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Existence of solutions for elliptic systems with critical Sobolev exponent *

Pablo Amster, Pablo De Nápoli, & Maria Cristina Mariani

Abstract

We establish conditions for existence and for nonexistence of nontrivial solutions to an elliptic system of partial differential equations. This system is of gradient type and has a nonlinearity with critical growth.

1 Introduction

The purpose of this work is to extend some results known for the quasilinear elliptic equation

$$\begin{gathered} -\Delta u = u^{p-1} + \lambda u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

to the general system

$$\begin{gathered} -\Delta u_i = f_i(u) + \sum_{j=1}^{n} a_{ij} u_j \quad \text{in } \Omega \\ u_i = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

First we recall some results for the single equation (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$. If $2 < p < 2^* = 2N/(N-2)$ (the critical Sobolev exponent), then (1.1) has a nontrivial solution if and only if $\lambda < \lambda_1(\Omega)$, the first eigenvalue of $-\Delta$. This is proved by applying the Mountain Pass Theorem for finding nontrivial critical points for the following functional in the Sobolev space $H_0^1(\Omega)$.

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \frac{1}{p}\int_\Omega F(u) \tag{1.3}$$

where $F(u) = |u|^p$. Then by the compact imbedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, φ satisfies the Palais-Smale condition (PS). However when $p = 2^*$, φ may not satisfy the Palais-Smale condition (PS) due to the lack of compactness of the above imbedding. For $\lambda \leq 0$, a Pohozaev identity shows that there are no

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nontrivial solutions when Ω is star shaped. For the case $0 < \lambda < \lambda_1(\Omega)$, Brezis a nd Nirenberg [1] proved the existence of at least one nontrivial solution when $N \geq 4$. Their proof relies in the fact that φ satisfies $(PS)_c$ (Palais-Smale at level c) if $c < c^* = S^{N/2}/N$, where

$$S = \inf_{u \in D_0^{1,2}(\mathbb{R}^N), \|u\|_{2^*}=1} \|\nabla u\|_2^2$$

which is the best constant in the Sobolev inequality. Moreover, when the value of S and the optimal functions are explicitly known, it is possible to prove that if

$$S_\lambda = \inf_{u \in H_0^1(\Omega), \|u\|_{2^*}=1} \|\nabla u\|_2^2 + \lambda \|u\|_2^2$$

then $S_\lambda < S$ for $\lambda > 0$. Then, using the Mountain Pass Theorem a critical value $c < c^*$ is obtained. For a detailed exposition see [12].

Quasilinear elliptic systems have been studied by several authors [4, 5, 6]. For gradient type systems such as (1.2), Boccardo and de Figueiredo [2] used variational arguments to show the existence of nontrivial solutions. They proved existence of solutions for the problem

$$\begin{gathered} -\Delta_p u = F_u(x,u,v) \quad \text{in } \Omega \\ -\Delta_q v = F_v(x,u,v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where F is superlinear and subcritical. In this article, we study the critical case $p = q = 2^*$.

The general problem of finding a condition on the matrix $A = (a_{ij})$ for which (1.2) admits a nontrivial solution is still an open question. In this paper, we present some results toward the solution of this question. For A symmetric with $\|A\| < \lambda_1(\Omega)$, we prove that the method presented in [1] can be applied. More precisely, we define appropriate numbers $S_{F,A}$ and S_F such that if $S_{F,A} < S_F$ then (1.2) admits a solution. Furthermore, we show cases where this inequality holds. We prove also that in some particular cases the condition $\|A\| < \lambda_1(\Omega)$ is necessary. We conclude this paper by showing that Pohoazev's nonexistence result may be generalized to problem (1.2) when A is symmetric and negative definite. We remark that the symmetry of A can be considered as a natural condition, since the proof of existence is based on the variational structure of the problem.

Before we state our results, we recall the following definitions [8].

$$D^{1,2}(\mathbb{R}^N, \mathbb{R}^n) = \{u = (u_1, \dots, u_n) \in L^{2^*}(\mathbb{R}^N, \mathbb{R}^n) : \nabla u_i \in L^2(\mathbb{R}^N, \mathbb{R}^N)\}$$

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. We shall say that:
i) A is nonnegative ($A \geq 0$) if $a_{ij} \geq 0$ for all i, j.
ii) A is reducible if by a simultaneous permutation of rows and columns, it may be written in the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B and D are square matrices. Throughout this article, the Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$.

