$ and $1 < p < \infty$. Then, the following two conditions are equivalent:*

- (i) $u \in W^{1,p}(a, b; X)$;
- (ii) *There exists a constant $C > 0$ such that $\int_a^{b-\delta} \|u(t+\delta) - u(t)\|_X^p \leq C\delta^p$ for every $\delta \in (0, b-a]$.*

The implication (i) \Rightarrow (ii) is still valid for $p = 1$. Moreover, (ii) holds if u is of bounded variation on $[a, b]$ and X is not necessarily reflexive.

Here and in what follows, L^p and $W^{k,p}$ denote the usual function and Sobolev spaces, respectively. Now, in order to state the next result, let us consider two real Hilbert spaces V and H , such that V is densely and continuously embedded into H . If H is identified with its own dual, then we have $V \subset H \subset V'$, algebraically and topologically. We have denoted by V' the dual of V . Denote also by $\langle \cdot, \cdot \rangle$ the pairing between V and V' , i.e., $\langle v, v^* \rangle := v^*(v)$, for $v \in V, v^* \in V'$. For $v^* \in H' \equiv H$, $\langle v, v^* \rangle$ reduces to the scalar product in H of v and v^* . Following [10, Chapter 1], for some $-\infty < a < b < \infty$ we set

$$W(a, b) := \{u \in L^2(a, b; V); u' \in L^2(a, b; V')\},$$

where u' is the distributional derivative of u , as a V -valued distribution.

Theorem 2.2 *Every $u \in W(a, b)$ has a representative $u_1 \in C([a, b]; H)$ and so u can be identified with such a continuous function. Furthermore, if $u, \tilde{u} \in W(a, b)$, then the function $t \mapsto \langle u(t), \tilde{u}(t) \rangle$ is absolutely continuous on $[a, b]$ and*

$$\frac{d}{dt} \langle u(t), \tilde{u}(t) \rangle = \langle u(t), \tilde{u}'(t) \rangle + \langle \tilde{u}(t), u'(t) \rangle \quad \text{for a.a. } t \in (a, b),$$

and, hence, in particular,

$$\frac{d}{dt} \|u(t)\|_H^2 = 2\langle u(t), u'(t) \rangle \quad \text{for a.a. } t \in (a, b).$$

Finally, we recall the following theorem due to H. Attouch and A. Damlamian [1] which will be needed for the derivation of higher regularity of the solution of the BVP:

Theorem 2.3 *Let $A(t) = \partial\phi(t, \cdot)$ for 0 , where $\phi(t, \cdot) : H \rightarrow (-\infty, +\infty]$ are proper, convex, and lower semi-continuous functions, with a domain of definition $D(\phi(t, \cdot)) = D$ which is independent of t . Here H is a real Hilbert space. Assume further that there exists a nondecreasing function $\gamma : [0, T] \rightarrow \mathbb{R}$ and some real constants C_1, C_2 such that*

$$\phi(t, v) \leq \phi(s, v) + [\gamma(t) - \gamma(s)] \cdot [\phi(s, v) + C_1 \|v\|_H^2 + C_2] \quad (2.1)$$

for all $v \in D$ and $0 \leq s \leq t \leq T$. Then, for every $u_0 \in D$ and $f \in L^2(0, T; H)$, there exists a unique solution $u \in W^{1,2}(0, T; H)$ of the equation $u'(t) + A(t)u(t) = f(t)$ for a.a. $t \in (0, T)$ with the initial condition $u(0) = u_0$. Moreover, there exists a function $h \in L^1(0, T)$ such that

$$\phi(t, u(t)) \leq \phi(s, u(s)) + \int_s^t h(r) dr \quad \text{for all } 0 \leq s \leq t \leq T. \quad (2.2)$$

Here, we denote by $\partial\phi(t, \cdot)$ the subdifferential of the function $\phi(t, \cdot)$. In what follows, we shall use the theory of evolution equations associated with monotone operators in Hilbert spaces. For details we refer to [5, 7, 12].

