*Electronic Journal of Differential Equations*, Vol. 2002(2002), No. 63, pp. 1–28.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

# ROBUST EXPONENTIAL ATTRACTORS FOR SINGULARLY PERTURBED PHASE-FIELD TYPE EQUATIONS

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Abstract. In this article, we construct robust (i.e. lower and upper semicontinuous) exponential attractors for singularly perturbed phase-field type equations. Moreover, we obtain estimates for the symmetric distance between these exponential attractors and that of the limit Cahn-Hilliard equation in terms of the perturbation parameter. We can note that the continuity is obtained without time shifts as it is the case in previous results.

## Introduction

In this article, we are interested in the study of the asymptotic behavior of phase-field type equations. The corresponding equations consist of a system of two parabolic equations involving two unknowns, namely the temperature $u(t,x)$ at point $x$ and time $t$ of a substance which can appear in two different phases (e.g. liquid-solid) and a phase-field function $\phi(t,x)$, also called order parameter, which describes the current phase at $x$ and $t$. Such models were introduced in order to study the evolution of interfaces in phase transitions. They have also led to other models of phase transitions and motion of interfaces as singular limits (e.g. the Stefan, Hele-Shaw and Cahn-Hilliard models). We refer the interested reader to [6, 7, 8, 9, 10, 19, 20, 21, 25, 27] and the references therein for more details.

The long time behavior of such models was extensively studied in [2, 3, 4, 5, 10, 11, 12, 13, 17]. In particular, the existence of the global attractor and exponential attractors is obtained in [3, 4, 5]. Furthermore, the upper semicontinuity of the global attractor for a singularly perturbed phase-field model is proved in [12] (see also [11] for a logarithmic nonlinearity) for two limit equations, namely the viscous Cahn-Hilliard and Cahn-Hilliard equations. The lower semicontinuity of the global attractor was studied in [10], but only in one space dimension. In that case, the authors do not need any assumption on the hyperbolicity of the stationary solutions, as it is usually the case to obtain the lower semicontinuity of the global attractor for dynamical systems which possess a global Lyapunov function [1, 22].

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1991 *Mathematics Subject Classification.* 35B40, 35B45.
*Key words and phrases.* Phase-field equations, exponential attractors, upper and lower semicontinuity.

Submitted April 18, 2002. Published July 4, 2002.
This research was partially supported by INTAS project 00-899.

In [14, 15], we constructed families of robust (i.e. upper and lower semicontinuous) exponential attractors for singularly perturbed viscous Cahn-Hilliard equations and damped wave equations. We can note that these results are not based on the study of stationary solutions and their unstable manifolds, as it is the case for regular global attractors [1, 22]; in particular, this allows us to obtain explicit estimates on the different constants (appearing e.g. in the estimate for the symmetric distance between the exponential attractors of the perturbed and unperturbed problems, see [14], [15] and below).

Our aim in this article is to obtain a similar result for singularly perturbed phase-field equations. Actually, we consider a more general system of equations, which does not possess a global Lyapunov function, by adding a nonlinear term in the equation for the temperature $u$.

In Section 1, we derive uniform estimates which are necessary for the study of the singular limit. Then, in Section 2, we study the asymptotic expansion of the solutions with respect to the singular perturbation parameter $\varepsilon$ and obtain estimates on the difference of solutions which are essential for our construction of exponential attractors. Finally, in Section 3, we construct a family of continuous exponential attractors for our problem and obtain in particular an explicit estimate for the symmetric distance between the exponential attractors of the perturbed and unperturbed equations in terms of the perturbation parameter $\varepsilon$ (see Theorem 3.1 below). The case of Neumann boundary conditions is briefly addressed in Section 4.

**Setting of the problem.** We consider the following system of singularly perturbed reaction-diffusion equations:

$$\begin{gathered}\delta\partial_t\phi = \Delta_x\phi - f_1(\phi) + u + g_1, \quad \phi\big|_{\partial\Omega} = 0,\\ \varepsilon\partial_t u + \partial_t\phi = \Delta_x u - f_2(u) + g_2, \quad u\big|_{\partial\Omega} = 0,\\ \phi\big|_{t=0} = \phi_0, \quad u\big|_{t=0} = u_0,\end{gathered} \tag{0.1}$$

where $\Omega$ is a bounded regular domain of $\mathbb{R}^3$, $(\phi(t,x), u(t,x))$ is an unknown pair of functions, $\Delta_x$ is the Laplacian with respect to the variable $x$, $g_i = g_i(x) \in L^2(\Omega)$, $i = 1, 2$, are given external forces and $\delta$ and $\varepsilon > 0$ are given constants.

We assume that the nonlinear terms $f_i$ belong to $C^3(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, and satisfy the following dissipativity conditions:

$$\begin{aligned} &f_1(v).v \ge -C, \quad C \ge 0\\ &f_1'(v) \ge -K, \quad K \ge 0\\ &f_2'(v) \ge 0, \quad f_2(0) = 0.\end{aligned} \tag{0.2}$$

Finally, we assume that the initial data $(\phi_0, u_0)$ belongs to the phase space $\Phi$, defined by

$$\Phi := \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right). \tag{0.3}$$

**Remark 0.1.** *Taking* $f_2 \equiv 0$ *in* (0.1)*, we recover the phase-field system considered in* [2, 3, 4, 5, 10, 11, 12, 13, 17].

## 1. Uniform a priori estimates

In this section, we derive several uniform (with respect to $\varepsilon \ll 1$) estimates for the solutions of problem (0.1) which are necessary for the study of the singular limit $\varepsilon \to 0$. We start with the following lemma.

**Lemma 1.1.** *Let the above assumptions hold and let the pair* $(\phi(t), u(t)) \in C(\mathbb{R}, \Phi)$ *be a solution of* (0.1). *Then, the following estimate is valid:*

$$\begin{aligned} &\|\nabla_x\phi(t)\|_{L^2}^2 + \varepsilon\|u(t)\|_{L^2}^2 + (F_1(\phi(t)),1) + \\ &\quad + \int_t^{t+1} \left(\|\partial_t\phi(s)\|_{L^2}^2 + \|\nabla_x u(s)\|_{L^2}^2 + (f_2(u(s)),u(s))\right) ds \le \\ &\quad \le C\left(\|\nabla_x\phi(0)\|_{L^2}^2 + \varepsilon\|u(0)\|_{L^2}^2 + (F_1(\phi(0)),1)\right) e^{-\gamma t} + \\ &\quad + C\left(\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2\right), \end{aligned} \tag{1.1}$$

*where* $F_1(v) := \int_0^v f_1(s)\,ds$, $(\cdot,\cdot)$ *denotes the standard inner product in* $L^2(\Omega)$ *and the positive constants* $C_1$, $C_2$ *and* $\gamma$ *are independent of* $\varepsilon$.

Proof. Taking the inner product in $L^2(\Omega)$ of the first equation of (0.1) by $\partial_t\phi(t)$ and of the second equation by $u(t)$ and summing the relations that we obtain, we have

$$\begin{aligned} &\partial_t[\delta\|\nabla_x\phi(t)\|_{L^2}^2 + 2(F_1(\phi(t)),1) + \varepsilon\|u(t)\|_{L^2}^2 - 2(g_1,\phi(t))] + \\ &\quad + 2\delta\|\partial_t\phi(t)\|_{L^2}^2 + 2\|\nabla_x u(t)\|_{L^2}^2 + 2(f_2(u(t)),u(t)) - 2(g_2,u(t)) = 0. \end{aligned} \tag{1.2}$$

Taking now the inner product in $L^2(\Omega)$ of the first equation of (0.1) by $2\beta\phi(t)$, where $\beta$ is a sufficiently small positive number, and summing the relation that we obtain with equation (1.2), we find

$$\partial_t E(t) + \gamma E(t) = h(t), \tag{1.3}$$

where

$$E(t) := \delta\|\nabla_x\phi(t)\|_{L^2}^2 + 2(F_1(\phi(t)),1) + \varepsilon\|u(t)\|_{L^2}^2 - 2(g_1,\phi(t)) + \beta\delta\|\phi(t)\|_{L^2}^2,$$

$0 < \gamma < \beta$ is another small positive parameter which will be fixed below and

$$\begin{aligned} h(t) &:= (\gamma\delta - 2\beta)\|\nabla_x\phi(t)\|_{L^2}^2 + 2\gamma\left(F_1(\phi(t)) - f_1(\phi(t))\phi(t),1\right) + \\ &\quad + 2(\gamma-\beta)(f_1(\phi(t)),\phi(t)) - 2\delta\|\partial_t\phi(t)\|_{L^2}^2 - 2\|\nabla_x u(t)\|_{L^2}^2 - \\ &\quad - 2(f_2(u(t)),u(t)) + 2(g_2,u(t)) + \gamma\varepsilon\|u(t)\|_{L^2}^2 + 2(\beta-\gamma)(g_1,\phi(t)) + \\ &\quad + \beta\delta\gamma\|\phi(t)\|_{L^2}^2 + 2\beta(u(t),\phi(t)). \end{aligned} \tag{1.4}$$

It follows from conditions (0.2) that

$$f_1(v).v + K|v|^2 \ge F_1(v), \quad \forall v \in \mathbb{R}, \tag{1.5}$$

(see e.g. [26]). Consequently, it is possible to fix the small positive parameters $\beta$ and $\gamma$ (which are independent of $0 < \varepsilon < 1$) such that the following estimate holds:

$$h(t) \le C_1\left(1 + \|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2\right), \tag{1.6}$$

where $C_1$ is independent of $\varepsilon$. Applying now Gronwall's inequality to relation (1.3) and using estimate (1.6) and equation (1.2), we find estimate (1.1) and Lemma 1.1 is proved. □

The next lemma gives uniform (with respect to $\varepsilon$) estimates of $(\phi, u)$ in the space $H^2(\Omega) \times H^1(\Omega)$.

**Lemma 1.2.** *Let the above assumptions hold. Then, the following estimate is valid for a solution $(\phi(t), u(t))$ of equation (0.1):*

$$\|\phi(t)\|_{H^2}^2+\|\partial_t\phi(t)\|_{L^2}^2+\|u(t)\|_{H^1}^2+\int_t^{t+1}\left(\|\partial_t\phi(s)\|_{H^1}^2+\varepsilon\|\partial_t u(s)\|_{L^2}^2\right)\,ds\le \\ \le Q(\|\phi(0)\|_{H^2}^2+\|u(0)\|_{H^1}^2)e^{-\gamma t}+Q(\|g_1\|_{L^2}^2+\|g_2\|_{L^2}^2), \quad (1.7)$$

*where the constant $\gamma>0$ and the monotonic function $Q$ are independent of $\varepsilon>0$.*

Proof. We set $\psi(t):=\partial_t\phi(t)$. Then, this function satisfies

$$\delta\partial_t\psi=\Delta_x\psi-f_1'(\phi)\psi+\partial_t u,\ \ \psi(0)=\delta^{-1}\left(\Delta_x\phi(0)-f_1(\phi(0))+u(0)+g_1\right). \quad (1.8)$$

Taking the inner product in $L^2(\Omega)$ of the equation by $\psi(t)$ and of the second equation of (0.1) by $\partial_t u$ and summing the relations that we obtain, we have

$$\partial_t[\delta\|\partial_t\phi(t)\|_{L^2}^2+\|\nabla_x u(t)\|_{L^2}^2+2(F_2(u(t)),1)-2(g_2,u(t))]+ \\ +[\delta\|\partial_t\phi(t)\|_{L^2}^2+\|\nabla_x u(t)\|_{L^2}^2+2(F_2(u(t)),1)-2(g_2,u(t))]+ \\ +2\|\nabla_x\psi(t)\|_{L^2}^2+2\varepsilon\|\partial_t u(t)\|_{L^2}^2=h_1(t), \quad (1.9)$$

where $F_2(v):=\int_0^v f_2(s)\,ds$ and

$$h_1(t):=[\delta\|\partial_t\phi(t)\|_{L^2}^2+\|\nabla_x u(t)\|_{L^2}^2+2(F_2(u(t)),1)-2(g_2,u(t))]+ \\ +2(g_1,\partial_t\phi(t))-2(f_1'(\phi(t))\partial_t\phi(t),\partial_t\phi(t)). \quad (1.10)$$

Analogously to (1.5), we have

$$f_2(v).v\ge F_2(v). \quad (1.11)$$

Furthermore, thanks to (0.2) and (1.11), we find

$$h_1(t)\le C_1\left(1+\|g_1\|_{L^2}^2+\|g_2\|_{L^2}^2\right)+ \\ +C_2\left((f_2(u(t)),u(t))+\|\partial_t\phi(t)\|_{L^2}^2+\|\nabla_x u(t)\|_{L^2}^2\right). \quad (1.12)$$

Applying Gronwall's inequality to relation (1.9) and using estimates (1.1) and (1.12), we obtain

$$\|\partial_t\phi(t)\|_{L^2}^2+\|u(t)\|_{H^1}^2+\int_t^{t+1}\left(\|\partial_t\phi(s)\|_{H^1}^2+\varepsilon\|\partial_t u(s)\|_{L^2}^2\right)\,ds\le \\ \le Q(\|\phi(0)\|_{H^2}^2+\|u(0)\|_{H^1}^2)e^{-\gamma t}+Q(\|g_1\|_{L^2}^2+\|g_2\|_{L^2}^2), \quad (1.13)$$

for appropriate constant $\gamma>0$ and monotonic function $Q$ which are independent of $\varepsilon$. There now remains to estimate the $H^2$-norm of $\phi(t)$. To this end, we rewrite the first equation of (0.1) in the form

$$\Delta_x\phi(t)-f_1(\phi(t))=h_2(t),\quad \phi(t)\big|_{\partial\Omega}=0, \quad (1.14)$$

where $h_2(t):=\delta\partial_t\phi(t)-u(t)-g_1$. Indeed, according to estimate (1.13), we have

$$\|h_2(t)\|_{L^2}^2\le Q(\|\phi(0)\|_{H^2}^2+\|u(0)\|_{H^1}^2)e^{-\gamma t}+Q(\|g_1\|_{L^2}^2+\|g_2\|_{L^2}^2). \quad (1.15)$$

Taking then the inner product in $L^2(\Omega)$ of equation (1.14) by $\Delta_x\phi(t)$ and using (0.2), we obtain

$$\|\Delta_x\phi(t)\|_{L^2}^2\le 2K\|\nabla_x\phi(t)\|_{L^2}^2+2\|h_2(t)\|_{L^2}^2. \quad (1.16)$$

Inserting finally estimates (1.15) and (1.13) into the right-hand side of (1.16), we derive the necessary estimate for the $H^2$-norm of $\phi(t)$ and Lemma 1.2 is proved. □

We are now in a position to derive a priori estimates for the solutions of (0.1) in the phase space $\Phi$.