Statement of results

Theorem 1.1 *Let $p = 2^* = 2N/(N-2)$. Let $[\cdot]$ be a norm on $\mathbb{R}^n$ such that $F(u) = [u]^p$ is differentiable. Define $f_i = \frac{1}{p}\partial_i F$, and assume that $A \in \mathbb{R}^{n\times n}$ is symmetric, with $\|A\| < \lambda_1(\Omega)$. Set*

$$S_F = \inf_{u\in D^{1,2}(\mathbb{R}^N,\mathbb{R}^n),\int_{\mathbb{R}^N} F(u)=1} \sum_{i=1}^n \int_{\mathbb{R}^N} |\nabla u_i|^2,$$

$$S_{F,A}(\Omega) = \inf_{u\in H^1(\Omega,\mathbb{R}^n),\int_\Omega F(u)=1} \sum_{i=1}^n \int_\Omega |\nabla u_i|^2 - \int_\Omega \langle Au, u\rangle$$

Then: 1) S_F is attained by a function $u \in D^{1,2}(\mathbb{R}^N, \mathbb{R}^n)$.
2) If $S_{F,A}(\Omega) < S_F$ then (1.2) admits at least one nontrivial weak solution.

As a consequence of this theorem, we have the existence of solutions for the following case.

Corollary 1.2 *Let p, A and f_i satisfy the conditions of the Theroem 1.1 with $[u] = |u|_q = (\sum_{i=1}^n |u_i|^q)^{1/q}$ for some $q \geq 2$. Moreover, assume that $N \geq 4$ and that $a_{ii} > 0$ for some i. Then (1.2) has a nontrivial weak solution.*

Theorem 1.3 *Let us assume that (1.2) admits a nonnegative nontrivial solution $u \in H_0^1(\Omega, \mathbb{R}^n)$, and that $f_i(u) \geq 0$, with $f_i(u) > 0$ for $u > 0$. We denote by $\mu_{\min}$ and $\mu_{\max}$ the smallest and the largest eigenvalues of A, respectively. Then*

1) *If A is symmetric and positive definite, then $\mu_{\min} < \lambda_1(\Omega)$.*

2) *If $A \geq 0$ is irreducible, then $\mu_{\max} < \lambda_1(\Omega)$.*

3) *If $a_{ij} > 0$ for every i, j, and A is symmetric, then $\|A\| < \lambda_1(\Omega)$.*

Using a Pohozaev-type identity [10] we shall prove as in [11] the following nonexistence result.

Theorem 1.4 *Let $F \in C^1(\mathbb{R}^n)$ be homogeneous of degree $p = 2^* = 2N/(N-2)$ and define $f_i = \frac{1}{p}\partial_i F$. Assume that A is symmetric and negative definite, and that Ω is star shaped. Then $u = 0$ is the unique classical solution of (1.2).*

2 The Brezis-Lieb Lemma

We shall use the following version of the Brezis-Lieb lemma [3].

Lemma 2.1 *Assume that* $F \in C^1(\mathbb{R}^n)$ *with* $F(0) = 0$ *and* $\left|\frac{\partial F}{\partial u_i}\right| \le C|u|^{p-1}$. *Let* $(u_k) \subset L^p(\Omega)$, *(*$1 \le p < \infty$*). If* (u_k) *is bounded in* $L^p(\Omega)$ *and* $u_k \to u$ *a.e. on* Ω, *then*

$$\lim_{k\to\infty} \Big(\int_\Omega F(u_k) - F(u_k - u) \Big) = \int_\Omega F(u)$$

Proof We first remark that $u \in L^p(\Omega)$ and $\|u\|_p \le \liminf \|u_k\|_p < \infty$. We claim that for a fixed $\varepsilon > 0$ there exists $c(\varepsilon)$ such that for $a, b \in \mathbb{R}^n$, it holds

$$|F(a+b) - F(a)| \le \varepsilon |a|^p + c(\varepsilon)|b|^p \tag{2.1}$$

Indeed, writing

$$|F(a+b) - F(a)| = \left| \sum_{i=1}^n \int_0^1 \frac{\partial F}{\partial u_i}(a+bt) b_i dt \right| \le C \sum_{i=1}^n \int_0^1 |a+bt|^{p-1} |b_i| dt$$

and using that $xy \le c(\widetilde{\varepsilon}) x^p + \widetilde{\varepsilon} y^{p'}$ $(x, y > 0)$ we obtain

$$|F(a+b) - F(a)| \le C \sum_{i=1}^n \int_0^1 (\widetilde{\varepsilon}|a+bt|^p + c(\widetilde{\varepsilon})|b_i|^p)\, dt$$