3 High Regularity of Solutions

If β_1, β_2 are maximal monotone mappings, then existence and uniqueness of the solution to the BVP is well known. The most important results, even for n dimensions, were established by H. Brezis [6, 7]. Our problem can be expressed as a Cauchy problem in $L^2(0, 1)$, associated with a subdifferential and, hence, existence is available (see, e.g., Brezis's theorem in [12, p. 56], where the regularizing effect of the subdifferential on the initial data is pointed out). For a more general problem than ours see [12, Chapter 3, §3], where the existence to a higher order, one-dimensional, parabolic equation is discussed.

Theorem 3.1 *Assume that*

$$f \in W^{1,1}(0, T; L^2(0, 1)); \tag{3.1}$$

$$\beta_1, \beta_2 \text{ are both maximal monotone}; \tag{3.2}$$

$$y_0 \in H^2(0, 1); \tag{3.3}$$

and that the following first-order compatibility conditions are fulfilled:

$$y'_0(0) \in \beta_1(y_0(0)), \quad -y'_0(1) \in \beta_2(y_0(1)). \tag{3.4}$$

Then the BVP (1.1), (1.2), (1.3) has a unique strong solution

$$y \in W^{1,\infty}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^2(0, 1)) \cap W^{1,2}(0, T; H^1(0, 1)). \tag{3.5}$$

Proof: Let $H = L^2(0, 1)$ with

$$\langle p, q \rangle = \int_0^1 p(x)q(x) dx, \quad \|p\|_H^2 = \int_0^1 p(x)^2 dx.$$

We define the operator $A : D(A) \subseteq H \rightarrow H$ as

$$Ap = -p'' + gp$$

on the domain of definition

$$D(A) = \{p \in H^2(0, 1); p(0) \in D(\beta_1), \\ p(1) \in D(\beta_2), p'(0) \in \beta_1(p(0)), -p'(1) \in \beta_2(p(1))\}.$$

Then the BVP may be written as the Cauchy problem in H ,

$$y'(t) + Ay(t) = f(t), \quad 0 < t < T, \\ y(0) = y_0, \tag{3.6}$$

where $y(t) := y(\cdot, t)$ and $f(t) := f(\cdot, t)$. The operator A is maximal monotone. Moreover, A is the subdifferential of an appropriate proper, convex, lower semi-continuous function (see [12, Chapter 3, §3]). By the existence and uniqueness result [see, e.g., [12, Theorem 2.1, p. 48]], it follows that there exists a unique strong solution of problem (3.6), $y \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; H^2(0, 1))$.

Now, using (3.6)₁, i.e. Eq. (1.1), we deduce that

$$y_t(x, t + \delta) - y_t(x, t) - [y_{xx}(x, t + \delta) - y_{xx}(x, t)] \\ + g[y(x, t + \delta) - y(x, t)] = f(x, t + \delta) - f(x, t),$$

where $\delta \in (0, T)$. If we multiply the last equality by $y(\cdot, t + \delta) - y(\cdot, t)$ scalarly and then integrate on $[0, T - \delta]$, we arrive at the inequality

$$\int_0^{T-\delta} \|y_x(\cdot, t + \delta) - y_x(\cdot, t)\|_H^2 dt \\ \leq \frac{1}{2} \|y(\cdot, \delta) - y_0\|_H^2 + \int_0^{T-\delta} \|f(\cdot, t + \delta) - f(\cdot, t)\|_H \|y(\cdot, t + \delta) - y(\cdot, t)\|_H dt.$$

Since $y \in W^{1,\infty}(0, T; H)$, there exists a positive constant C_1 such that for all $t \in [0, T - \delta]$:

$$\|y(\cdot, t + \delta) - y(\cdot, t)\|_H \leq C_1 \delta. \quad (3.8)$$

Now, combining (3.7), (3.8), (3.1), and Theorem 2.1 for $p = 1$, we can easily see that

$$\int_0^{T-\delta} \|y_x(\cdot, t + \delta) - y_x(\cdot, t)\|_H^2 dt \leq C_2 \delta^2, \quad (3.9)$$

with some positive constant C_2 . Taking again into account Theorem 2.1, it follows by (3.9) that $y_x \in W^{1,2}(0, T; H)$, which concludes the proof of the theorem.