**Lemma 1.3.** *Let the above assumptions hold. Then, the following estimate holds, for every solution* $(\phi(t), u(t))$ *of problem* (0.1)*:*

$$\|\phi(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \varepsilon^2\|\partial_t u(t)\|_{L^2}^2 \le \\ \le Q(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\alpha t} + Q(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \quad (1.17)$$

*where the positive constant* $\alpha$ *and the monotonic function* $Q$ *are independent of* $\varepsilon$.

Proof. We rewrite the second equation of system (0.1) in the form

$$\varepsilon\partial_t u - \Delta_x u + f_2(u) = h(t) := g_2 - \partial_t\phi(t). \quad (1.18)$$

Rescaling now the time variable ($t := \varepsilon\tau$), we have

$$\partial_\tau u - \Delta_x u + f_2(u) = \tilde{h}(\tau) := h(\varepsilon\tau), \quad u\big|_{\tau=0} = u_0. \quad (1.19)$$

Moreover, it follows from (1.7) that

$$\|\tilde{h}(\tau)\|_{L^2} \le Q(\|\phi(0)\|_{H^2}^2 + \|u(0)\|_{H^1}^2)e^{-\gamma\varepsilon\tau} + Q(\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2), \quad (1.20)$$

where $Q$ and $\alpha$ are independent of $\varepsilon$. Applying the standard maximum principle to equation (1.19), using the fact that $f_2(u).u \ge 0$ and noting that $H^2 \subset C$ (since $n = 3$), we obtain the estimate

$$\|u(\tau)\|_{L^\infty} \le C\|u(0)\|_{H^2}e^{-\beta\tau} + C\sup_{s\in[0,\tau]}\left\{e^{-\beta(\tau-s)}\|\tilde{h}(s)\|_{L^2}\right\}, \quad (1.21)$$

for appropriate positive constants $\beta$ and $C$ (see e.g. [18] for details). Inserting estimate (1.20) into the right-hand side of (1.21) and returning to the time variable $t$, we find

$$\|u(t)\|_{L^\infty} \le Q(\|\phi(0)\|_{H^2}^2 + \|u(0)\|_{H^2}^2)e^{-\gamma t} + Q(\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2), \quad (1.22)$$

where the constant $\gamma$ and the function $Q$ are independent of $\varepsilon$.

Let us now derive a uniform estimate for the $H^2$-norm of $u(t)$. To this end, we introduce the functions $G_i = G_i(x) := (-\Delta_x)^{-1}g_i$, $i = 1, 2$, and split the solution $(\phi(t), u(t))$ as follows:

$$\phi(t) := G_1 + \phi_1(t) + \phi_2(t), \quad u(t) := G_2 + u_1(t) + u_2(t) + u_3(t), \quad (1.23)$$

where $u_1(t)$ solves

$$\varepsilon\partial_t u_1 = \Delta_x u_1, \quad u_1\big|_{t=0} = u_0 - G_2, \quad (1.24)$$

the function $u_2(t)$ is solution of

$$\varepsilon\partial_t u_2 = \Delta_x u_2 - \partial_t\phi_1,\ u_2\big|_{t=0} = 0, \quad (1.25)$$

with

$$\delta\partial_t\phi_1 = \Delta_x\phi_1, \quad \phi_1\big|_{t=0} = \phi_0 - G_1, \quad (1.26)$$

and the function $u_3(t)$ solves

$$\varepsilon\partial_t u_3 - \Delta_x u_3 = h_3(t) := -\partial_t\phi_2(t) - f_2(u(t)), \quad u_3\big|_{t=0} = 0, \quad (1.27)$$

with

$$\delta\partial_t\phi_2 - \Delta_x\phi_2 = h_4(t) := u(t) - f_1(\phi(t)), \quad \phi_2\big|_{t=0} = 0. \quad (1.28)$$

Obviously, $G_i \in H^2(\Omega)$ and

$$\|G_i\|_{H^2} \le C\|g_i\|_{L^2}, \quad i = 1, 2. \quad (1.29)$$

Moreover, since $-\Delta_x$ generates an analytic semigroup in $H^2(\Omega)$, then

$$\|u_1(t)\|_{H^2} \le Ce^{-\gamma t/\varepsilon}\left(\|u_0\|_{H^2} + \|g_2\|_{L^2}\right), \tag{1.30}$$

where the constants $C$ and $\gamma$ are independent of $\varepsilon$. Let us then estimate $u_2(t)$. To this end, we note that

$$u_2(t) = \frac{\delta}{\delta - \varepsilon}\left(\phi_1(t) - \tilde{u}_0(t)\right), \tag{1.31}$$

for $\varepsilon \ll 1$, where the function $\tilde{u}_0(t)$ solves the problem

$$\varepsilon\partial_t\tilde{u}_0 = \Delta_x\tilde{u}_0, \quad \tilde{u}_0\big|_{t=0} = \phi_0 - G_1.$$

Analogously to (1.30), we have

$$\|u_2(t)\|_{H^2} \le Ce^{-\beta t}\left(\|\phi_0\|_{H^2} + \|g_1\|_{L^2}\right), \tag{1.32}$$

where $C$ and $\beta$ are independent of $\varepsilon$. So, there only remains to estimate $u_3(t)$. To this end, we note that, due to estimate (1.7) and due to the fact that $H^2 \subset C$, the function $h_4$ defined in (1.28) satisfies

$$\|h_4(t)\|_{H^1} \le Q(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\gamma t} + Q(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \tag{1.33}$$

for appropriate $\gamma$ and $Q$ which are independent of $\varepsilon$. Applying the parabolic regularity theorem (see e.g. [18]) to equation (1.28), we obtain

$$\|\partial_t\phi_2(t)\|_{H^{1-\beta}} \le Q_\beta(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\gamma t} + Q_\beta(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \tag{1.34}$$

where $0 < \beta < 1$ and $\gamma$ and $Q_\beta$ are independent of $\varepsilon$. Consequently, according to (1.7), (1.22) and (1.34), we have the following estimate for the function $h_3(t)$ in the right-hand side of (1.17):

$$\|h_3(t)\|_{H^{1-\beta}} \le Q_\beta(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\gamma t} + Q_\beta(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \tag{1.35}$$

for appropriate $\gamma$ and $Q_\beta$ which are independent of $\varepsilon$. Applying now the standard parabolic regularity theorem to equation (1.27) and rescaling the time as above ($t := \varepsilon\tau$) in order to eliminate the dependence on $\varepsilon$ (analogously to (1.18)–(1.22)), we deduce from (1.35) that

$$\|u_3(t)\|_{H^2} \le Q(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\gamma t} + Q(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \tag{1.36}$$

where the positive constant $\gamma$ and the monotonic function $Q$ are independent of $\varepsilon$. Combining (1.29), (1.30), (1.32) and (1.36), we finally have

$$\|u(t)\|_{H^2} \le Q(\|\phi(0)\|_{H^2} + \|u(0)\|_{H^2})e^{-\gamma t} + Q(\|g_1\|_{L^2} + \|g_2\|_{L^2}), \tag{1.37}$$

for some new positive constant $\gamma$ and monotonic function $Q$ which are independent of $\varepsilon$. Thus, the uniform estimate for the $H^2$-norm of $u(t)$ is obtained. The uniform estimate for the $L^2$-norm of $\varepsilon\partial_t u(t)$ is an immediate corollary of (1.7), (1.37) and of the second equation in (0.1). This finishes the proof of Lemma 1.3. $\square$

**Lemma 1.4.** *Let the above assumptions hold. Then, for every* $(\phi_0, u_0) \in \Phi$, *problem* (0.1) *has a unique solution* $(\phi(t), u(t)) \in C(\mathbb{R}, \Phi)$ *which satisfies estimate*

(1.17). *Moreover, for any solutions* $(\phi_i(t), u_i(t)) \in \Phi$, $i = 1, 2$, *the following inequality holds:*

$$\|\phi_1(t) - \phi_2(t)\|_{H^1}^2 + \varepsilon\|u_1(t) - u_2(t)\|_{L^2}^2 + \\ + \int_t^{t+1} \left(\|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_{L^2}^2 + \|\nabla_x u_1(s) - \nabla_x u_2(s)\|_{L^2}^2\right) ds \leq \\ \leq Ce^{Lt}\left(\|\phi_1(0) - \phi_2(0)\|_{H^1}^2 + \varepsilon\|u_1(0) - u_2(0)\|_{L^2}^2\right), \quad (1.38)$$

*where the constants* $C$ *and* $L$ *depend on* $\|\phi_i(0)\|_{H^2}$ *and on* $\|u_i(0)\|_{H^2}$, *but are independent of* $\varepsilon$.

Proof. The existence of a solution can be proved in a standard way, based on a priori estimate (1.17) and on the Leray-Schauder fixed point theorem (see e.g. [18]). So, there remains to deduce estimate (1.38). To this end, we set $v(t) := \phi_1(t) - \phi_2(t)$ and $w(t) := u_1(t) - u_2(t)$. These functions satisfy the equations

$$\begin{gathered} \delta\partial_t v = \Delta_x v - l_1(t)v + w, \quad v\big|_{t=0} = \phi_1(0) - \phi_2(0), \quad v\big|_{\partial\Omega} = 0, \\ \varepsilon\partial_t w + \partial_t v = \Delta_x w - l_2(t)w, \quad w\big|_{t=0} = u_1(0) - u_2(0), \quad w\big|_{\partial\Omega} = 0, \end{gathered} \quad (1.39)$$

where

$$l_1(t) := \int_0^1 f_1'(s\phi_1(t) + (1-s)\phi_2(t))\,ds, \quad l_2(t) := \int_0^1 f_2'(su_1(t) + (1-s)u_2(t))\,ds.$$

It now follows from estimates (1.7) and (1.17) and from the embedding $H^2 \subset C$ that

$$\|l_1(t)\|_{H^2} + \|\partial_t l_1(t)\|_{L^2} + \|l_2(t)\|_{H^2} \leq \\ \leq L := Q(\|(\phi_1(0), u_1(0))\|_\Phi + \|(\phi_2(0), u_2(0))\|_\Phi), \quad (1.40)$$

for a monotonic function $Q$ which is independent of $\varepsilon$. Moreover, due to our assumptions on $f_2'$, we have

$$l_2(t) \geq 0. \quad (1.41)$$

Multiplying now the first equation of (1.39) by $\partial_t v(t)$ and the second one by $w(t)$, integrating over $\Omega$ and summing the relations that we obtain, we find, taking into account estimates (1.40) and (1.41)

$$\partial_t[\|\nabla_x v(t)\|_{L^2}^2 + \varepsilon\|w(t)\|_{L^2}^2] + 2\delta\|\partial_t v(t)\|_{L^2}^2 + 2\|\nabla_x w(t)\|_{L^2}^2 \leq \\ \leq L^2\delta^{-1}\|v(t)\|_{L^2}^2 + \delta\|\partial_t v(t)\|_{L^2}^2. \quad (1.42)$$

Applying Gronwall's inequality to this relation, we derive estimate (1.38) and Lemma 1.4 is proved. □

**Corollary 1.5.** *Let the above assumptions hold. Then, for every* $\varepsilon > 0$, *problem* (0.1) *defines a semigroup* $S_t^\varepsilon$ *in the phase space* $\Phi$ *by*

$$S_t^\varepsilon : \Phi \to \Phi, \quad S_t^\varepsilon(\phi_0, u_0) = (\phi(t), u(t)), \quad (1.43)$$

*where the function* $(\phi(t), u(t))$ *solves* (0.1).

Let us now consider the limit equation of (0.1) (i.e. $\varepsilon = 0$ in (0.1)):

$$\begin{gathered} \delta\partial_t\bar\phi_0 = \Delta_x\bar\phi_0 - f_1(\bar\phi_0) + \bar u_0 + g_1, \quad \bar\phi_0\big|_{t=0} = \phi_0, \quad \bar\phi_0\big|_{\partial\Omega} = 0, \\ \partial_t\bar\phi_0 = \Delta_x\bar u_0 - f_2(\bar u_0) + g_2, \quad \bar u_0\big|_{\partial\Omega} = 0. \end{gathered} \quad (1.44)$$

We note that, in contrast to the case $\varepsilon > 0$, the values of $(\bar\phi_0(t), \bar u_0(t))$ are not independent in that case. Indeed, it follows from (1.44) that

$$\delta\Delta_x \bar u_0(t) - \delta f_2(\bar u_0(t)) - \bar u_0(t) = \Delta_x \bar\phi_0(t) - f_1(\bar\phi_0(t)) + g_1 - \delta g_2. \tag{1.45}$$

Moroever, as shown in the following proposition, the value of $\bar u_0(t)$ is uniquely defined by (1.45), if the value $\bar\phi_0(t)$ is known.

**Proposition 1.6.** *Let the above assumptions hold. Then, the nonlinear operator in the left-hand side of* (1.45) *is invertible in* $H^2(\Omega) \cap H^1_0(\Omega)$*, i.e. there exists a nonlinear* $C^1$*-operator*

$$\mathcal{L} \in C^1(H^2(\Omega) \cap H^1_0(\Omega), H^2(\Omega) \cap H^1_0(\Omega)), \tag{1.46}$$

*such that* (1.45) *is equivalent to*

$$\bar u_0(t) = \mathcal{L}(\bar\phi_0(t)). \tag{1.47}$$

This proposition is an immediate corollary of the condition $f_2'(v) \ge 0$ (which provides the invertibility of the operator in the left-hand side of (1.45)) and of standard elliptic estimates.