Moreover, as $(x+y)^p \le 2^{p-1}(x^p + y^p)$ $(x, y > 0)$, we obtain:

$$|F(a+b) - F(a)| \le 2^{p-1} C \sum_{i=1}^n \int_0^1 \widetilde{\varepsilon}(|a|^p + t^p|b|^p) + c(\widetilde{\varepsilon})|b_i|^p dt$$

and (2.1) follows. Letting $a = u_k(x) - u(x)$, $b = u(x)$ we obtain

$$|F(u_k) - F(u_k - u)| \le \varepsilon |u_k - u|^p + c(\varepsilon)|u|^p$$

We introduce the functions:

$$f_k^\varepsilon = (|F(u_k) - F(u_k - u) - F(u)| - \varepsilon |u_k - u|^p)^+$$

As $|F(u)| \le K|u|^p$, then $|f_k^\varepsilon| \le (K + c(\varepsilon))|u|^p$. By Lebesgue theorem $\int_\Omega f_k^\varepsilon \to 0$. Since $|F(u_k) - F(u_k - u) - F(u)| \le f_k^\varepsilon + \varepsilon |u_k - u|^p$, we obtain

$$\limsup_{k\to\infty} \int_\Omega |F(u_k) - F(u_k - u) - F(u)| \le \varepsilon c$$

with $c = \sup_k \|u_k - u\|_p^p < \infty$. Letting $\varepsilon \to 0$, the result follows.

Remark In particular, this result holds for F is homogeneous of degree p.

3 Proofs of results

For the proof of part 1) of Theorem 1.1 we shall use the Lemma 3.1 below, which is a version of the concentration compactness lemma in [9].

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function homogeneous of degree $p = 2^*$, such that $F(u) > 0$ if $u \neq 0$. By homogeneity, it is easy to see that

$$S_F = \inf_{u \in D^{1,2}(\mathbb{R}^N,\mathbb{R}^n), u\neq 0} \frac{\sum_{k=1}^n \int_{\mathbb{R}^N} |\nabla u_k|^2}{\left(\int_{\mathbb{R}^N} F(u)\right)^{2/2^*}}$$

Lemma 3.1 *Let $(u^{(i)}) \subset D_0^{1,2}(\mathbb{R}^N, \mathbb{R}^n)$ be a sequence such that:*

i) $u^{(i)} \to u$ *weakly in* $D^{1,2}(\Omega)$

ii) $|\nabla(u_k^{(i)} - u_k)| \to \mu_k$ *in* $M(\mathbb{R}^n)$ *weak* * *for* $k = 1, ..., n$.

iii) $F(u^{(i)} - u) \to \nu$ *in* $M(\mathbb{R}^n)$ *weak**

iv) $u^{(i)} \to u$ *a.e. on* $\mathbb{R}^N$

and define: $\mu = \sum_{k=1}^n \mu_k$,

$$\nu^\infty = \lim_{R\to\infty} \Big(\limsup_{i\to\infty} \int_{|x|\geq R} F(u^{(i)})dx \Big),$$
$$\mu_k^\infty = \lim_{R\to\infty} \Big(\limsup_{i\to\infty} \int_{|x|\geq R} |\nabla u_k^{(i)}|^2 dx \Big)$$

Then:

$$\|\nu\|^{2/2^*} \leq \frac{1}{S_F} \|\mu\| \tag{3.1}$$

$$(\nu^\infty)^{2/2^*} \leq \frac{1}{S_F} \sum_{k=1}^n \mu_k^\infty \tag{3.2}$$

$$\limsup_{i\to\infty} |\nabla u_k^{(i)}|_2^2 = |\nabla u|_2^2 + \|\mu_k\| + \mu_k^\infty \quad \textit{for } k = 1, ..., n \tag{3.3}$$

$$\limsup_{i\to\infty} \int_\Omega F(u^{(i)}) = \int_\Omega F(u) + \|\nu\| + \nu_\infty \tag{3.4}$$

Moreover, if $u = 0$ and equality holds in (3.1), then $\mu = 0$ or μ is concentrated at a single point.