Remark 3.1 In fact, the results similar to those above are known [6]. However, it seems that, for multivalued β_1 and β_2 , no improvement of the above regularity is possible, although we could not find an appropriate counter-example yet. But an improvement of regularity is possible for single-valued and smooth β_1 and β_2 .

Theorem 3.2 *Assume that*

$$f \in W^{2,1}(0, T; L^2(0, 1)), \quad f(\cdot, 0) \in H^2(0, 1); \quad (3.10)$$

β_1, β_2 are both defined on \mathbb{R} , single-valued, and satisfy

$$\beta_1, \beta_2 \in W_{loc}^{2,\infty} \mathbb{R}, \quad \beta_1' \geq 0, \beta_2' \geq 0. \quad (3.11)$$

Moreover,

$$y_0 \in H^4(0, 1). \quad (3.12)$$

In addition, we require (3.4) (where the inclusions must be replaced by equalities) as well as the following second order compatibility conditions:

$$z_0'(0) = \beta_1'(y_0(0))z_0(0), \quad -z_0'(1) = \beta_2'(y_0(1))z_0(1), \quad (3.13)$$

where z_0 is defined as

$$z_0(x) = f(x, 0) + y_0''(x) - g y_0(x), \quad 0 \leq x \leq 1. \quad (3.14)$$

Then the solution y of the BVP belongs to

$$W^{2,2}(0, T; H^1(0, 1)) \cap W^{2,\infty}(0, T; L^2(0, 1))$$

Proof: Obviously, all the conditions of Theorem 3.1 are fulfilled and so the BVP has a unique solution y satisfying (3.5). Let us denote $V = H^1(0, 1)$ and $V' = (H^1(0, 1))'$ (the dual space). We will first show that $y_t \in W^{1,2}(0, T; V')$. To this end, it suffices to prove (cf. Theorem 2.1) that there exists a positive constant C such that for every $\delta \in (0, T]$:

$$\int_0^{T-\delta} \|y_t(\cdot, t + \delta) - y_t(\cdot, t)\|_{V'}^2 dt \leq C \delta^2. \quad (3.15)$$

Indeed, we have for a.a. $t \in (0, T - \delta)$ and all $\phi \in V$ the equation

$$\begin{aligned} & \langle y_t(\cdot, t + \delta) - y_t(\cdot, t), \phi \rangle \\ &= \langle y_{xx}(\cdot, t + \delta) - y_{xx}(\cdot, t), \phi \rangle \\ & \quad - g \langle y(\cdot, t + \delta) - y(\cdot, t), \phi \rangle + \langle f(\cdot, t + \delta) - f(\cdot, t), \phi \rangle \\ &= - \langle y_x(\cdot, t + \delta) - y_x(\cdot, t), \phi' \rangle - [\beta_2(y(1, t + \delta)) - \beta_2(y(1, t))] \phi(1) \\ & \quad - [\beta_1(y(0, t + \delta)) - \beta_1(y(0, t))] \phi(0) \\ & \quad - g \langle y(\cdot, t + \delta) - y(\cdot, t), \phi \rangle + \langle f(\cdot, t + \delta) - f(\cdot, t), \phi \rangle, \end{aligned} \tag{3.16}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 -duality pairing between V and V' . Taking into account (3.5) and (3.11), which in particular implies that β_1, β_2 are Lipschitzian on bounded sets, one obtains from (3.16) the estimate

$$\|y_t(\cdot, t + \delta) - y_t(\cdot, t)\|_{V'}^2 \leq C_3 \{ \|y(\cdot, t + \delta) - y(\cdot, t)\|_V^2 + \|f(\cdot, t + \delta) - f(\cdot, t)\|_H^2 \}, \tag{3.17}$$