Thus, the solution $(\bar\phi_0(t), \bar u_0(t))$ of problem (1.44) exists only for initial data $(\phi_0, u_0)$ that belong to the infinite dimensional submanifold $\mathbb{L}$ of the phase space $\Phi$ defined by

$$\mathbb{L} := \{(\phi_0, u_0) \in \Phi, \quad u_0 = \mathcal{L}(\phi_0)\} \subset \Phi. \tag{1.48}$$

**Lemma 1.7.** *Let the above assumptions hold. Then, for every* $(\phi_0, u_0) \in \mathbb{L}$*, problem* (1.44) *has a unique solution* $(\bar\phi_0(t), \bar u_0(t)) \in \mathbb{L}$*, for* $t \ge 0$*, which satisfies the estimate*

$$\begin{aligned} \|\bar\phi_0(t)\|^2_{H^2} + \|\partial_t\bar\phi_0(t)\|^2_{L^2} + \|\bar u_0(t)\|^2_{H^2} + \int_t^{t+1} \|\partial_t\bar\phi_0(s)\|^2_{H^1}\,ds \le \\ \le Q(\|\bar\phi_0(0)\|^2_{H^2})e^{-\gamma t} + Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}), \end{aligned} \tag{1.49}$$

*for a positive constant* $\gamma$ *and a monotonic function* $Q$*. Consequently, equation* (1.44) *defines a semigroup* $S^0_t$ *on the manifold* $\mathbb{L}$ *by*

$$S^0_t : \mathbb{L} \to \mathbb{L}, \quad S^0_t(\phi_0, u_0) := (\bar\phi_0(t), \bar u_0(t)), \tag{1.50}$$

*where the function* $(\bar\phi_0(t), \bar u_0(t))$ *solves* (1.44)*.*

Proof. Since estimates (1.7) and (1.17) are uniform with respect to $\varepsilon$, then, passing to the limit $\varepsilon \to 0$ in equations (0.1), we obtain a solution $(\bar\phi_0(t), \bar u_0(t))$ for problem (1.44) which satisfies (1.49). The uniqueness of this solution can be proved exactly as in Lemma 1.4. □

In the sequel, we will also need the estimates for $\partial_t \bar u_0$ and $\partial_t^2 \bar u_0$ that are given in the following lemma.

**Lemma 1.8.** *Let the above assumptions hold. Then, the following estimate is valid for the solution* $(\bar\phi_0(t), \bar u_0(t))$ *of problem* (1.44)*:*

$$\begin{aligned} \|\partial_t \bar u_0(t)\|^2_{L^2} + \int_t^{t+1} \left(\|\partial_t \bar u_0(s)\|^2_{H^1} + \|\partial_t^2 \bar u_0(s)\|^2_{H^{-1}}\right) ds \le \\ \le Q(\|\bar\phi_0(0)\|^2_{H^2})e^{-\gamma t} + Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}), \end{aligned} \tag{1.51}$$

*for a positive constant* $\gamma$ *and a monotonic function* $Q$*.*

Proof. Let us first derive estimate (1.51) for the first derivative $\partial_t\bar u_0(t)$. To this end, we differentiate relation (1.45) with respect to $t$ and split $\partial_t\bar u_0$ as follows:

$$\partial_t\bar u_0(t)=\delta^{-1}\partial_t\bar\phi_0(t)+\psi_0(t). \tag{1.52}$$

After straightforward substitutions, we find

$$\begin{aligned}\delta\Delta_x\psi_0(t)-\delta f_2'(\bar u_0(t))\psi_0(t)-\psi_0(t)&=\\ &=(f_2'(\bar u_0(t))-f_1'(\bar\phi_0(t))+\delta^{-1})\partial_t\bar\phi_0(t):=\Psi(t).\end{aligned} \tag{1.53}$$

It then follows from (1.49) that

$$\|\Psi(t)\|_{L^2}\le Q(\|\bar\phi_0(0)\|_{H^2})e^{-\gamma t}+Q(\|g_1\|_{L^2}+\|g_2\|_{L^2}),$$

and, consequently, due to the assumption $f_2'\ge 0$, it follows from (1.53) (using standard elliptic estimates) that

$$\|\psi_0(t)\|_{H^2}\le Q(\|\bar\phi_0(0)\|_{H^2})e^{-\gamma t}+Q(\|g_1\|^2_{L^2}+\|g_2\|^2_{L^2}). \tag{1.54}$$

Estimates (1.49) and (1.54) imply the part of (1.51) for $\partial_t\bar u_0$. So, there remains to estimate $\partial_t^2\bar u_0$ only. To this end, we differentiate the first equation of (1.44) with respect to $t$:

$$\delta\partial_t^2\bar\phi_0(t)=\Delta_x\partial_t\bar\phi_0(t)-f_1'(\bar\phi_0(t))\partial_t\bar\phi_0(t)+\delta^{-1}\partial_t\bar\phi_0(t)+\psi_0(t), \tag{1.55}$$

and obtain, using (1.49) and (1.54)

$$\int_t^{t+1}\|\partial_t^2\bar\phi_0(s)\|^2_{H^{-1}}\,ds\le Q(\|\bar\phi_0(0)\|^2_{H^2})e^{-\gamma t}+Q(\|g_1\|^2_{L^2}+\|g_2\|^2_{L^2}). \tag{1.56}$$

Differentiating now equation (1.53) with respect to $t$ and setting $\theta_0(t):=\partial_t\psi_0(t)$, we have

$$\begin{aligned}\delta\Delta_x\theta_0-\delta f_2'(\bar u_0)\theta_0-\theta_0&=\Big[(f_2'(\bar u_0)-f_1'(\bar\phi_0)+\delta^{-1})\partial_t^2\bar\phi_0\Big]+\\ +\Big[(\delta^{-1}f_2''(\bar u_0)-f_1''(\bar\phi_0))(\partial_t\bar\phi_0)^2\Big]&+\Big[f_2''(\bar u_0)(\delta\psi_0+2\partial_t\bar\phi_0)\psi_0\Big]:=\\ &:=I_1(t)+I_2(t)+I_3(t).\end{aligned} \tag{1.57}$$

Multiplying (1.57) by $\theta_0(t)$, integrating over $\Omega$ and noting that $f_2'\ge 0$, we obtain the inequality

$$\begin{aligned}\delta\|\nabla_x\theta_0(t)\|^2_{L^2}+\|\theta_0(t)\|^2_{L^2}&\le\\ &\le|\,(I_1(t),\theta_0(t))\,|+|\,(I_2(t),\theta_0(t))\,|+|\,(I_3(t),\theta_0(t))\,|.\end{aligned} \tag{1.58}$$

Let us estimate each term in the right-hand side of (1.58). Using Schwarz' inequality and the embeddings $H^2\subset C$ and $H^1\subset L^6$, we have

$$\begin{aligned}&|\,(I_1(t),\theta_0(t))\,|\le\\ &\le C\|\partial_t^2\bar\phi_0(t)\|_{H^{-1}}\|\nabla_x[(f_2'(\bar u_0(t))-f_1'(\bar\phi_0(t))+\delta^{-1})\theta_0(t)]\|_{L^2}\le\\ &\le Q(\|\bar\phi_0(t)\|_{H^2})\|\partial_t^2\bar\phi_0(t)\|_{H^{-1}}\|\nabla_x\theta_0(t)\|_{L^2}\le\\ &\le\frac{\delta}{4}\|\nabla_x\theta_0(t)\|^2_{L^2}+Q_1(\|\bar\phi_0(t)\|_{H^2})\|\partial_t^2\bar\phi_0(t)\|^2_{H^{-1}},\end{aligned} \tag{1.59}$$

where $Q$ and $Q_1$ are appropriate monotonic functions (here, we implicitly used formula (1.47) in order to estimate $\|\bar{u}_0(t)\|_{H^2}$ through $\|\bar{\phi}_0(t)\|_{H^2}$). Thanks to Hölder's inequality, we can estimate the second term:

$$\begin{aligned}|\,(I_2(t),\theta_0(t))\,| &\le Q(\|\bar{\phi}_0(t)\|_{H^2})\|\partial_t\bar{\phi}_0(t)\|_{L^2}\|\partial_t\bar{\phi}_0(t)\|_{L^3}\|\theta_0(t)\|_{L^6} \le \\ &\le Q_1(\|\bar{\phi}_0(t)\|_{H^2})\|\partial_t\bar{\phi}_0(t)\|^2_{L^2}\|\partial_t\bar{\phi}_0(t)\|^2_{H^1} + \frac{\delta}{4}\|\nabla_x\theta_0(t)\|^2_{L^2}. \quad (1.60)\end{aligned}$$

Finally, using estimates (1.49) and (1.54), we have

$$|\,(I_3(t),\theta_0(t))\,| \le \|\theta_0(t)\|^2_{L^2} + Q(\|\bar{\phi}_0(0)\|_{H^2})e^{-\gamma t} + Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}). \quad (1.61)$$

Inserting estimates (1.59)-(1.61) into (1.58), integrating the inequality that we obtain over $[t,t+1]$ and using estimates (1.49) and (1.54) again, we find

$$\int_t^{t+1}\|\theta_0(s)\|^2_{H^1}\,ds \le Q(\|\bar{\phi}_0(0)\|^2_{H^2})e^{-\gamma t} + Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}), \quad (1.62)$$

for a positive constant $\gamma$ and a monotonic function $Q$. There now remains to recall that $\partial_t^2\bar{u}_0 := \delta^{-1}\partial_t^2\bar{\phi}_0 + \theta_0$ and that the appropriate estimate for $\partial_t^2\bar{\phi}_0$ is given by (1.56) to finish the proof of the lemma. □

## 2. Estimates on the difference of solutions

In this section, we derive several estimates on the difference of two solutions of problem (0.1) which are of fundamental significance for our study of exponential attractors.

We start with computing the first terms of the asymptotic expansions of the solution $(\phi(t),u(t))$ of problem (0.1) as $\varepsilon \to 0$. To this end, following the general procedure (see e.g. [24]), we introduce the fast variable $\tau := \frac{t}{\varepsilon}$ and expand the solution as follows:

$$\phi(t) = \phi_0(t,\tau) + \varepsilon\phi_1(t,\tau) + \cdots, \quad u(t) = u_0(t,\tau) + \varepsilon u_1(t,\tau) + \cdots, \quad (2.1)$$

where the functions $u_i(t,\tau)$ are of the form

$$u_i(t,\tau) := \bar{u}_i(t) + \tilde{u}_i(\tau), \quad \phi_i(t,\tau) := \bar{\phi}_i(t) + \tilde{\phi}_i(\tau), \quad (2.2)$$

and satisfy the additional conditions

$$\lim_{\tau\to\infty}\tilde{u}_i(\tau) = \lim_{\tau\to\infty}\tilde{\phi}(\tau) = 0. \quad (2.3)$$

Inserting these expansions into system (0.1) and assuming that the $u_i(t,\tau)$ are independent of $\varepsilon$, we can obtain the recurrent equations for $u_i(t,\tau)$ and $\phi_i(t,\tau)$. Indeed, at order $\varepsilon^{-1}$, it follows from the first equation of (0.1) that

$$\partial_\tau\tilde{\phi}_0(\tau) = 0 \ \text{ and, consequently, } \tilde{\phi}_0(\tau) \equiv 0.$$

At order $\varepsilon^0$, we obtain

$$\delta\partial_\tau\tilde{\phi}_1(\tau) = \tilde{u}_0(\tau), \quad \delta\partial_t\bar{\phi}_0(t) = \Delta_x\bar{\phi}_0(t) - f_1(\bar{\phi}_0) + \bar{u}_0 + g_1.$$

Analogously, we deduce from the second equation of (0.1) that

$$\partial_t\bar{\phi}_0(t) = \Delta_x\bar{u}_0(t) - f_2(\bar{u}_0(t)) + g_2,$$

and

$$\partial_\tau\tilde{u}_0(\tau) = \Delta_x\tilde{u}_0(\tau) - [f_2(\bar{u}_0(0) + \tilde{u}_0(\tau)) - f_2(\bar{u}_0(0))] - \partial_\tau\tilde{\phi}_1(\tau).$$

Expanding now the initial data for $(\phi(t), u(t))$, we have

$$\bar{\phi}_0(0) = \phi(0), \quad \bar{\phi}_1(0) + \tilde{\phi}_1(0) = 0, \quad \tilde{u}(0) = u(0) - \bar{u}_0(0).$$

Thus, the function $(\bar{\phi}_0(t), \bar{u}_0(t))$ solves equation (1.44) with initial data $\bar{\phi}_0(0) = \phi(0)$, i.e.

$$(\bar{\phi}_0(t), \bar{u}_0(t)) = S^0_t(\phi(0), \mathcal{L}(\phi(0))), \tag{2.4}$$

and the first boundary layer term $\tilde{u}_0(\tau)$ can be found as a solution of the following problem:

$$\begin{aligned} \partial_\tau \tilde{u}_0(\tau) = \Delta_x \tilde{u}_0(\tau) - [f_2(\bar{u}_0(0) + \tilde{u}_0(\tau)) - f_2(\bar{u}_0(0))] - \delta^{-1}\tilde{u}_0(\tau), \\ \tilde{u}_0(0) = u(0) - \mathcal{L}(\phi(0)), \quad \tilde{u}_0\big|_{\partial\Omega} = 0. \end{aligned} \tag{2.5}$$

Then, the boundary layer term $\tilde{\phi}_1(\tau)$ is given by

$$\tilde{\phi}_1(\tau) = \delta^{-1} \int_\tau^\infty \tilde{u}_0(s)\, ds. \tag{2.6}$$

We restrict ourselves to the first boundary layer term in the asymptotic expansions (2.1) only and estimate the rest (which is in fact sufficient for our purposes). To be more precise, we seek for a solution of equations (0.1) of the form

$$\phi(t) := \bar{\phi}_0(t) + \varepsilon\tilde{\phi}(t/\varepsilon) + \varepsilon\widehat{\phi}(t), \quad u(t) := \bar{u}_0(t) + \tilde{u}(t/\varepsilon) + \varepsilon\widehat{u}(t), \tag{2.7}$$

where $(\bar{\phi}_0(t), \bar{u}_0(t))$ solves the limit problem (1.44), the boundary layer term $\tilde{u}(\tau)$ solves

$$\begin{aligned} \partial_\tau \tilde{u}(\tau) = \Delta_x \tilde{u}(\tau) - [f_2(\bar{u}_0(\varepsilon\tau) + \tilde{u}(\tau)) - f_2(\bar{u}_0(\varepsilon\tau))] - \delta^{-1}\tilde{u}(\tau), \\ \tilde{u}(0) = u(0) - \mathcal{L}(\phi(0)), \quad \tilde{u}\big|_{\partial\Omega} = 0, \end{aligned} \tag{2.8}$$

and the boundary layer term $\tilde{\phi}(\tau)$ is defined by

$$\tilde{\phi}(\tau) = \delta^{-1} \int_\tau^\infty \tilde{u}(s)\, ds. \tag{2.9}$$

Equation (2.8) on $\tilde{u}(\tau)$ differs slightly from equation (2.5) for the function $\tilde{u}_0(\tau)$ (the term $\bar{u}_0(0)$ is replaced by $\bar{u}_0(t) := \bar{u}_0(\varepsilon\tau)$). We note however that the difference $\tilde{u}(\tau) - \tilde{u}_0(\tau)$ is of order $\varepsilon^1$ and, consequently, can be interpreted as a part of the rest in the asymptotic expansions (2.1).