Proof of Theorem 1.1 Part 1) Let $(u^{(i)}) \subset D^{1,2}(\mathbb{R}^N, \mathbb{R}^n)$ be a minimizing sequence for S_F, i.e.,

$$\int_{\mathbb{R}^N} F(u^{(i)}) = 1, \qquad \sum_{k=1}^n \int_{\mathbb{R}^N} |\nabla u_k^{(i)}|^2 \to S_F$$

Using (3.1)-(3.3), we deduce, as in [12, Theorem 1.41], the existence of a sequence $(y_i, \lambda_i) \in \mathbb{R}^N \times \mathbb{R}$ such that $\lambda_i^{(N-2)/2} u(\lambda_i x + y_i)$ has a convergent subsequence. In particular there exists a minimizer for S_F.

To prove the second part of Theorem 1, we shall use the following version of the Mountain Pass Lemma [12].

Theorem 3.2 (Ambrosetti-Rabinowitz) *Let X be a Hilbert space, φ be an element of $C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\| > r$, $b = \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e)$. Then for each $\varepsilon > 0$ there exists $u \in X$ such that $c - \varepsilon \leq \varphi(u) \leq c + \varepsilon$ and $\|\varphi'(u)\| \leq \varepsilon$ where*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

Letting $\varepsilon = 1/k$, we get a Palais-Smale sequence at level c; i.e., a sequence $(u^{(k)}) \subset X$ such that

$$\varphi(u^{(k)}) \to c, \quad \varphi'(u^{(k)}) \to 0$$

We shall apply this result to the functional

$$\varphi(u) = \frac{1}{2} \int_\Omega \sum_{i=1}^n |\nabla u_i|^2 - \frac{1}{2^*} \int_\Omega F(u) - \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij} u_i u_j$$

in the Sobolev space $X = H_0^1(\Omega, \mathbb{R}^n)$. As $\|A\| < \lambda_1(\Omega)$, we may define on X the norm

$$\|u\| = \left(\int_\Omega \sum_{i=1}^n |\nabla u_i|^2 - \int_\Omega \sum_{i,j=1}^n a_{ij} u_i u_j \right)^{1/2}$$

which is equivalent to the usual norm. By standard arguments $\varphi \in C^1(X)$ and

$$\langle \varphi'(u), h \rangle = \sum_{i=1}^n \int_\Omega \nabla u_i \cdot \nabla h_i - \sum_{i=1}^n \int_\Omega f_i(u) h_i - \int_\Omega \sum_{i,j=1}^n a_{ij} h_i u_j$$

It follows that the critical points of φ are weak solutions of the system.

To ensure that the value c given by the mountain pass theorem is indeed a critical value we need to prove the following lemma.

Lemma 3.3 *Let F be homogeneous of degree 2^*. Then any $(PS)_c$ sequence with $c < c^* = \left(\frac{1}{2} - \frac{1}{2^*}\right) \frac{S_F^{N/2}}{N}$ has a convergent subsequence.*

Proof Let $(u^{(k)}) \subset H_0^1(\Omega, \mathbb{R}^n)$ be a $(PS)_c$ sequence. First we show that it is bounded.

$$\langle \varphi'(u^{(k)}), u^{(k)} \rangle = \|u^{(k)}\|^2 - \sum_{i=1}^n \int_\Omega f_i(u^{(k)}) u_i^{(k)}$$

Since F is homogeneous of degree 2^*, we have that $\sum_{i=1}^n f_i(u^{(k)})u_i^{(k)} = F(u^{(k)})$. Then

$$\frac{1}{2}\|u^{(k)}\|^2 = \varphi(u^{(k)}) + \frac{1}{2^*}\int_\Omega F(u^{(k)}) = \varphi(u^{(k)}) + \frac{1}{2^*}\big(\|u^{(k)}\|^2 - \langle\varphi'(u^{(k)}), u^{(k)}\rangle\big)$$

Hence, for $k \geq k_0$ we have

$$\big(\frac{1}{2} - \frac{1}{2^*}\big)\|u^{(k)}\|^2 \leq C + \varepsilon\|u^{(k)}\|$$

and we conclude that $\|u^{(k)}\|$ is bounded.

We may assume that $u^{(k)} \to u$ weakly in $H_0^1(\Omega)^n$, $u^{(k)} \to u$ in $L^2(\Omega)^n$, and $u^{(k)} \to u$ a.e..