where C_3 is some positive constant. Now, (3.5), (3.10), (3.17) and Theorem 2.1 lead us to the desired estimate (3.15). Therefore $z := y_t \in W^{1,2}(0, T; V')$ and, thus, we can differentiate with respect to t the equation

$$\begin{aligned} & \langle y_t(\cdot, t), \phi \rangle + \langle y_x(\cdot, t), \phi' \rangle + \beta_1(y(0, t)) \phi(0) + \beta_2(y(1, t)) \phi(1) + g \langle y(\cdot, t), \phi \rangle \\ &= \langle f(\cdot, t), \phi \rangle \end{aligned}$$

to obtain

$$\begin{aligned} & \langle z_t(\cdot, t), \phi \rangle + \langle z_x(\cdot, t), \phi' \rangle + g_1(t) z(0, t) \phi(0) + g_2(t) z(1, t) \phi(1) + g \langle z(\cdot, t), \phi \rangle \\ &= \langle f_t(\cdot, t), \phi \rangle \text{ for all } \phi \in V, \text{ and a.a. } t \in (0, T), \end{aligned} \tag{3.18}$$

where

$$g_1(t) := \beta_1'(y(0, t)), \quad g_2(t) := \beta_2'(y(1, t)).$$

In addition,

$$z(\cdot, 0) = z_0, \tag{3.19}$$

with z_0 defined by (3.14). Consequently, z is the unique solution of the problem (3.18), (3.19) in the class of y_t . Indeed, if we take in (3.18), (3.19) $f_t \equiv 0$, $z_0 \equiv 0$, and $\phi := z(\cdot, t)$, then, according to Theorem 2.2, we have

$$\frac{d}{dt} \|z(\cdot, t)\|_H^2 \leq 0 \quad \text{for a.a. } t \in (0, T),$$

which clearly implies that $z \equiv 0$. Note that $z = y_t$ satisfies the linear problem

$$\begin{aligned} & z_t - z_{xx} + gz = f_t, \quad \text{in } D_T, \\ & z(x, 0) = z_0(x), \quad 0 < x < 1, \\ & z_x(0, t) = g_1(t) z(0, t), \quad -z_x(1, t) = g_2(t) z(1, t), \quad 0 < t < T \end{aligned} \tag{3.20}$$

formally. By Theorem 3.1 and the assumption (3.11) it follows that $g_1, g_2 \in H^1(0, T)$, and that $g_1 \geq 0, g_2 \geq 0$ in $[0, 1]$. Actually, problem (3.20) has a

unique strong solution. To show this, let us use the fact that this problem can be expressed as the following time-dependent Cauchy problem in $H = L^2(0, 1)$:

$$\begin{aligned} z'(t) + B(t)z(t) &= f_t(\cdot, t) \quad \text{for } 0 < t < T, \\ z(0) &= z_0, \end{aligned} \quad (3.21)$$

where $z(t) := z(\cdot, t)$ and $B(t) : D(B(t)) \subset H \rightarrow H$ is defined by

$$B(t)p = -p'' + gp$$

on the domain of definition

$$D(B(t)) = \{p \in H^2(0, 1); p'(0) = g_1(t)p(0), -p'(1) = g_2(t)p(1)\}.$$

$B(t)$ is maximal monotone for every $t \in [0, T]$ and, even more, $B(t)$ is the sub-differential of the function $\varphi(t, \cdot) : H \rightarrow (-\infty, +\infty]$, given by

$$\varphi(t, p) = \begin{cases} (1/2)\{ \int_0^1 p'(x)^2 dx + g \int_0^1 p(x)^2 dx \\ + g_1(t)p(0)^2 + g_2(t)p(1)^2 \} & \text{for } p \in H^1(0, 1), \\ +\infty & \text{for } p \in H \setminus H^1(0, 1) \end{cases}$$