The next lemma shows that the function $\tilde{u}(\tau)$, solution of equation (2.8), is indeed a boundary layer term.

**Lemma 2.1.** *Let the above assumptions hold. Then, the solution $\tilde{u}(\tau)$ of problem (2.8) satisfies the estimate*

$$\|\tilde{u}(\tau)\|_{H^2} + \|\partial_\tau \tilde{u}(\tau)\|_{L^2} \le Q\Big(\|(\phi(0), u(0))\|_\Phi\Big)\|\tilde{u}(0)\|_{H^2} e^{-\gamma\tau}, \tag{2.10}$$

*where $\gamma > 0$ is a positive constant and $Q$ is a monotonic function that are both independent of $\varepsilon$.*

Proof. We set $\tilde v(\tau) := \tilde u(\tau)^2$. Then, due to the assumption $f_2' \ge 0$, this function satisfies the inequation

$$\partial_\tau \tilde v(\tau) - \Delta_x \tilde v(\tau) - 2\delta^{-1}\tilde v(\tau) \le 0, \quad \tilde v(0) = \tilde u(0)^2,$$

and, consequently, due to the comparison principle, we have

$$\|\tilde u(\tau)\|_{L^\infty} \le C\|\tilde u(0)\|_{L^\infty} e^{-\gamma t}. \tag{2.11}$$

Having estimate (2.11) for the $L^\infty$-norm of $\tilde u(\tau)$ and estimates (1.49) and (1.51) for $\bar u_0(t)$, we deduce (2.10) by applying standard parabolic regularity arguments to equation (2.8) and Lemma 2.1 is proved. □

We are now in a position to estimate the rest $(\widehat\phi(t), \widehat u(t))$ in expansions (2.7).

**Lemma 2.2.** *Let the above assumptions hold. Then, the rest* $(\widehat\phi(t), \widehat u(t))$ *in the asymptotic expansions* (2.7) *enjoys the following estimate:*

$$\|\widehat\phi(t)\|_{H^2} + \|\widehat u(t)\|_{H^2} + \|\partial_t\widehat\phi(t)\|_{L^2} + \varepsilon\|\partial_t \widehat u(t)\|_{L^2} \le Ce^{Lt}, \tag{2.12}$$

*where the constants $C$ and $L$ depend on $\|(\phi(0), u(0))\|_\Phi$, but are independent of $\varepsilon$.*

Proof. The functions $\widehat\phi(t)$ and $\widehat u(t)$ satisfy the equations

$$\begin{gathered}
\delta\partial_t\widehat\phi = \Delta_x\widehat\phi - \frac{1}{\varepsilon}\Big[f_1(\bar\phi_0 + \varepsilon\tilde\phi + \varepsilon\widehat\phi) - f_1(\bar\phi_0)\Big] + \widehat u + \Delta_x\tilde\phi,\\
\varepsilon\partial_t\widehat u = \Delta_x \widehat u - \frac{1}{\varepsilon}\Big[f_2(\bar u_0 + \tilde u + \varepsilon\widehat u) - f_2(\bar u_0 + \tilde u)\Big] - \partial_t\widehat\phi - \partial_t \bar u_0,\\
\widehat\phi\big|_{t=0} = -\tilde\phi(0), \quad \widehat u\big|_{t=0} = 0.
\end{gathered} \tag{2.13}$$

We first note that, according to (2.9) and (2.10), we have

$$\|\tilde\phi(\tau)\|_{H^2} \le Q\Big(\|(\phi(0), u(0))\|_\Phi\Big)\|\tilde u(0)\|_{H^2} e^{-\gamma\tau}, \tag{2.14}$$

where $Q$ is independent of $\epsilon$, and, consequently, the initial data in (2.13) is uniformly bounded in $H^2(\Omega)$ as $\varepsilon \to 0$.

Multiplying the first equation of (2.13) by $\widehat\phi(t)$ and integrating over $\Omega$, we have, noting that $f_1' \ge -K$

$$\delta\partial_t\|\widehat\phi(t)\|^2_{L^2} + \frac{3}{2}\|\nabla_x\widehat\phi(t)\|^2_{L^2} \le 2K\|\widehat\phi(t)\|^2_{L^2} + C\left(\|\widehat u(t)\|^2_{L^2} + \|\nabla_x\tilde\phi(\frac{t}{\varepsilon})\|^2_{L^2}\right). \tag{2.15}$$

We now differentiate the first equation of (2.13) with respect to $t$, multiply the relation that we obtain by $\partial_t\widehat\phi(t)$ and integrate over $\Omega$ to find

$$\begin{aligned}
\delta\partial_t\|\partial_t\widehat\phi(t)\|^2_{L^2} + 2\|\nabla_x\partial_t\widehat\phi(t)\|^2_{L^2} - 2(\partial_t\widehat\phi(t), \partial_t\widehat u(t)) \le 2K\|\partial_t\widehat\phi(t)\|^2_{L^2} -\\
- \frac{2}{\varepsilon}\Big([f_1'(\bar\phi_0 + \varepsilon\tilde\phi + \varepsilon\widehat\phi) - f_1'(\bar\phi_0)]\partial_t\bar\phi_0, \partial_t\widehat\phi\Big) - 2(f_1'(\bar\phi_0 + \varepsilon\tilde\phi + \varepsilon\widehat\phi)\partial_t\tilde\phi, \partial_t\widehat\phi) +\\
+ \|\partial_t\Delta_x\tilde\phi\|_{L^2}\Big(1 + \|\partial_t\widehat\phi(t)\|^2_{L^2}\Big).
\end{aligned} \tag{2.16}$$

Since the functions $\tilde{\phi}$ and $\varepsilon\widehat{\phi}$ are uniformly bounded (with respect to $\varepsilon$) in $H^2(\Omega)$ and $\partial_t\bar{\phi}_0$ is bounded in $L^2(\Omega)$ (see (1.17), (1.51) and (2.14)), it follows that

$$\frac{2}{\varepsilon}\left([f_1'(\bar{\phi}_0) - f_1'(\bar{\phi}_0 + \varepsilon\tilde{\phi} + \varepsilon\widehat{\phi})]\partial_t\bar{\phi}_0, \partial_t\widehat{\phi}\right) \le C\left((1 + |\widehat{\phi}|)|\partial_t\bar{\phi}_0|, |\partial_t\widehat{\phi}|\right) \le \\ \le C\left(1 + \|\partial_t\widehat{\phi}(t)\|^2_{L^2} + \|\widehat{\phi}(t)\|^2_{L^2}\right) + \frac{1}{2}\|\nabla_x\widehat{\phi}(t)\|^2_{L^2} + \|\nabla_x\partial_t\widehat{\phi}(t)\|^2_{L^2}, \quad (2.17)$$

where the constant $C$ depends on $\|(\phi_0, u_0)\|_\Phi$, but is independent of $\varepsilon$. Analogously, we have

$$2|(f_1'(\bar{\phi}_0 + \varepsilon\tilde{\phi} + \varepsilon\widehat{\phi})\partial_t\tilde{\phi}, \partial_t\widehat{\phi})| \le C\|\partial_t\tilde{\phi}\|_{H^2}(1 + \|\partial_t\widehat{\phi}(t)\|^2_{L^2}), \quad (2.18)$$

where $C$ is independent of $\varepsilon$. Inserting estimates (2.17) and (2.18) into estimate (2.16) and summing the relation that we obtain with inequality (2.15), we find

$$\delta\partial_t\left(\|\widehat{\phi}(t)\|^2_{L^2} + \|\partial_t\widehat{\phi}(t)\|^2_{L^2} + 1\right) + \|\nabla_x\partial_t\widehat{\phi}(t)\|^2_{L^2} + \|\nabla_x\widehat{\phi}(t)\|^2_{L^2} - \\ - 2\left(\partial_t\widehat{\phi}(t), \partial_t\widehat{u}(t)\right) \le \\ \le C\left(1 + \|\partial_t\tilde{\phi}\|_{H^2}\right)\left(1 + \|\widehat{\phi}(t)\|^2_{L^2} + \|\partial_t\widehat{\phi}(t)\|^2_{L^2} + \|\widehat{u}(t)\|^2_{L^2}\right), \quad (2.19)$$

where the constant $C$ depends on $\|(\phi_0, u_0)\|_\Phi$, but is independent of $\varepsilon$.

Multiplying now the second equation of (2.13) by $\partial_t\widehat{u}(t)$ and integrating over $\Omega$, we have

$$\partial_t\left(\|\nabla_x u(t)\|^2_{L^2} - 2(\partial_t\bar{u}_0(t), \widehat{u}(t))\right) + 2(\partial_t\widehat{\phi}(t), \partial_t\widehat{u}(t)) + \varepsilon\|\partial_t\widehat{u}(t)\|^2_{L^2} \le \\ \le -\frac{2}{\varepsilon}\left([f_2(\bar{u}_0 + \tilde{u} + \varepsilon\widehat{u}) - f_2(\bar{u}_0 + \tilde{u})], \partial_t\widehat{u}(t)\right) - \\ - \|\partial_t^2\bar{u}_0\|_{H^{-1}}(1 + \|\widehat{u}(t)\|^2_{H^1}). \quad (2.20)$$

In order to transform (2.20), we use the following identity:

$$\frac{1}{\varepsilon}\left([f_2(\bar{u}_0 + \tilde{u} + \varepsilon\widehat{u}) - f_2(\bar{u}_0 + \tilde{u})], \partial_t\widehat{u}(t)\right) = \\ = \partial_t\left[\frac{1}{\varepsilon^2}\left(F_2(\bar{u}_0 + \tilde{u} + \varepsilon\widehat{u}) - F_2(\bar{u}_0 + \tilde{u}) - \varepsilon f_2(\bar{u}_0 + \tilde{u})\widehat{u}, 1\right)\right] - \\ - \left[\frac{1}{\varepsilon^2}\left(f_2(\bar{u}_0 + \tilde{u} + \varepsilon\widehat{u}) - f_2(\bar{u}_0 + \tilde{u}_0) - \varepsilon f_2'(\bar{u}_0 + \tilde{u})\widehat{u}, \partial_t\bar{u}_0 + \partial_t\tilde{u}\right)\right] := \\ := \partial_t\Theta_\varepsilon(t) - \theta_\varepsilon(t), \quad (2.21)$$

where $F_2(v) := \int_0^v f_2(s)\,ds$. We now note that, due to the assumption $f_2'(v) \ge 0$ and due to the condition $\widehat{u}(0) = 0$, we have

$$\Theta_\varepsilon(t) \ge 0, \quad \Theta_\varepsilon(0) = 0. \quad (2.22)$$

Moreover, arguing in a standard way, we can obtain the following estimate for $\theta_\varepsilon(t)$:

$$|\theta_\varepsilon(t)| \le C\left(|\widehat{u}(t)|^2, |\partial_t\tilde{u}| + |\partial_t\bar{u}_0|\right) \le C_1(\|\partial_t\tilde{u}\|_{L^2} + 1)\|\widehat{u}(t)\|^2_{H^1}, \quad (2.23)$$

where the constants $C$ and $C_1$ depend on $\|(\phi_0, u_0)\|_\Phi$, but are independent of $\varepsilon$. Inserting identity (2.21) and inequality (2.23) into relation (2.20) and summing the relation that we obtain with inequality (2.19), we finally find

$$\begin{aligned}\partial_t\Big[\delta\|\widehat{\phi}(t)\|^2_{L^2}+\delta\|\partial_t\widehat{\phi}(t)\|^2_{L^2}+\|\widehat{u}(t)\|^2_{H^1}-2(\partial_t\bar{u}_0(t),\widehat{u}(t))+2\Theta_\varepsilon(t)+C_2\Big]\le\\ \le C_3\Big(1+\|\partial_t\tilde{\phi}(t/\varepsilon)\|_{H^2}+\|\partial_t\tilde{u}(t/\varepsilon)\|_{L^2}+\|\partial_t^2\bar{u}_0(t)\|^2_{H^{-1}}\Big)\times\\ \times\Big[\delta\|\widehat{\phi}(t)\|^2_{L^2}+\delta\|\partial_t\widehat{\phi}(t)\|^2_{L^2}+\|\widehat{u}(t)\|^2_{H^1}-2(\partial_t\bar{u}_0(t),\widehat{u}(t))+2\Theta_\varepsilon(t)+C_2\Big],\end{aligned}$$

where the constants $C_2$ and $C_3$ depend on $\|(\phi_0,u_0)\|_\Phi$, but are independent of $\varepsilon$. Moreover, the constant $C_2$ can be chosen such that the expression in square brackets in the right-hand side of the above inequality is positive (it is possible to do so thanks to estimates (1.51) and (2.22)). Applying Gronwall's inequality to this relation and noting that (1.51) and (2.10) yield the estimate

$$\int_t^{t+1}\Big(\|\partial_t\tilde{\phi}(s/\varepsilon)\|_{H^2}+\|\partial_t\tilde{u}(s/\varepsilon)\|_{L^2}+\|\partial_t^2\bar{u}_0(s)\|^2_{H^{-1}}\Big)\,ds\le C_4, \tag{2.24}$$

where $C_4$ is independent of $\varepsilon$, we find the estimate

$$\|\widehat{\phi}(t)\|^2_{L^2}+\|\partial_t\widehat{\phi}(t)\|^2_{L^2}+\|\widehat{u}(t)\|^2_{H^1}\le C_5e^{L_1t}, \tag{2.25}$$

where the constants $C_5$ and $L_1$ depend on $\|(\phi_0,u_0)\|_\Phi$, but are independent of $\varepsilon$.