Since $(u^{(k)})$ is bounded in $L^{2^*}(\Omega)$, $f(u^{(k)})$ is bounded in $L^{2N/(N+2)}(\Omega)$. So we may assume that $f(u^{(k)}) \to f(u)$ weakly in $L^{2N/(N+2)}$. It follows that u is a critical point of φ, i.e. u is a weak solution of the system. We deduce that

$$\langle\varphi'(u), u\rangle = \|u\|^2 - \int_\Omega \sum_{i=1}^n f_i(u)u_i = 0$$

Moreover,

$$\varphi(u) = \frac{\|u\|^2}{2} - \frac{1}{2^*}\int_\Omega F(u) = \frac{\|u\|^2}{2} - \frac{1}{2^*}\int_\Omega \sum_{i=1}^n f_i(u)u_i = \big(\frac{1}{2} - \frac{1}{2^*}\big)\|u\|^2 \geq 0$$

Writing $v^{(k)} = u - u^{(k)}$, by Lemma 2.1 we have

$$\int_\Omega F(u^{(k)}) = \int_\Omega F(u) + \int_\Omega F(v^{(k)}) + o(1)$$

and then

$$\varphi(u^{(k)}) = \frac{1}{2}\|u - v^{(k)}\|^2 - \frac{1}{2^*}\int_\Omega F(u^{(k)}) - \frac{1}{2^*}\int_\Omega F(v^{(k)}) + o(1)$$

As $v^{(k)} \to 0$ weakly,

$$\|u^{(k)}\|^2 = \|u - v^{(k)}\|^2 = \|u\|^2 + \|v^{(k)}\|^2 + o(1)$$

and then we obtain

$$\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v^{(k)}\|^2 - \frac{1}{2^*}\int_\Omega F(u) - \frac{1}{2^*}\int_\Omega F(v^{(k)}) \to c \tag{3.5}$$

On the other hand we also know that $\langle\varphi'(u^{(k)}), u^{(k)}\rangle \to 0$ and

$$\begin{aligned}\langle\varphi'(u^{(k)}), u^{(k)}\rangle &= \|u^{(k)}\|^2 - \int_\Omega \sum_{i=1}^n f_i(u^{(k)})u_i^{(k)} = \|u^{(k)}\|^2 - 2^*\int_\Omega F(u^{(k)}) \\ &= \|u\|^2 + \|v^{(k)}\|^2 - 2^*\int_\Omega F(u) - 2^*\int_\Omega F(v^{(k)}) + o(1) \to 0\end{aligned}$$

Hence,

$$\|v^{(k)}\|^2 - 2^* \int_\Omega F(v^{(k)}) \to 2^* \int_\Omega F(u) - \|u\|^2 = 0$$

We may therefore assume that $\|v^{(k)}\|^2 \to b$, $2^* \int_\Omega F(v^{(k)}) \to b$. From (3.5), we deduce that

$$\varphi(u) + \Big(\frac{1}{2} - \frac{1}{2^*}\Big)b = c$$

and since $\varphi(u) \geq 0$,

$$\Big(\frac{1}{2} - \frac{1}{2^*}\Big)b \leq c$$

We claim that $b = 0$. Indeed, since $u^{(k)} \to 0$ in $L^2(\Omega, \mathbb{R}^n)$, it follows that $\sum_{i=1}^n \int_\Omega |\nabla(v^{(k)})_i|^2 \to b$. On the other hand,

$$\sum_{i=1}^n \int_\Omega |\nabla(v^{(k)})_i|^2 \geq S_F \Big(\int_\Omega F(v^{(k)}) \Big)^{2/2^*}$$

and, letting $k \to \infty$, $b \geq S_F b^{2/2^*}$. It follows that $b = 0$ or $b \geq S_F^{N/2}$. In this last case,

$$c^* = (\frac{1}{2} - \frac{1}{2^*})\frac{S_F^{N/2}}{N} \leq (\frac{1}{2} - \frac{1}{2^*})\frac{b}{N} \leq c,$$

a contradiction. Hence, $b = 0$ and $v^{(k)} \to 0$ strongly.

Proof of Theorem 1.1 part 2) In the same way of [12, Theorem 1.45], it suffices to apply the Mountain Pass Theorem with a value $c < c^*$. We shall find the maximum of the function $h : [0, 1] \to \mathbb{R}$ given by

$$h(t) = \varphi(tv) = \Big(\frac{t^2}{2} \|v\|^2 - \frac{t^{2^*}}{2^*} \int_\Omega F(v)\Big) = A\frac{t^2}{2} - Bt^{2^*}$$

Since $2^* - 2 = \frac{4}{N-2}$, we obtain the critical point

$$t_0 = (\frac{A}{2^* B})^{(N-2)/4}$$

with

$$h(t_0) = (\frac{A}{2^* B})^{N/2}(\frac{2^*}{2} - 1)B > 0$$

Then, it is easy to conclude that $c < c^*$.