(see, e. g., [12, Chapter 3, §3]). For every $t \in [0, T]$, the effective domain is $D(\varphi(t, \cdot)) = H^1(0, 1)$, i.e. it is independent on t . Now, we are going to show that the condition (2.1) of Theorem 2.3 is satisfied. To this end, let $p \in H^1(0, 1)$ and $0 \leq s \leq t \leq T$. We have

$$\begin{aligned} \varphi(t, p) - \varphi(s, p) &= \frac{g_1(t) - g_1(s)}{2} p(0)^2 + \frac{g_2(t) - g_2(s)}{2} p(1)^2 \\ &= \frac{1}{2} p(0)^2 \int_s^t g_1'(\tau) d\tau + \frac{1}{2} p(1)^2 \int_s^t g_2'(\tau) d\tau \\ &\leq \frac{K}{2} (\|p\|_H^2 + \|p'\|_H^2) \int_s^t (|g_1'(\tau)| + |g_2'(\tau)|) d\tau, \end{aligned}$$

where K is a positive constant due to the continuous embedding of $H^1(0, 1)$ into $C[0, 1]$. Therefore,

$$\varphi(t, p) \leq \varphi(s, p) + [\gamma(t) - \gamma(s)] \cdot [\varphi(s, p) + \frac{1}{2} \|p\|_H^2],$$

where

$$\gamma(t) = K \int_0^t (|g_1'(\tau)| + |g_2'(\tau)|) d\tau.$$

By Theorem 2.3 one obtains that problem (3.21) has a unique solution

$$z \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1)).$$

This solution z satisfies problem (3.18), (3.19) and, hence, it coincides with y_t . So, we have already proved that

$$y \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; H^1(0, 1)).$$

We are now going to show that

$$z_t = y_{tt} \in L^\infty(0, T; H) \cap L^2(0, T; H^1(0, 1)).$$

To this end, we proceed in a standard manner by starting from the estimates

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|z(\cdot, t+h) - z(\cdot, t)\|_H^2 + \|z_x(\cdot, t+h) - z_x(\cdot, t)\|_H^2 \\ & + [g_1(t+h)z(0, t+h) - g_1(t)z(0, t)] \cdot [z(0, t+h) - z(0, t)] \\ & + [g_2(t+h)z(1, t+h) - g_2(t)z(1, t)] \cdot [z(1, t+h) - z(1, t)] \\ & \leq \|f_t(\cdot, t+h) - f_t(\cdot, t)\|_H \cdot \|z(\cdot, t+h) - z(\cdot, t)\|_H \end{aligned}$$

for a.a. $0 \leq t < t+h \leq T$, and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dh} \|z(\cdot, h) - z_0\|_H^2 + \|z_x(\cdot, h) - z'_0\|_H^2 + [g_1(h)z(0, h) \\ & - g_1(0)z_0(0)] \cdot [z(0, h) - z_0(0)] + [g_2(h)z(1, h) - g_2(0)z_0(1)] \cdot [z(1, h) - z_0(1)] \\ & \leq \|f_h(\cdot, h) - B(0)z_0\|_H \cdot \|z(\cdot, h) - z_0\|_H \quad \text{for a.a. } h \in (0, T). \end{aligned}$$

Since $y \in W^{1,\infty}(0, T; H^1(0, 1))$ and $\beta_1, \beta_2 \in W^{2,\infty}_{loc}(\mathbb{R})$, both the functions g_1, g_2 are Lipschitz continuous. So, taking also into account that $g_1 \geq 0$ and $g_2 \geq 0$ in $[0, 1]$, we get from the above inequalities