Estimate (2.12) can be deduced from estimate (2.25), based on standard parabolic regularity arguments, exactly as in the proof of Lemma 1.3, which finishes the proof of Lemma 2.2. □

Let us now formulate several useful corollaries of estimate (2.12).

**Corollary 2.3.** *Let the above assumptions hold. We also assume that $(\phi(t),u(t))$ is solution of equation* (0.1) *and $(\bar{\phi}_0(t),\bar{u}_0(t))$ is solution of the limit problem* (1.44), *with $\bar{\phi}_0(0)=\phi(0)$. Then, the following estimate is valid:*

$$\begin{aligned}\|\phi(t)-\bar{\phi}_0(t)\|_{H^2}+\|u(t)-\bar{u}_0(t)\|_{H^2}+\|\partial_t\phi(t)-\partial_t\bar{\phi}_0(t)\|_{L^2}+\\ +\varepsilon\|\partial_tu(t)-\partial_t\bar{u}_0(t)\|_{L^2}\le C\Big(\|u(0)-\mathcal{L}(\phi(0))\|_{H^2}e^{-\gamma\frac{t}{\varepsilon}}+\varepsilon e^{Lt}\Big),\end{aligned} \tag{2.26}$$

*where $\gamma>0$ is a positive constant depending only on $\Omega$ and the constants $C$ and $L$ depend on $\|(\phi(0),u(0))\|_\Phi$, but are independent of $\varepsilon$.*

Indeed, estimate (2.26) is an immediate corollary of the asymptotic expansions (2.7) and of estimates (2.10), (2.12) and (2.14).

**Corollary 2.4.** *Let the above assumptions hold and let $(\phi(t),u(t))$ be solution of problem* (0.1). *Then, the following estimates hold:*

$$\|\partial_tu(t)\|_{L^2}\le Q\left(\|(\phi(0),u(0))\|_\Phi\right)\left(1+\frac{1}{\varepsilon}\|u(0)-\mathcal{L}(\phi(0))\|_{H^2}e^{-\gamma\frac{t}{\varepsilon}}\right), \tag{2.27}$$

*and*

$$\|u(t)-\mathcal{L}(\phi(t))\|_{H^2}\le Q\left(\|(\phi(0),u(0))\|_\Phi\right)\Big(\varepsilon+\|u(0)-\mathcal{L}(\phi(0))\|_{H^2}e^{-\gamma\frac{t}{\varepsilon}}\Big), \tag{2.28}$$

*where the constant $\gamma>0$ and the function $Q$ are independent of $\varepsilon$.*

Proof. Without loss of generality, we can derive estimates (2.27) and (2.28) for $t \le 1$ only. Now, estimate (2.27) is an immediate corollary of (2.26) and (1.51). So, there only remains to deduce estimate (2.28). To this end, we recall that, by definition of the operator $\mathcal{L}$, we have $\bar u_0(t) = \mathcal{L}(\bar\phi_0(t))$ and, consequently

$$\|u(t) - \mathcal{L}(\phi(t))\|_{H^2} \le \|u(t) - \bar u_0(t)\|_{H^2} + \|\mathcal{L}(\phi(t)) - \mathcal{L}(\bar\phi_0(t))\|_{H^2}. \tag{2.29}$$

Estimate (2.28) is now a corollary of (2.26), (2.29) and of Proposition 1.6. □

**Remark 2.5.** *Let the function $\tilde U(\tau)$ be solution of the problem*

$$\partial_\tau \tilde U = \Delta_x \tilde U - \delta^{-1}\tilde U, \quad \tilde U\big|_{t=0} = u(0) - \mathcal{L}(\phi(0)), \tag{2.30}$$

*i.e. $\tilde U(\tau) := e^{-(-\Delta_x+\delta^{-1}I)\tau}(u(0) - \mathcal{L}(\phi(0)))$. Then, it is not difficult to verify that the quantity $\tilde u(t/\varepsilon) - \tilde U(t/\varepsilon)$ is of order $\varepsilon^1$ as $\varepsilon \to 0$ and, consequently, the boundary layer term in expansions (2.7) can be simplified as follows:*

$$u(t) = \bar u_0(t) + e^{-(-\Delta_x+\delta^{-1}I)\frac{t}{\varepsilon}}[u(0) - \bar u_0(0)] + O(\varepsilon). \tag{2.31}$$

We are now able to verify the uniform (with respect to $\varepsilon$) Lipschitz continuity of the semigroups $S_t^\varepsilon$ associated with problem (0.1) in the phase space $\Phi$.

**Lemma 2.6.** *Let the assumptions of Lemma 1.1 hold and let $(\phi_1(t), u_1(t))$ and $(\phi_2(t), u_2(t))$ be two solutions of problem (0.1) with initial data in $\Phi$. Then, the following estimate is valid:*

$$\begin{aligned}\|\phi_1(t) - \phi_2(t)\|^2_{H^2} + \|u_1(t) - u_2(t)\|^2_{H^2} + \|\partial_t\phi_1(t) - \partial_t\phi_2(t)\|^2_{L^2} + \\ + \varepsilon^2\|\partial_t u_1(t) - \partial_t u_2(t)\|^2_{L^2} \le \\ \le Ce^{Lt}\left(\|\phi_1(0) - \phi_2(0)\|^2_{H^2} + \|u_1(0) - u_2(0)\|^2_{H^2}\right),\end{aligned} \tag{2.32}$$

*where the constants $C$ and $L$ depend on $\|\phi_i(0)\|_{H^2}$ and on $\|u_i(0)\|_{H^2}$, but are independent of $\varepsilon$.*

Proof. We set $v(t) := \phi_1(t) - \phi_2(t)$ and $w(t) := u_1(t) - u_2(t)$. These functions satisfy equation (1.39). Moreover, due to estimate (2.27) as well as (1.40) and (1.41), we also have the uniform estimate

$$\int_t^{t+1} \|\partial_t l_2(s)\|_{L^2}\, ds \le L. \tag{2.33}$$

Differentiating now the first equation of (1.39) with respect to $t$, multiplying by $\partial_t v(t)$, summing the relation that we obtain with the second equation of (1.39) multiplied by $\partial_t w(t)$ and integrating over $\Omega$, we obtain

$$\begin{aligned}\partial_t[\delta\|\partial_t v(t)\|^2_{L^2} + \|\nabla_x w(t)\|^2_{L^2} + (l_2(t)w(t), w(t))] + 2\|\nabla_x\partial_t v(t)\|^2_{L^2} \le \\ \le -2(l_1(t)\partial_t v(t), \partial_t v(t)) - 2(\partial_t l_1(t)v(t), \partial_t v(t)) + (\partial_t l_2(t)w(t), w(t)),\end{aligned} \tag{2.34}$$

where

$$l_1(t) := \int_0^1 f_1'(s\phi_1(t) + (1-s)\phi_2(t))\, ds, \quad l_2(t) := \int_0^1 f_2'(su_1(t) + (1-s)u_2(t))\, ds.$$

Estimates (2.34), (1.40) and (1.41) imply that

$$\begin{aligned}\partial_t[\delta\|\partial_t v(t)\|^2_{L^2} + \|\nabla_x w(t)\|^2_{L^2} + (l_2(t)w(t), w(t))] + 2\|\nabla_x\partial_t v(t)\|^2_{L^2} \le \\ \le C\left(1 + \|\partial_t l_2(t)\|_{L^2}\right)[\delta\|\partial_t v(t)\|^2_{L^2} + \|\nabla_x w(t)\|^2_{L^2} + (l_2(t)w(t), w(t)) + \\ + 2\|\nabla_x\partial_t v(t)\|^2_{L^2}] + C\|v(t)\|^2_{L^2},\end{aligned} \tag{2.35}$$

where the constant $C$ depends on $\|(\phi_i(0),u_i(0))\|_\Phi$, but is independent of $\varepsilon$. Applying Gronwall's inequality to relation (2.35) and taking into account inequalities (2.33) and (1.38), we find

$$\|\partial_t v(t)\|^2_{L^2}+\|w(t)\|^2_{H^1}\le Ce^{Lt}\left(\|v(0)\|^2_{H^2}+\|w(0)\|^2_{H^1}\right), \tag{2.36}$$

where the constants $C$ and $L$ depend on $\|(\phi_i(0),u_i(0))\|_\Phi$, but are independent of $\varepsilon$.

Estimate (2.32) is a corollary of (2.36) and of standard parabolic regularity arguments (see the proof of Lemma 1.3). This finishes the proof of Lemma 2.6. □

To conclude this section, we finally derive standard smoothing estimates for the difference of solutions of (0.1) which are necessary for our construction of exponential attractors.

**Lemma 2.7.** *Let the assumptions of Lemma 1.1 hold and let $(\phi_1(t),u_1(t))$ and $(\phi_2(t),u_2(t))$ be two solutions of problem* (0.1) *with initial data in $\Phi$. Then, the following estimate is valid:*

$$\begin{aligned}\|\phi_1(t)-\phi_2(t)\|^2_{H^3}&+\|u_1(t)-u_2(t)\|^2_{H^3}\le\\ &\le Ce^{Lt}\frac{t+1}{t}\left(\|\phi_1(0)-\phi_2(0)\|^2_{H^2}+\|u_1(0)-u_2(0)\|^2_{H^2}\right),\quad t>0,\end{aligned} \tag{2.37}$$

*where the constants $C$ and $L$ depend on $\|\phi_i(0)\|_{H^2}$ and on $\|u_i(0)\|_{H^2}$, but are independent of $\varepsilon$.*

Proof. We split the solution $(v(t),w(t))$ of problem (1.39) into a sum of two functions

$$v(t):=v_1(t)+v_2(t),\quad w(t):=w_1(t)+w_2(t), \tag{2.38}$$

where the function $(v_1(t),w_1(t))$ solves

$$\begin{gathered}\delta\partial_t v_1-\Delta_x v_1=H_1(t):=w(t)-l_1(t)v(t),\quad v_1\big|_{t=0}=0,\\ \varepsilon\partial_t w_1+\partial_t v_1-\Delta_x w_1=H_2(t):=-l_2(t)w(t),\quad w_1\big|_{t=0}=0,\end{gathered} \tag{2.39}$$

with

$$l_1(t):=\int_0^1 f_1'(s\phi_1(t)+(1-s)\phi_2(t))\,ds,\quad l_2(t):=\int_0^1 f_2'(su_1(t)+(1-s)u_2(t))\,ds,$$

and the function $(v_2(t),w_2(t))$ solves

$$\begin{gathered}\delta\partial_t v_2-\Delta_x v_2=0,\quad v_2\big|_{t=0}=v(0),\\ \varepsilon\partial_t w_2+\partial_t v_2-\Delta_x w_2=0,\quad w_2\big|_{t=0}=w(0).\end{gathered} \tag{2.40}$$

It follows from estimates (2.32) and from the assumption $f_i\in C^3$, $i=1,2$, that

$$\|H_1(t)\|_{H^2}+\|H_2(t)\|_{H^2}\le Ce^{Lt}\left(\|v(0)\|_{H^2}+\|w(0)\|_{H^2}\right), \tag{2.41}$$

and, consequently, due to standard parabolic regularity arguments (see the proof of Lemma 1.3), we have

$$\|v_1(t)\|_{H^3}+\|w_1(t)\|_{H^3}\le Ce^{Lt}\left(\|v(0)\|_{H^2}+\|w(0)\|_{H^2}\right), \tag{2.42}$$

where the constants $C$ and $L$ depend on $\|(\phi_i(0),u_i(0))\|_\Phi$, but are independent of $\varepsilon$.

The solution $(v_2(t),w_2(t))$ of problem (2.40) can be easily found by using the analytic semigroups theory (see [16] and [23]). More precisely, we have

$$v_2(t)=e^{-A\frac{t}{\delta}}v(0),\quad w_2(t)=e^{-A\frac{t}{\varepsilon}}w(0)+\frac{\delta}{\delta-\varepsilon}\left(e^{-A\frac{t}{\delta}}-e^{-A\frac{t}{\varepsilon}}v(0)\right), \tag{2.43}$$

where $A := -\Delta_x$, associated with Dirichlet boundary conditions. A standard smoothing estimate for analytic semigroups (see e.g. [16]), applied to (2.43), implies that

$$\|v_2(t)\|^2_{H^3} + \|w_2(t)\|^2_{H^3} \le Ct^{-1}e^{-\gamma t}\left(\|v(0)\|^2_{H^2} + \|w(0)\|^2_{H^2}\right), \quad t > 0, \tag{2.44}$$

where $C$ and $\gamma > 0$ are independent of $\varepsilon$. Combining estimates (2.42) and (2.44), we derive (2.37) and Lemma 2.7 is proved. □

**Remark 2.8.** *We recall that estimates* (2.32) *and* (2.37) *hold uniformly with respect to $\varepsilon > 0$. Consequently, passing to the limit $\varepsilon \to 0$ in these estimates, we see that the same estimates remain valid for the difference of solutions of the limit problem* (1.39).

## 3. Robust exponential attractors

In this section, we construct a uniform family of exponential attractors $\mathcal{M}_\varepsilon$ in $\Phi$ for problem (0.1) which converges as $\varepsilon \to 0$ to the limit exponential attractor $\mathcal{M}_0$ of problem (1.44). To be more precise, the main result of this section is the following theorem.

**Theorem 3.1.** *Let assumptions (0.2) hold. Then, there exists a family of compact sets $\mathcal{M}_\varepsilon \subset \Phi$, $\varepsilon \in [0,1]$, such that*

*1. These sets are semi-invariant with respect to the flows $S^\varepsilon_t$ associated with problem* (0.1), *i.e.*

$$S^\varepsilon_t \mathcal{M}_\varepsilon \subset \mathcal{M}_\varepsilon. \tag{3.1}$$

*2. The fractal dimension of the sets $\mathcal{M}_\varepsilon$ is finite and uniformly bounded with respect to $\varepsilon$:*

$$\dim_F(\mathcal{M}_\varepsilon, \Phi) \le C < \infty, \tag{3.2}$$

*where $C$ is independent of $\varepsilon$.*

*3. These sets attract exponentially the bounded subsets of $\Phi$, i.e. there exists a positive constant $\alpha > 0$ and a monotonic function $Q$ which are independent of $\varepsilon$ such that, for every bounded subset $B$ in the phase space $\Phi$, we have*

$$\operatorname{dist}_\Phi(S^\varepsilon_t B, \mathcal{M}_\varepsilon) \le Q(\|B\|_\Phi)e^{-\alpha t}, \quad \varepsilon \in [0,1], \tag{3.3}$$

*where* $\operatorname{dist}_\Phi$ *denotes the nonsymmetric Hausdorff distance between sets in $\Phi$ (for $\varepsilon = 0$, we should take $B \subset \mathbb{L}$).*

*4. The symmetric Hausdorff distance between the limit attractor $\mathcal{M}_0$ and the attractors $\mathcal{M}_\varepsilon$ enjoys the following estimate:*

$$\operatorname{dist}_{sym,\Phi}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\kappa, \tag{3.4}$$

*where the constants $C > 0$ and $0 < \kappa < 1$ are independent of $\varepsilon$ and can be computed explicitly.*

The proof of this result is based on the following abstract result for exponential attractors of singularly perturbed discrete maps.