Now we consider the special case

$$[u] = |u|_q = \Big(\sum_{i=1}^n |u_i|^q\Big)^{1/q} \quad \text{and} \quad F_q(u) = \Big(\sum_{i=1}^n |u_i|^q\Big)^{2^*/q}$$

for proving Corollary 1.2.

Lemma 3.4 *$S_F(\Omega) = S$ for $q \geq 2$ where S is the best constant for the Sobolev inequality with $n = 1$.*

Proof Suppose first that $q \geq 2^*$, then we have the estimate

$$\begin{aligned}
\Big[\int_\Omega \Big(\sum_{i=1}^n |u_i|^q\Big)^{2^*/q}\Big]^{2/2^*} &\leq \Big[\int_\Omega \sum_{i=1}^n |u_i|^{2^*}\Big]^{2/2^*} = \Big[\sum_{i=1}^n \int_\Omega |u_i|^{2^*}\Big]^{2/2^*} \\
&\leq \Big[\sum_{i=1}^n (S^{-1}\int_\Omega |\nabla u_i|_2^2)^{2^*/2}\Big]^{2/2^*} \leq \sum_{i=1}^n S^{-1}\int_\Omega |\nabla u_i|^2
\end{aligned}$$

It follows that $S_F \geq S$. For $2 \leq q \leq 2^*$ we use Minkowski inequality:

$$\begin{aligned}
\Big[\int_\Omega \Big(\sum_{i=1}^n |u_i|^q\Big)^{2^*/q}\Big]^{2/2^*} &= \Big\{\Big[\int_\Omega \Big(\sum_{i=1}^n |u_i|^q\Big)^{2^*/q}\Big]^{q/2^*}\Big\}^{2/q} \\
&\leq \Big[\Big(\sum_{i=1}^n \int_\Omega |u_i|^{2^*}\Big)^{q/2^*}\Big]^{2/q} \\
&= \sum_{i=1}^n \Big(\int_\Omega |u_i|^{2^*}\Big)^{2/2^*} \leq \sum_{i=1}^n S^{-1}\int_\Omega |\nabla u_i|^2
\end{aligned}$$

The inequality $S_F \leq S$ is verified easily taking functions of the form $u = (u_1, 0, ..., 0)$.

Proof of Corollary 1.2 First we note that by the 2^*-homogeneity of F, S_F does not depend on Ω. Taking $u(x) = U(x)e_i$, where

$$U(x) = \frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}$$

is the function that attains Sobolev's best constant in one dimension [12, Theorem 1.42], it follows that S_F is achieved when $\Omega = \mathbb{R}^N$ (where $N \geq 4$). By translation invariance of the problem, S_F is also achieved with $u_\varepsilon(x) = U_\varepsilon(x)\cdot e_i$, for

$$U_\varepsilon(x) = \varepsilon^{(2-N)/2}U(x/\varepsilon)$$

We shall see that $S_{F,A} < S_F$. Indeed, we may assume that $0 \in \Omega$ and choose i such that $a_{ii} > 0$. Then, if we define $u(x) = v_\varepsilon(x)e_i$, with $v_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, and ψ a smooth function with compact support in Ω such that $\psi \equiv 1$ in $B(0,\rho)$, we obtain as in [12, Lemma 1.46]:

$$\frac{\int_\Omega \sum_{i=1}^n |\nabla u_i|^2 - \int_\Omega \langle Au, u\rangle}{\Big(\int_\Omega F(u)\Big)^{2/p}} = \frac{\int_\Omega |\nabla u_\varepsilon|^2 - a_{ii}\int_\Omega u_\varepsilon^2}{\Big(\int_\Omega |u_\varepsilon|^p\Big)} < S$$

for ε small enough.

Proof of Theorem 1.3: Necessary conditions for the existence of non-negative solutions We recall the following theorem in [8].

Lemma 3.5 (Perron-Frobenius Theorem) *Let $A \geq 0$ be irreducible. Then A has a positive simple eigenvalue $\mu_{\max}$ such that $|\mu| \leq \mu_{\max}$ for any μ eigenvalue of A. Furthermore, there exists an eigenvector of $\mu_{\max}$ with positive coordinates.*

Now we are able to prove Theorem 1.3.