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|z(\cdot, t+h) - z(\cdot, t)\|_H^2 + \|z_x(\cdot, t+h) - z_x(\cdot, t)\|_H^2 \\ & \leq \|f_t(\cdot, t+h) - f_t(\cdot, t)\|_H \cdot \|z(\cdot, t+h) - z(\cdot, t)\|_H \\ & \quad + L_1 h |z(0, t+h) - z(0, t)| + L_2 h |z(1, t+h) - z(1, t)|, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dh} \|z(\cdot, h) - z_0\|_H^2 + \|z_x(\cdot, h) - z'_0\|_H^2 \\ & \leq \|f_h(\cdot, h) - B(0)z_0\|_H \cdot \|z(\cdot, h) - z_0\|_H \\ & \quad + L_1 h |z(0, h) - z_0(0)| + L_2 h |z(1, h) - z_0(1)|. \end{aligned} \tag{3.23}$$

Now, taking into account the continuous embedding of $H^1(0, 1)$ into $C[0, 1]$ and the inequality $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ for all $a, b \geq 0, \varepsilon > 0$, we obtain from (3.22) and (3.23) the two estimates

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|z(\cdot, t+h) - z(\cdot, t)\|_H^2 + \|z_x(\cdot, t+h) - z_x(\cdot, t)\|_H^2 \\ & \leq \|f_t(\cdot, t+h) - f_t(\cdot, t)\|_H \cdot \|z(\cdot, t+h) - z(\cdot, t)\|_H \\ & \quad + \frac{1}{2} (\|z_x(\cdot, t+h) - z_x(\cdot, t)\|_H^2 + \|z(\cdot, t+h) - z(\cdot, t)\|_H^2) + \frac{1}{2} C_4 h^2, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dh} \|z(\cdot, h) - z_0\|_H^2 + \|z_x(\cdot, h) - z'_0\|_H^2 \\ & \leq \|f_h(\cdot, h) - B(0)z_0\|_H \cdot \|z(\cdot, h) - z_0\|_H \\ & \quad + \frac{1}{2} (\|z_x(\cdot, h) - z'_0\|_H^2 + \|z(\cdot, h) - z_0\|_H^2) + \frac{1}{2} C_5 h^2, \end{aligned} \quad (3.25)$$

where C_4 and C_5 are some positive constants. Therefore,

$$\begin{aligned} & \frac{d}{dt} (e^{-t} \|z(\cdot, t+h) - z(\cdot, t)\|_H^2) + e^{-t} \|z_x(\cdot, t+h) - z_x(\cdot, t)\|_H^2 \\ & \leq 2e^{-t} \|f_t(\cdot, t+h) - f_t(\cdot, t)\|_H \cdot \|z(\cdot, t+h) - z(\cdot, t)\|_H + C_4 h^2, \end{aligned} \quad (3.26)$$

and

$$\frac{d}{dh} (e^{-h} \|z(\cdot, h) - z_0\|_H^2) \leq C_5 h^2 + 2e^{-h} \|f_h(\cdot, h) - B(0)z_0\|_H \cdot \|z(\cdot, h) - z_0\|_H. \quad (3.27)$$

If we drop the second term in the left-hand side of (3.26) and integrate the resulting inequality over $[0, t]$, then, by using a Gronwall type lemma (see, e.g., [12, p. 47]) we arrive at the estimate

$$\begin{aligned} \|z(\cdot, t+h) - z(\cdot, t)\|_H & \leq C_6 \{ \|z(\cdot, h) - z_0\|_H + h \\ & \quad + \int_0^t \|f_s(\cdot, s+h) - f_s(\cdot, s)\|_H ds \}. \end{aligned} \quad (3.28)$$

Now, from (3.25) we obtain in a similar way

$$\|z(\cdot, h) - z_0\|_H \leq C_7 (h^{3/2} + \int_0^h \|f_s(\cdot, s) - B(0)z_0\|_H ds). \quad (3.29)$$

Finally, (3.28) and (3.29) imply that $z_t \in L^\infty(0, T; H)$. Using this conclusion together with (3.24), we get, by using of Theorem 2.1, that $z_x \in W^{1,2}(0, T; H)$. This concludes the proof.