**Proposition 3.2.** *Let $B_\varepsilon \subset \Phi$, $\varepsilon \in [0,1]$, be a family of closed and bounded subsets of a Banach space $\Phi$ and let $S^\varepsilon : B_\varepsilon \to B_\varepsilon$ be a family of maps which satisfies the following properties:*

*1. There exists another Banach space $\Phi_1$, which is compactly embedded into $\Phi$, such that, for every $b^1_\varepsilon, b^2_\varepsilon \in B_\varepsilon$, the following estimate holds:*

$$\|S^\varepsilon b^1_\varepsilon - S^\varepsilon b^2_\varepsilon\|_{\Phi_1} \le K\|b^1_\varepsilon - b^2_\varepsilon\|_\Phi, \tag{3.5}$$

*where the constant $K$ is independent of $\varepsilon$.*

*2. There exist nonlinear 'projectors' $\Pi_\varepsilon : B_\varepsilon \to B_0$ such that, for every $b_\varepsilon \in B_\varepsilon$*

$$\|S^\varepsilon_{(k)} b_\varepsilon - S^0_{(k)} \Pi_\varepsilon b_\varepsilon\|_\Phi \le \varepsilon L^k, \quad k \in \mathbb{N}, \tag{3.6}$$

*where $S_{(k)}$ denotes the $k$th iteration of $S$ and the constant $L$ is independent of $\varepsilon$.*

*Then, the maps $S^\varepsilon$ possess a family of exponential attractors $\mathcal{M}^d_\varepsilon$ which satisfies (3.1), (3.2), (3.4) uniformly with respect to $\varepsilon$ and such that*

$$\operatorname{dist}_\Phi(S^\varepsilon_{(k)} B_\varepsilon, \mathcal{M}^d_\varepsilon) \le C e^{-\gamma k}, \tag{3.7}$$

*where $C$ and $\gamma > 0$ are also independent of $\varepsilon$ and can be computed explicitly.*

The proof of this proposition is given in [15] in a more general setting.

Proof of Theorem 3.1. We apply the abstract result of Proposition 3.2 to our situation. To this end, we define the sets $B_\varepsilon \subset \Phi$ for $\varepsilon \ne 0$ by

$$B_\varepsilon = B := \{(\phi_0, u_0) \in \Phi,\ \|(\phi_0, u_0)\|^2_\Phi \le 2Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2})\}, \tag{3.8}$$

where the function $Q$ is defined in (1.17), and, for $\varepsilon = 0$, we set

$$B_0 = \{(\phi_0, u_0) \in \Phi,\ \|\phi_0\|^2_{H^2} \le 2Q(\|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}),\ u_0 = \mathcal{L}(\phi_0)\}. \tag{3.9}$$

Then, it follows from the uniform estimate (1.17) that there exists a time $T > 0$ which is independent of $\varepsilon$ such that

$$S^\varepsilon_T B_\varepsilon \subset B_\varepsilon, \quad \varepsilon \in [0, 1]. \tag{3.10}$$

We now set

$$S^\varepsilon := S^\varepsilon_T, \quad \varepsilon \in [0, 1], \tag{3.11}$$

and verify that the operators (3.11) satisfy all the assumptions of Proposition 3.2. Indeed, according to (3.10), the maps $S^\varepsilon$ are well defined on $B_\varepsilon$. Estimate (3.5), with $\Phi_1 := H^3(\Omega) \times H^3(\Omega)$, is an immediate corollary of Lemma 2.7. So, there remains to verify (3.5). To this end, we define the nonlinear projector $\Pi_\varepsilon$ by

$$\Pi_\varepsilon : B_\varepsilon \to B_0, \quad \Pi_\varepsilon(\phi_0, u_0) := (\phi_0, \mathcal{L}(\phi_0)). \tag{3.12}$$

Then, estimate (3.6) is an immediate corollary of (2.26) (in which the boundary layer term disappears since $t = T > 0$ and $T$ is independent of $\varepsilon$). Thus, all the assumptions of Proposition 3.2 are satisfied for the family of maps (3.11) and, consequently, these maps possess a family of discrete exponential attractors $\mathcal{M}^d_\varepsilon$ which satisfies (3.1), (3.2), (3.4) and (3.7).

We now define the desired family $\mathcal{M}_\varepsilon$ of exponential attractors by the standard expression:

$$\mathcal{M}_\varepsilon := \cup_{t \in [1, T+1]} S^\varepsilon_t \mathcal{M}^d_\varepsilon. \tag{3.13}$$

The semi-invariance (3.1) is then an immediate corollary of the semi-invariance of $\mathcal{M}^d_\varepsilon$ and of definition (3.13). The exponential attraction (3.3) follows from the fact that the $B_\varepsilon$ are uniform (with respect to $\varepsilon$) absorbing sets for $S^\varepsilon_t$ (due to (1.17)) and from the uniform Lipschitz continuity (2.32). Estimate (3.4) for the symmetric distance is also a corollary of an analogous result for the discrete exponential attractors and of estimates (2.26) and (2.32). We note that the boundary layer term in (2.26) also disappears, due to estimate (2.28), since

$$\mathcal{M}_\varepsilon \subset S^\varepsilon_1 B_\varepsilon. \tag{3.14}$$

Thus, there only remains to verify estimate (3.3) for the fractal dimension. To this end, we need the following lemma.

**Lemma 3.3.** *Let assumptions (0.2) hold. Then, the solution $(\phi(t), u(t))$ of equation (0.1) is Hölder continuous with respect to $t$, with Hölder exponent $1/3$ if $t \geq 1$, i.e., for every $t \geq 1$ and $0 \leq s \leq 1$, we have*

$$\|\phi(t+s) - \phi(t)\|_{H^2} + \|u(t+s) - u(t)\|_{H^2} \leq Q(\|(\phi_0, u_0)\|_\Phi) s^{1/3}, \tag{3.15}$$

*for an appropriate monotonic function $Q$ which is independent of $\varepsilon$.*

Proof. According to estimates (2.27) and (1.17) and since $t \geq 1$, we have

$$\|\phi(t+s) - \phi(t)\|_{L^2} + \|u(t+s) - u(t)\|_{L^2} \leq Q(\|(\phi_0, u_0)\|_\Phi) s^1, \tag{3.16}$$

for an appropriate monotonic function $Q$ which is independent of $\varepsilon$. In order to derive (3.15) from (3.16), we note that, due to estimate (2.37) and due to a standard interpolation inequality

$$\|\phi(t+s)-\phi(t)\|_{H^2} + \|u(t+s)-u(t)\|_{H^2} \leq \|\phi(t+s)-\phi(t)\|_{L^2}^{1/3} \|\phi(t+s)-\phi(t)\|_{H^3}^{2/3} + \\ + \|u(t+s) - u(t)\|_{L^2}^{1/3} \|u(t+s) - u(t)\|_{H^3}^{2/3} \leq Q_1(\|(\phi_0, u_0)\|_\Phi) s^{1/3},$$

which completes the proof of Lemma 3.3. □

The Lipschitz continuity (2.32) of $S_t^\varepsilon$ with respect to the initial data $(\phi_0, u_0)$, together with the Hölder continuity (3.15), imply that

$$\dim_F(\mathcal{M}_\varepsilon, \Phi) \leq \dim_F(\mathcal{M}_\varepsilon^d, \Phi) + 3, \tag{3.17}$$

and Theorem 3.1 is proved. □

## 4. The case of Neumann boundary conditions

In this concluding section, we briefly consider system (0.1) of phase-field equations associated with the Neumann boundary conditions

$$\partial_n \phi\big|_{\partial\Omega} = \partial_n u\big|_{\partial\Omega} = 0. \tag{4.1}$$

We first note that, if the nonlinear function $f_2$ is strictly monotone, i.e.

$$f_2'(v) \geq \alpha, \quad \forall v \in \mathbb{R}, \tag{4.2}$$

for some strictly positive constant $\alpha$ (which improves the nonstrict monotonicity assumption (0.2) 3.), then, repeating word by word the proofs given above for the case of Dirichlet boundary conditions, we easily extend all the results of Sections 1-3 to the case of Neumann boundary conditions. That is the reason why we consider only the case where assumption (4.2) is violated below. More precisely, we assume that $f_2 \equiv 0$ and $g_2 \equiv 0$. Then, system (0.1) reads

$$\begin{gathered} \delta\partial_t\phi = \Delta_x\phi - f(\phi) + u + g, \quad \partial_n\phi\big|_{\partial\Omega} = 0, \\ \varepsilon\partial_t u + \partial_t\phi = \Delta_x u, \quad \partial_n u\big|_{\partial\Omega} = 0, \\ \phi\big|_{t=0} = \phi_0, \quad u\big|_{t=0} = u_0, \end{gathered} \tag{4.3}$$

which corresponds to the standard phase-field system. We assume that $g \in L^2(\Omega)$ and that the nonlinear function $f \in C^3(\mathbb{R}, \mathbb{R})$ satisfies the assumptions

$$1.\ f(v).v \geq \mu|v|^2 - \mu', \quad 2.\ f'(v) \geq -K, \quad \text{for every } v \in \mathbb{R}, \tag{4.4}$$

with $\mu > 0$ and $\mu'$, $K \geq 0$.

The main difference between system (4.3) and system (0.1) with Dirichlet boundary conditions is the existence of a conservation law. Indeed, integrating the second equation of (4.3) over $\Omega$, we have

$$\varepsilon\partial_t\langle u(t)\rangle + \partial_t\langle\phi(t)\rangle = 0, \quad \text{where} \quad \langle v\rangle := \frac{1}{|\Omega|}\int_\Omega v(x)\,dx. \tag{4.5}$$

Integrating then (4.5) with respect to $t$, we obtain the conservation law mentioned above:

$$\varepsilon\langle u(t)\rangle + \langle\phi(t)\rangle = \varepsilon\langle u(0)\rangle + \langle\phi(0)\rangle := I_0(u_0,\phi_0), \quad t\in\mathbb{R}_+. \tag{4.6}$$

Therefore, we cannot expect the existence of the global dissipative estimate (1.17) for the solutions of (4.3) in the phase space $\Phi$. Nevertheless, we will show in this section that all the results obtained above for Dirichlet boundary conditions remain valid (after minor changes) for Neumann boundary conditions. To this end, we need to modify the phase space $\Phi$ for problem (4.3) by fixing explicitly the bounds for the possible values of the conserved integral $I_0$, namely, for every $M>0$, we define the phase space $\Phi_M$ for problem (4.3) as follows:

$$\Phi_M := \{(\phi_0,u_0)\in H^2(\Omega)\times H^2(\Omega), \\ \partial_n\phi_0\big|_{\partial\Omega} = \partial_n u_0\big|_{\partial\Omega} = 0, \quad |I_0(u_0,\phi_0)|\le M\}. \tag{4.7}$$

The following theorem gives a dissipative estimate for the solutions of (4.3) in the phase space $\Phi_M$, similar to that given in Lemma 1.3.

**Theorem 4.1.** *Let assumption* (4.4) *hold. Then, for every* $M>0$ *and every* $(\phi_0,u_0)\in\Phi_M$*, problem* (4.3) *possesses a unique solution* $(\phi(t),u(t))$ *which satisfies the following estimate:*

$$\|\phi(t)\|^2_{H^2} + \|u(t)\|^2_{L^2} + \varepsilon^2\|\partial_t u(t)\|^2_{L^2} \le \\ \le Q_M(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q_M(\|g\|_{L^2}), \tag{4.8}$$

*where* $\alpha>0$ *and the monotonic function* $Q_M$ *depend on* $M$*, but are independent of* $\varepsilon$*.*

Proof. Let us first derive the analogue of estimate (1.1) for equation (4.3). Taking the scalar product of the first equation of (4.3) by $\partial_t\phi(t)+\beta\phi(t)$, of the second equation by $u(t)$ and summing the relations that we obtain, we have (analogously to (1.3))

$$\partial_t E(t) + \gamma E(t) = h(t), \tag{4.9}$$

where $\beta$ and $\gamma$ are small positive numbers such that $\beta>\gamma$,

$$E(t) := \delta\|\nabla_x\phi(t)\|^2_{L^2} + 2(F(\phi(t)),1) + \varepsilon\|u(t)\|^2_{L^2} - \\ - 2(g,\phi(t)) + \beta\delta\|\phi(t)\|^2_{L^2}, \tag{4.10}$$

$F(v)=\int_0^v f(s)\,ds$ and the function $h(t)$ is defined by

$$h(t) := (\gamma\delta-2\beta)\|\nabla_x\phi(t)\|^2_{L^2} + 2\gamma(F(\phi(t)) - f(\phi(t))\phi(t),1) - \\ - 2(\beta-\gamma)(f(\phi(t)),\phi(t)) - 2\delta\|\partial_t\phi(t)\|^2_{L^2} - 2\|\nabla_x u(t)\|^2_{L^2} + \gamma\varepsilon\|u(t)\|^2_{L^2} + \\ + 2(\beta-\gamma)(g,\phi(t)) + \beta\gamma\delta\|\phi(t)\|^2_{L^2} + 2\beta(u(t),\phi(t)). \tag{4.11}$$

We now transform the last term in the right-hand side of (4.11) as follows, using the conservation law (4.6):

$$\begin{aligned}2\beta(u,\phi) = 2\beta(u-|\Omega|\langle u\rangle,\phi) + 2\beta|\Omega|^2\langle\phi\rangle\langle u\rangle = \\ = 2\beta(u-|\Omega|\langle u\rangle,\phi) - 2\beta|\Omega|^2\varepsilon\langle u\rangle^2 + 2\beta I_0(u_0,\phi_0)|\Omega|^2\langle u\rangle. \quad (4.12)\end{aligned}$$

Inserting this relation into the right-side of (4.11) and using inequalities (1.6), (4.4) and the following analogue of Friedrichs' inequality:

$$\|u-|\Omega|\langle u\rangle\|_{L^2}^2 \le C_\Omega\|\nabla_x u\|_{L^2}^2, \quad (4.13)$$

for sufficiently small (but independent of $\varepsilon$) constants $\gamma$ and $\beta$, we obtain the estimate

$$\begin{aligned}h(t) \le -\frac{1}{2}(2\beta-\gamma\delta)\|\phi(t)\|_{H^1}^2 - (\beta-\gamma)(f(\phi(t)),\phi(t)) - \delta\|\partial_t\phi(t)\|_{L^2}^2 - \\ - \|\nabla_x u(t)\|_{L^2}^2 - 2(\beta-\gamma)\varepsilon|\Omega|^2\langle u(t)\rangle^2 + C\left(1+\|g\|_{L^2}^2\right) + 2\beta|\Omega|^2\langle u(t)\rangle, \quad (4.14)\end{aligned}$$

for some constant $C$ that is independent of $\varepsilon$ (in contrast to the case of Dirichlet boundary conditions, we now need assumption (4.4) 1. *with strictly positive* constant $\mu$ because, in the case of Neumann boundary conditions, the term $\|\nabla_x\phi(t)\|_{L^2}^2$ does not bound the $L^2$-norm of $\phi$ and we obtain the estimate for this norm from the third term in the right-hand side of (4.11)).