Suppose that the system has a nonnegative nontrivial solution. If e_1 is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, then $e_1 \in C^\infty(\overline{\Omega})$ and $e_1(x) > 0 \, \forall x \in \Omega$. Then

$$\lambda_1 \int_\Omega u_i e_1 = \int_\Omega -\Delta u_i e_1 = \int_\Omega f_i(u) e_1 + \sum_{j=1}^n a_{ij} \int_\Omega u_j e_1$$

If $z_i = \int_\Omega u_i e_1$, then

$$\lambda_1 z \geq Az,$$

and the inequality between the i-th components is strict if $u_i \neq 0$ for some i. Since $z \geq 0$, and $z_i > 0$ for some i, we obtain

$$\lambda_1 |z|^2 > \langle Az, z \rangle .$$

Since A is symmetric and positive definite,

$$\lambda_1 |z|^2 > \mu_{\min} |z|^2 \quad \textit{and} \quad \lambda_1 > \mu_{\min}.$$

This proves the first claim of the theorem.

For $A \geq 0$ and irreducible, let v be the eigenvector of A^t corresponding to $\mu_{\max}$, then from the Perron-Frobenius Theorem [8], $v_i > 0$ for any i and

$$\lambda_1 \langle z, v \rangle > \langle Az, v \rangle = \left\langle z, A^t v \right\rangle = \mu_{\max} \langle z, v \rangle$$

and since $\langle z, v \rangle > 0$, it follows that $\lambda_1 > \mu_{\max}$ and the second claim is proved.

Finally when $A \geq 0$ is symmetric, we have $\mu_{\max} = \|A\|$, and the proof is complete. □

Proof of Theorem 1.4 The proof of Theorem 1.4 consists of the next lemma and the next corollary.

Lemma 3.6 *Suppose that $u \in C^2(\overline{\Omega}, \mathbb{R}^n)$ is a classical solution of the gradient elliptic system*

$$\begin{gathered} -\Delta u_i = g_i(u) \quad \textit{in } \Omega \\ u = 0 \textit{ on } \quad \partial\Omega \end{gathered}$$

where $g_i = \frac{\partial G}{\partial u_i}$, $G \in C^1(\mathbb{R}^n)$, $G(0) = 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary. Then for a fixed y,

$$\sum_{k=1}^n \int_{\partial\Omega} |\nabla u_k|^2 (x-y) \cdot n(x) dS = 2N \int_\Omega G(u) dx - (N-2) \sum_{k=1}^n \int_\Omega g_k(u) u_k dx$$

Proof Multiply the k-th equation by $(x-y)\cdot\nabla u_k = \sum_{i=1}^N (x_i-y_i)\frac{\partial u_k}{\partial x_i}$ and integrate by parts, then we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N (x_i-y_i)\frac{\partial u_k}{\partial x_i} g_k(u) \\ = \int_\Omega |\nabla u_k|^2 + \int_\Omega \sum_{i,j=1}^n (x_i-y_i)\frac{\partial u_k}{\partial x_j}\frac{\partial^2 u_k}{\partial x_i x_j} - \int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x)dS\end{aligned}$$

Hence,

$$\int_\Omega \sum_{i=1}^N (x_i-y_i)\frac{\partial u_k}{\partial x_i} g_k(u) = \big(1-\frac{N}{2}\big)\int_\Omega |\nabla u_k|^2 - \frac{1}{2}\int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x)dS$$

Adding this identities for $k=1,2,\dots n$,

$$\begin{aligned}\int_\Omega \sum_{i=1}^N (x_i-y_i)\sum_{k=1}^n \frac{\partial u_k}{\partial x_i} g_k(u) \\ = \big(1-\frac{N}{2}\big)\sum_{k=1}^n \int_\Omega |\nabla u_k|^2 - \frac{1}{2}\sum_{k=1}^n \int_{\partial\Omega} |\nabla u|^2 (x-y)\cdot n(x)dS\end{aligned}$$

By the chain rule we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N (x_i-y_i)\sum_{k=1}^N \frac{\partial u_k}{\partial x_i} g_k(u) &= \int_\Omega \sum_{i=1}^N (x_i-y_i)\frac{\partial G(u)}{\partial x_i} \\ &= -N\int_\Omega G(u) + \sum_{i=1}^n \int_{\partial\Omega} G(u)(x_i-y_i)\cdot n_i(x)dS\,.\end{aligned}$$