Remark 3.2 If f in Theorem 3.2 is assumed to be more regular with respect to x , then y is also more regular with respect to x , because

$$y_{tt} = y_{xxt} - gy_t + f_t = y_{xxxx} - gy_{xx} + f_{xx} - gy_t + f_t.$$

On the other hand, by reasoning similarly as in the proof of Theorem 3.2, one can show that $y \in W^{3,2}(0, T; H^1(0, 1)) \cap W^{3,\infty}(0, T; L^2(0, 1))$ under appropriate assumptions on the smoothness of β_1 , β_2 , y_0 , f and compatibility. The proof needs a slight modification since the boundary conditions corresponding to (3.20) are now inhomogeneous. Fortunately, the inhomogeneous terms there are H^1 -functions and so Theorem 2.3 is again applicable, with a slight change of ϕ . The corresponding t -dependent operator is nonlinear because its domain is an affine subset of $L^2(0, 1)$. Here we will not further present these details. Of course, higher regularity with respect to x can also be obtained at the expense

of additional regularity and compatibility assumptions. Actually, our procedure can be repeated as many times as we want to and so any regularity of the solution with respect to t and x can be reached under sufficient smoothness of the data and compatibility conditions. More precisely, the $H^k(D_T)$ -regularity can be shown for every k and, thus, the $C^k(\overline{D_T})$ -regularity can be obtained as well for every k , due to Sobolev's embedding theorem.

Remark 3.3 Theorem 3.2 is formulated in such a way that the next level of regularity can be obtained. Note, however, that this is a parabolic problem so that, in order to get, for instance,

$$y \in W^{2,2}(0, T; L^2(0, 1)) \cap W^{1,\infty}(0, T; H^1(0, 1)),$$

we can relax our assumptions. More precisely, it suffices to assume that

$$f \in W^{1,2}(0, T; L^2(0, 1)), \quad f(\cdot, 0) \in H^1(0, 1);$$

β_1 and β_2 satisfy (3.11); $y_0 \in H^3(0, 1)$, and y_0 satisfies (3.4) with equalities instead of inclusions.

Remark 3.4 Here, we do not discuss nonlinear cases of the equation (1.1), since this would require a special treatment. However, let us point out some immediate extensions of the above results.

Assume that the linear term gy of Equation (1.1) is replaced by a nonlinear one, say $g(y)$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a single-valued, continuous and nondecreasing function. Then Theorem 3.1 is still valid. If, in addition, g is a $W_{loc}^{2,\infty}(\mathbb{R})$ -function, then Theorem 3.2 is also valid for this nonlinear case with

$$z_0(x) = f(x, 0) + y_0''(x) - g(y_0(x)).$$

Indeed, (3.17) remains true, because g is Lipschitzian on bounded sets and y takes values in a bounded set. So, again, $z = y_t \in W^{1,2}(0, T; V')$. The remainder of the proof of Theorem 3.2 continues with slight modifications. In particular, in Eq. (3.20)₁ the term gz must be replaced by $g'(y)z$ and $B(t)$ becomes

$$B(t)p = -p'' + g'(y(\cdot, t))p,$$

with the same domain of definition as before. This means that in the definition of the energy function $\phi(t, p)$, the term $(1/2)g \int_0^1 p(x)^2 dx$ should be replaced by $(1/2) \int_0^1 g'(y(x, t))p(x)^2 dx$. Finally, here γ can be chosen in the specific form

$$\gamma(t) = K \int_0^t (|g_1'(\tau)| + |g_2'(\tau)| + \|g''(y(\cdot, \tau))y_\tau(\cdot, \tau)\|_{L^\infty(0,1)}) d\tau.$$

Then all the conclusions can be derived by similar arguments as above.

On the other hand, if the (1.2) are inhomogeneous and (1.1) is nonlinear, then a transformation like (1.4) would lead us to an evolution equation where the spatial operator becomes time-dependent. Instead, we still can keep the inhomogeneous form of (1.2) and associate with our problem an energy functional $\phi(t, \cdot)$ which also includes the inhomogeneous terms.

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