So, there remains to estimate the last term in the right-hand side of (4.14). To this end, we integrate the first equation of (4.3) over $\Omega$ and express $\langle\phi(t)\rangle$ through $\langle u(t)\rangle$ by using the conservation law to obtain

$$\varepsilon\partial_t\langle u(t)\rangle + \langle u(t)\rangle = \langle f(\phi(t))\rangle - \langle g\rangle. \quad (4.15)$$

We also note that, due to (4.4) 1. and the continuity of the function $f$, we have

$$|\langle f(\phi(t))\rangle| \le \langle|f(\phi(t))|\rangle \le \nu(f(\phi(t)),\phi(t)) + C_\nu, \quad (4.16)$$

where the positive constant $\nu$ can be arbitrarily small. We now multiply (4.15) by $\kappa := \frac{2\beta|\Omega|^2 I_0(u_0,\phi_0)}{1-\gamma\varepsilon}$ and sum the relation that we obtain with (4.9). Then, according to (4.11)-(4.12) and (4.14)-(4.16), we have

$$\begin{aligned}\partial_t[\kappa\varepsilon\langle u(t)\rangle + E(t)] + \gamma[\kappa\varepsilon\langle u(t)\rangle + E(t)] + \\ + \gamma'\left(\|\phi(t)\|_{H^1}^2 + \|\nabla_x u(t)\|_{L^2}^2 + \|\partial_t\phi(t)\|_{L^2}^2\right) \le C(1+\|g\|_{L^2}^2), \quad (4.17)\end{aligned}$$

where all the constants are positive and are independent of $\varepsilon$. We also recall that, due to (4.10)

$$\begin{aligned}C_M^{-1}\left(\varepsilon\|u(t)\|_{L^2}^2 + (F(\phi(t)),1) + \delta\|\phi(t)\|_{H^1}^2 - 1 - \|g\|_{L^2}^2\right) \le \\ \le \kappa\varepsilon\langle u(t)\rangle + E(t) \le \\ \le C_M\left(\varepsilon\|u(t)\|_{L^2}^2 + (F(\phi(t)),1) + \delta\|\phi(t)\|_{H^1}^2 + 1 + \|g\|_{L^2}^2\right), \quad (4.18)\end{aligned}$$

for every $(\phi(t),u(t)) \in \Phi_M$. Here, the constant $C_M$ depends on $M$, but is independent of $\varepsilon$. Applying Gronwall's inequality to (4.17) and using (4.18), we have the following estimate (which is similar to that obtained in Lemma 1.1):

$$\varepsilon\|u(t)\|^2_{L^2} + (F(\phi(t)),1) + \delta\|\phi(t)\|^2_{H^1} + \\ + \int_t^{t+1} (\|\partial_t\phi(s)\|^2_{L^2} + \|\nabla_x u(s)\|^2_{L^2})\,ds \le \\ \le C_M\left(\varepsilon\|u(t)\|^2_{L^2} + (F(\phi(t)),1) + \delta\|\phi(t)\|^2_{H^1}\right)e^{-\alpha t} + C_M\left(1+\|g\|^2_{L^2}\right), \quad (4.19)$$

where the constant $C_M$ depends on $M$, but is independent of $\varepsilon$. Our aim is now to derive the analogue of estimate (1.7). Arguing as in the proof of Lemma 1.2, we have

$$\partial_t[\delta\|\partial_t\phi(t)\|^2_{L^2} + \|\nabla_x u(t)\|^2_{L^2}] + [\delta\|\partial_t\phi(t)\|^2_{L^2} + \|\nabla_x u(t)\|^2_{L^2}] + \\ + 2\|\partial_t\nabla_x\phi(t)\|^2_{L^2} + 2\varepsilon\|\partial_t u(t)\|^2_{L^2} \le [\delta\|\partial_t\phi(t)\|^2_{L^2} + \|\nabla_x u(t)\|^2_{L^2}] + \\ + 2(g,\partial_t\phi(t)) - 2(f'(\phi(t))\partial_t\phi(t),\partial_t\phi(t)).$$

Applying Gronwall's inequality to this relation and using (4.19), we obtain the estimate

$$\delta\|\partial_t\phi(t)\|^2_{L^2} + \|\nabla_x u(t)\|^2_{L^2} + \int_t^{t+1}(\|\partial_t\phi(s)\|^2_{H^1} + \varepsilon\|\partial_t u(s)\|^2_{L^2})\,ds \le \\ \le Q(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q(\|g\|_{L^2}), \quad (4.20)$$

where the function $Q$ depends on $M$, but is independent of $\varepsilon$. Multiplying then the first equation of (4.3) by $\Delta_x\phi(t)$, integrating by parts in $(u(t),\Delta_x\phi(t))$ and using estimate (4.20), we have, analogously to (1.14)-(1.16)

$$\|\phi(t)\|^2_{H^2} \le Q_1(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q_1(\|g\|_{L^2}), \quad (4.21)$$

for some function $Q_1$ which is independent of $\varepsilon$. Since $H^2 \subset C$, (4.21) implies the estimate

$$\|f(\phi(t))\|^2_{L^2} \le Q_2(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q_2(\|g\|_{L^2}), \quad (4.22)$$

for an appropriate function $Q_2$ which depends on $M$, but is independent of $\varepsilon$. Returning now to equation (4.15) and using (4.22), we find

$$\langle u(t)\rangle \le Q_3(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q_3(\|g\|_{L^2}) \quad (4.23)$$

(see (1.19)-(1.22)). Finally, estimates (4.20), (4.21) and (4.23) imply the analogue of estimate (1.7) for the case of Neumann boundary conditions:

$$\delta\|\partial_t\phi(t)\|^2_{L^2} + \|u(t)\|^2_{H^1} + \|\phi(t)\|^2_{H^2} + \\ + \int_t^{t+1}(\|\partial_t\phi(s)\|^2_{H^1} + \varepsilon\|\partial_t u(s)\|^2_{L^2})\,ds \le \\ \le Q(\|\phi(0)\|^2_{H^2} + \|u(0)\|^2_{H^2})e^{-\alpha t} + Q(\|g\|_{L^2}), \quad (4.24)$$

where the monotonic function $Q$ depends on $M$, but is independent of $\varepsilon$. Estimate (4.8) follows from (4.24), exactly as in Lemma 1.3, and Theorem 4.1 is proved. □

We now formulate the analogues of Lemma 2.6 and Lemma 2.7 for the difference of two solutions of (4.3) (since $f_2 \equiv 0$, we do not need estimate (2.27) in order to prove this result).

**Theorem 4.2.** *Let the above assumptions hold and let* $(\phi_1, u_1)$ *and* $(\phi_2, u_2)$ *be two solutions of* (4.3) *belonging to* $\Phi_M$*. Then, the following estimate is valid:*

$$\|\phi_1(t)-\phi_2(t)\|^2_{H^2}+\|u_1(t)-u_2(t)\|^2_{H^2}+ \\ +\varepsilon^2\|\partial_t u_1(t)-\partial_t u_2(t)\|^2_{L^2}\le \\ \le Ce^{Lt}\left(\|\phi_1(0)-\phi_2(0)\|^2_{H^2}+\|u_1(0)-u_2(0)\|^2_{H^2}\right), \quad (4.25)$$

*where the constants* $C$ *and* $L$ *depend on* $M$, $\|\phi_i(0)\|_{H^2}$ *and* $\|u_i(0)\|_{H^2}$*, but are independent of* $\varepsilon$*. Moreover, the following smoothing estimate holds:*

$$\|\phi_1(t)-\phi_2(t)\|^2_{H^3}+\|u_1(t)-u_2(t)\|^2_{H^3}\le \\ \le Ce^{Lt}\frac{t+1}{t}\left(\|\phi_1(0)-\phi_2(0)\|^2_{H^2}+\|u_1(0)-u_2(0)\|^2_{H^2}\right), \quad t>0. \quad (4.26)$$

Proof. We set $v(t):=\phi_1(t)-\phi_2(t)$ and $w(t):=u_1(t)-u_2(t)$. These functions satisfy

$$\partial_t v=\Delta_x v+w+G(t),\quad \varepsilon\partial_t w+\partial_t v=\Delta_x u,\quad \partial_n v\big|_{\partial\Omega}=\partial_n w\big|_{\partial\Omega}=0, \quad (4.27)$$

where $G(t):=\int_0^1 f'(s\phi_1(t)+(1-s)\phi_2(t))\,ds\cdot v(t)$. We note that system (4.27) also possesses a conservation law:

$$I_0(v(t),w(t)):=I_0(\phi_1(t),u_1(t))-I_0(\phi_2(t),u_2(t))\equiv const. \quad (4.28)$$

Moreover, obviously

$$|I_0(v(t),w(t))|^2\le C\left(\|v(0)\|^2_{H^2}+\|w(0)\|^2_{H^2}\right) \quad (4.29)$$

and, due to estimate (4.8) and the embedding $H^2\subset C$

$$\|G(t)\|^2_{L^2}+\|\partial_t G(t)\|^2_{L^2}\le C\left(\|\partial_t v(t)\|^2_{L^2}+\|v(t)\|^2_{L^2}\right), \quad (4.30)$$

where $C$ depends on $M$ and on the $H^2$-norm of the initial data, but is independent of $\varepsilon$. Interpreting now the function $G(t)$ in (4.27) as a nonautonomous external force, repeating word by word the proof of Theorem 4.1 and using estimates (4.29) and (4.30), we find estimate (4.25). Having estimate (4.25), we can prove the smoothing property (4.26) exactly as in Lemma 2.7 and Theorem 4.2 is proved. □

As in Section 1, we now study the limit problem (4.3) with $\varepsilon=0$:

$$\begin{gathered}\delta\partial_t\bar\phi_0=\Delta_x\bar\phi_0-f(\bar\phi_0)+\bar u_0+g,\quad \partial_t\bar\phi_0=\Delta_x\bar u_0,\\ \bar\phi_0\big|_{t=0}=\phi_0,\quad \partial_n\bar\phi_0\big|_{\partial\Omega}=\partial_n\bar u_0\big|_{\partial\Omega}=0.\end{gathered} \quad (4.31)$$

Again, the variables $(\bar\phi_0,\bar u_0)$ are not independent, but satisfy the relation

$$\delta\Delta_x\bar u_0(t)-\bar u_0(t)=\Delta_x\bar\phi_0(t)-f(\bar\phi_0)+g,\quad t\in\mathbb{R}_+ \quad (4.32)$$

(compare with (1.45)) and, consequently, there exists a nonlinear operator

$$\mathcal{L}\in C^1(H^2(\Omega),\{v\in H^2(\Omega),\ \partial_n v\big|_{\partial\Omega}=0\}), \quad (4.33)$$

such that

$$\bar u_0(t)=\mathcal{L}(\bar\phi_0(t)),\quad t\in\mathbb{R}_+, \quad (4.34)$$

for every solution $(\bar\phi_0(t),\bar u_0(t))$ of problem (4.31). Thus, problem (4.31) defines a semigroup in the infinite dimensional submanifold of $\Phi$ defined by

$$\mathbb{L}_M:=\{(\phi_0,u_0)\in H^2(\Omega),\quad u_0=\mathcal{L}(\phi_0),\quad \partial_n\phi_0\big|_{\partial\Omega}=0,\quad |\langle\phi_0\rangle|\le M\}. \quad (4.35)$$

The next theorem gives the analogue of Lemmas 1.5 and 1.6 for equation (4.31).