Since $G(u)=0$ on $\partial\Omega$,

$$-N\int_\Omega G(u) = (1-\frac{N}{2})\sum_{k=1}^N \int_\Omega |\nabla u_j|^2 - \frac{1}{2}\sum_{k=1}^N \int_{\partial\Omega} |\nabla u|^2 (x-y)\cdot n(x)dS$$

Finally

$$\int_\Omega |\nabla u_k|^2 = \int_\Omega g_k(u)u_k$$

and

$$\sum_{k=1}^N \int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x) = 2N\int_\Omega G(u) - (N-2)\sum_{k=1}^N \int_\Omega g_k(u)u_k$$

□

With the following corollary, we complete the proof of Theorem 1.4.

Corollary 3.7 *Assume that $F \in C^1(\mathbb{R}^n)$ is homogeneous of degree $p = 2^* = 2N/(N-2)$, with $F(0) = 0$. Further, assume that A is symmetric and negative definite, and that Ω is star shaped. Then the system*

$$\begin{gathered} -\Delta u_j = f_k(u) + \sum_{j=1}^{k} a_{jk} u_j \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

with $f_k = \frac{\partial F}{\partial u_k}$ admits only the trivial solution.

Proof Let $G(u) = F(u) + \frac{1}{2}\langle Au, u\rangle$. Since F is homogeneous of degree p,

$$\sum_{k=1}^{N} f_k(u) u_k = pF(u)$$

and

$$\sum_{k=1}^{N} \int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x) = [2N - p(N-2)] \int_{\Omega} F(u) + 2\sum_{k=1}^{N} \int_{\Omega} \langle Au, u\rangle$$

Since $p = 2N/(N-2)$,

$$\sum_{k=1}^{N} \int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x) = 2\sum_{k=1}^{N} \int_{\Omega} \langle Au, u\rangle$$

Now, because A is negative definite, $\langle Au, u\rangle \le M|u|^2$ where $M < 0$ and then

$$\sum_{k=1}^{N} \int_{\partial\Omega} |\nabla u_k|^2 (x-y)\cdot n(x) \le 2M \sum_{k=1}^{n} \int_{\Omega} |u|^2$$

Since Ω is star shaped, $(x-y)\cdot n(x) > 0$ on $\partial\Omega$, and we conclude that $u = 0$.

References

[1] H. Brezis and L. Nirenberg, *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents,* Communications on Pure and Applied Mathematics. Vol XXXVI , 437-477 (1983)

[2] L. Boccardo and D. de Figueiredo, *Some Remarks on a system of quasilinear elliptic equations*, to appear.

[3] H. Brezis and E. Lieb, *A relation between point wise convergence of functions and convergence of functionals,* Proc. A.M.S. vol. 48, No 3 (1993), pp. 486-499

[4] P. Clément, E. Mitidieri and R., Manásevich, *Positive solutions for a quasilinear system via blow up,* Comm. in Part. Diff. Eq., Vol. 18, No. 12, (1993).

[5] D. de Figueiredo, *Semilinear Elliptic Systems*, Notes of the course on semilinear elliptic systems of the EDP Chile-CIMPA Summer School (Universidad de la Frontera, Temuco, Chile, January 1999).

[6] D. de Figueiredo, *Semilinear Elliptic Systems: A survey of superlinear problems.* Resenhas IME-USP 1996, vol. 2, No. 4, pp. 373-391.

[7] D. Gilbarg, and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer- Verlag (1983).

[8] F. R. Grantmacher, *The Theory of matrices*, Chelsea (1959).

[9] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case.* Rev. Mat. Iberoamericana (1985), pp. 145-201 and 45-121.

[10] S. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$. Nonlinearity Doklady Akad. Nauk SSRR 165, (1965), pp. 36-9.

[11] P. Pucci and J. Serrin, *A general variational identity,* Indiana University Journal, Vol. 35, No. 3, (1986), 681-703.

[12] M. Willem, *Minimax Theorems*, Birkhauser (1986)

Pablo Amster (e-mail: pamster@dm.uba.ar)
Pablo De Nápoli (e-mail: pdenapo@dm.uba.ar)
Maria Cristina Mariani (e-mail: mcmarian@dm.uba.ar)
Departamento. de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires.
Pabellón I, Ciudad Universitaria (1428)
Buenos Aires, Argentina