**Theorem 4.3.** *Let the above assumptions hold. Then, for every* $(\phi_0, u_0) \in \mathbb{L}_M$*, problem* (1.44) *has a unique solution* $(\bar\phi_0(t), \bar u_0(t)) \in \mathbb{L}_M$*,* $t \ge 0$*, which satisfies the estimate*

$$\|\bar\phi_0(t)\|^2_{H^2} + \|\partial_t\bar\phi_0(t)\|^2_{L^2} + \|\bar u_0(t)\|^2_{H^2} + \int_t^{t+1} \|\partial_t\bar\phi_0(s)\|^2_{H^1}\,ds \le \\ \le Q(\|\bar\phi_0(0)\|^2_{H^2})e^{-\gamma t} + Q(\|g\|_{L^2}), \quad (4.36)$$

*for a positive constant* $\gamma$ *and a monotonic function* $Q$ *which depend on* $M$*. Moreover, estimates* (4.25) *and* (4.26) *remain valid for the difference of solutions of the limit problem* (4.31) *and the following analogue of estimate* (1.51) *holds:*

$$\|\partial_t\bar u_0(t)\|^2_{L^2} + \int_t^{t+1} \left(\|\partial_t\bar u_0(s)\|^2_{H^1} + \|\partial_t^2\bar u_0(s)\|^2_{H^{-1}}\right)\,ds \le \\ \le Q(\|\bar\phi_0(0)\|^2_{H^2})e^{-\gamma t} + Q(\|g\|_{L^2}), \quad (4.37)$$

*where* $H^{-1}(\Omega)$ *denotes here the dual of* $H^1(\Omega)$*.*

Proof. Since the constant $\alpha$ and the monotonic function $Q_M$ in (4.8) are independent of $\varepsilon$, then, passing to the limit $\varepsilon \to 0$, we have estimate (4.36). The estimates for the difference of solutions can be obtained similarly. Finally, estimate (4.37) can be verified exactly as in Lemma 1.8 and Theorem 4.3 is proved. □

We now extend the asymptotic expansions for $(\phi(t), u(t))$ as $\varepsilon \to 0$ (obtained in Section 2 for the case of Dirichlet boundary conditions) to the case of Neumann boundary conditions. We note that the formulae for the first boundary layer term are simpler now, since $f_2 \equiv 0$. Indeed, analogously to (2.8) and (2.9), we obtain the following system for $\tilde\phi(\tau)$ and $\tilde u(\tau)$, $\tau := \frac{t}{\varepsilon}$:

$$\begin{gathered}\delta\partial_\tau\tilde\phi(\tau) = u(\tau), \quad \partial_\tau\tilde u(\tau) = (\Delta_x - \delta^{-1})\tilde u(\tau), \quad \partial_n\tilde u\big|_{\partial\Omega} = 0,\\ \tilde u(0) = u(0) - \mathcal{L}(\phi(0)), \quad \lim_{\tau\to\infty}\tilde\phi(\tau) = 0.\end{gathered} \quad (4.38)$$

The solution $(\tilde\phi(\tau), \tilde u(\tau))$ can be expressed explicitly, using the analytic semigroups theory:

$$\begin{gathered}\tilde\phi(\tau) = (I - \delta\Delta_x)^{-1}e^{(\Delta_x - \delta^{-1}I)\tau}(u(0) - \mathcal{L}(\phi(0)),\\ \tilde u(\tau) := e^{(\Delta_x - \delta^{-1}I)\tau}(u(0) - \mathcal{L}(\phi(0)),\end{gathered} \quad (4.39)$$

where $\Delta_x$ is associated with Neumann boundary conditions. As in Section 2, we seek for asymptotic expansions for $(\phi(t), u(t))$ near $t = 0$ of the form

$$\phi(t) := \bar\phi_0(t) + \varepsilon\tilde\phi(\frac{t}{\varepsilon}) + \varepsilon\widehat\phi(t), \quad u(t) := \bar u_0(t) + \tilde u(\frac{t}{\varepsilon}) + \varepsilon\widehat u(t), \quad (4.40)$$

where $(\bar\phi_0(t), \bar u_0(t))$ is solution of (4.31) with $\bar\phi_0(0) := \phi(0)$ and $(\tilde\phi, \tilde u)$ is defined by (4.39). The following theorem is an analogue of Lemma 2.2.

**Theorem 4.4.** *Let the above assumptions hold. Then, the rest* $(\widehat\phi(t), \widehat u(t))$ *in the asymptotic expansions* (4.40) *enjoys the following estimate:*

$$\|\widehat\phi(t)\|_{H^2} + \|\widehat u(t)\|_{H^2} + \|\partial_t\widehat\phi(t)\|_{L^2} + \varepsilon\|\widehat u(t)\|_{L^2} \le Ce^{Lt}, \quad (4.41)$$

*where the constants* $C$ *and* $L$ *depend on* $\|\phi(0)\|_{H^2}$ *and* $\|u(0)\|_{H^2}$*, but are independent of* $\varepsilon$*.*

Proof. The functions $\tilde u(t)$ and $\tilde\phi(t)$ satisfy the equations

$$\begin{gathered}\delta\partial_t\widehat\phi=\Delta_x\widehat\phi-\frac{1}{\varepsilon}\Big[f(\bar\phi_0+\varepsilon\tilde\phi+\varepsilon\widehat\phi)-f(\bar\phi_0)\Big]+\widehat u+\Delta_x\tilde\phi,\\ \varepsilon\partial_t\widehat u=\Delta_x\widehat u-\partial_t\widehat\phi-\partial_t\bar u_0,\quad \phi\big|_{t=0}=-\tilde\phi(0),\quad \widehat u\big|_{t=0}=0,\\ \partial_n\widehat\phi\big|_{\partial\Omega}=\partial_n\widehat u\big|_{\partial\Omega}=0.\end{gathered}\tag{4.42}$$

Arguing as in (2.15)-(2.23) (with $f_2\equiv 0$), we obtain the estimate

$$\begin{gathered}\partial_t\Big[\delta\|\widehat\phi(t)\|^2_{L^2}+\delta\|\partial_t\widehat\phi(t)\|^2_{L^2}+\|\nabla_x\widehat u(t)\|^2_{L^2}-2(\partial_t\bar u_0(t),\widehat u(t))+C_1\Big]\le\\ \le C_2\left(1+\|\partial_t\tilde\phi(t/\varepsilon)\|_{H^2}+\|\partial_t\tilde u(t/\varepsilon)\|_{L^2}+\|\partial_t^2\bar u_0(t)\|^2_{H^{-1}}\right)\times\\ \times\Big[\delta\|\widehat\phi(t)\|^2_{L^2}+\delta\|\partial_t\widehat\phi(t)\|^2_{L^2}+\|\nabla_x\widehat u(t)\|^2_{L^2}-2(\partial_t\bar u_0(t),\widehat u(t))+C_1+\langle u(t)\rangle^2\Big],\end{gathered}\tag{4.43}$$

where we have the additional term $\langle u(t)\rangle^2$ in the right-hand side (which appears because of Friedrichs' inequality (4.13) for Neumann boundary conditions) and the constants $C_1$ and $C_2$ are independent of $\varepsilon$. In order to estimate this term, we integrate the first equation of (4.42) over $\Omega$:

$$\langle\widehat u(t)\rangle=\delta\langle\partial_t\widehat\phi(t)\rangle+\langle\frac{1}{\varepsilon}\Big[f(\bar\phi_0+\varepsilon\tilde\phi+\varepsilon\widehat\phi)-f(\bar\phi_0)\Big]\rangle.\tag{4.44}$$

Since, due to Theorems 4.1 and 4.3, the $L^\infty$-norms of $\phi(t):=\bar\phi_0(t)+\varepsilon\tilde\phi(t/\varepsilon)+\varepsilon\widehat\phi(t)$ and $\bar\phi_0(t)$ are uniformly (with respect to $\varepsilon$) bounded, it follows from (4.44) that

$$\langle u(t)\rangle^2\le C\left(1+\|\partial_t\widehat\phi(t)\|^2_{L^2}+\|\widehat\phi(t)\|^2_{L^2}\right),\tag{4.45}$$

where the constant $C$ is independent of $\varepsilon$. Applying now Gronwall's inequality to (4.43) and using (2.24) and (4.45), we have

$$\|\widehat\phi(t)\|^2_{L^2}+\|\partial_t\widehat\phi(t)\|^2_{L^2}+\|\widehat u(t)\|^2_{H^1}\le Ce^{Lt},\tag{4.46}$$

where the constants $C$ and $L$ are independent of $\varepsilon$. Estimate (4.41) can be deduced from (4.46) exactly as in Lemma 1.3. This finishes the proof of Theorem 4.4. □

**Corollary 4.5.** *Under the assumptions of Theorem 4.4, estimates* (2.26), (2.27) *and* (2.28) *remain valid (for the case of Neumann boundary conditions).*

Indeed, these estimates can be deduced from (4.46) exactly as in Corollaries 2.3 and 2.4.

We are now ready to construct a robust family of exponential attractors for problem (4.3) with Neumann boundary conditions. Since we have the dissipativity of system (4.3) in the phase spaces $\Phi_M$ only (for every fixed $M$; for Dirichlet boundary conditions, this property was valid in the whole space $\Phi$), it is natural to construct the exponential attractors $\mathcal M_\varepsilon^M$ for the semigroups

$$S_t^{\varepsilon,M}:\Phi_M\to\Phi_M,\quad S_t^{\varepsilon,M}(\phi_0,u_0):=(\phi(t),u(t))\tag{4.47}$$

(where $(\phi(t),u(t))$ is the corresponding solution of (4.3)) acting in the spaces $\Phi_M$. In that case, the exponential attractors $\mathcal M_\varepsilon^M$ depend obviously on $M$. We consider the following limit semigroup $S_t^{0,M}$ for (4.47):

$$S_t^{0,M}:\mathbb L_M\to\mathbb L_M,\quad S_t^{0,M}(\phi_0,u_0):=(\bar\phi_0(t),\bar u_0(t)),\tag{4.48}$$

associated with the limit problem (4.31) on the manifold $\mathbb{L}_M$ defined by (4.35).

The main result of this section is the following analogue of Theorem 3.1 for the case of Neumann boundary conditions.

**Theorem 4.6.** *Let the assumptions of Theorem 4.1 hold. Then, for every $M > 0$, there exists a family of compact sets $\mathcal{M}_\varepsilon^M \subset \Phi_M$, $\varepsilon \in [0,1]$, such that*

*1. These sets are semi-invariant with respect to the flows $S_t^{\varepsilon,M}$ associated with problem (4.3), i.e.*

$$S_t^{\varepsilon,M}\mathcal{M}_\varepsilon^M \subset \mathcal{M}_\varepsilon^M. \tag{4.49}$$

*2. The fractal dimension of the sets $\mathcal{M}_\varepsilon^M$ is finite and uniformly bounded with respect to $\varepsilon$:*

$$\dim_F(\mathcal{M}_\varepsilon^M, \Phi_M) \le C < \infty, \tag{4.50}$$

*where $C = C(M)$ is independent of $\varepsilon$.*

*3. These sets attract exponentially the bounded subsets of $\Phi_M$, i.e. there exist a positive constant $\alpha = \alpha(M) > 0$ and a monotonic function $Q = Q_M$ which are independent of $\varepsilon$ such that, for every bounded subset $B$ in the phase space $\Phi_M$, we have*

$$\operatorname{dist}_{\Phi_M}(S_t^{\varepsilon,M}B, \mathcal{M}_\varepsilon^M) \le Q(\|B\|_{\Phi_M})e^{-\alpha t}, \quad \varepsilon \in [0,1] \tag{4.51}$$

*(for $\varepsilon = 0$, we should take $B \subset \mathbb{L}_M$).*

*4. The symmetric Hausdorff distance between the limit attractor $\mathcal{M}_0^M$ and the attractors $\mathcal{M}_\varepsilon^M$ enjoys the following estimate:*

$$\operatorname{dist}_{sym,\Phi_M}(\mathcal{M}_\varepsilon^M, \mathcal{M}_0^M) \le C\varepsilon^\kappa, \tag{4.52}$$

*where the constants $C = C(M) > 0$ and $0 < \kappa = \kappa(M) < 1$ are independent of $\varepsilon$ and can be computed explicitly.*

Proof. As in the case of Dirichlet boundary conditions, the proof of this theorem is based on the abstract result given in Proposition 3.2 and coincides, up to minor changes, with that of Theorem 3.1. That is the reason why we only indicate these changes below and leave the details to the reader.

Instead of the absorbing sets $B_\varepsilon$ and $B_0$ defined by (3.8) and (3.9) respectively, we now consider, for every $M > 0$, the sets

$$B_\varepsilon^M := \{(\phi_0, u_0) \in \Phi_M, \quad \|(\phi_0, u_0)\|_\Phi^2 \le 2Q_M(\|g\|_{L^2})\}, \tag{4.53}$$

$$B_0^M := \{(\phi_0, u_0) \in \mathbb{L}_M, \quad \|\phi_0\|_{H^2}^2 \le 2Q_M(\|g\|_{L^2})\}, \tag{4.54}$$

where the function $Q_M$ is the same as in (4.8). We note that, in contrast to the case of Dirichlet boundary conditions, the sets $B_\varepsilon$ now *depend* on $\varepsilon$, since the conserved integral (4.5) depends explicitly on $\varepsilon$.

Then, these sets are indeed uniform (with respect to $\varepsilon$) absorbing sets for semigroups (4.47) and (4.48) (due to estimates (4.8) and (4.36) (thus, the analogue of (3.10) is also satisfied)). Moreover, condition (3.5) of Proposition 3.2 is satisfied for these semigroups, due to Theorem 4.2.

Let us now verify condition (3.6) of this proposition. To this end, we modify slightly the construction of the nonlinear projectors $\Pi_\varepsilon$ as follows:

$$\Pi_\varepsilon : B_\varepsilon^M \to B_0^M, \quad \Pi_\varepsilon(\phi_0, u_0) := (\phi_0 + \varepsilon\langle u_0\rangle, \mathcal{L}(\phi_0 + \varepsilon\langle u_0\rangle). \tag{4.55}$$

Since

$$|\langle \phi_0 + \varepsilon\langle u_0\rangle\rangle| = |I_0(\phi_0, u_0)| \le M,$$

projectors (4.55) are indeed well defined. Moreover, the analogue of condition (3.6) for our case now follows from estimate (2.26) (see Corollary 4.5), estimate (4.25) for the limit problem (4.31) and from the obvious estimate

$$\|\Pi_\varepsilon(\phi_0, u_0) - (\phi_0, \mathcal{L}(\phi_0))\|_\Phi \le \varepsilon C_M, \tag{4.56}$$

for every $(\phi_0, u_0) \in B_\varepsilon$.

Thus, we can apply Proposition 3.2 to our situation and we obtain the desired family of exponential attractors $\mathcal{M}_\varepsilon^{M,d}$ for the discrete semigroups $S_{nT}^{\varepsilon,M}$ acting on the absorbing sets (4.53) and (4.54). The existence of the exponential attractors for the continuous semigroups then follows exactly as in the proof of Theorem 3.1 and Theorem 4.6 is proved. □

**Remark 4.7.** *We recall that the exponential attractors $\mathcal{M}_\varepsilon^M$ constructed in Theorem 4.6 depend on $M$. Moreover, all the constants in estimates* (4.50)-(4.52) *also depend a priori on $M$. It is possible to prove, however, that, under natural assumptions on $f$, there exists a positive constant $M_0 \gg 1$ that is independent of $\varepsilon$ such that every solution of equation* (4.3) *with initial data satisfying*

$$|I_0(\phi_0, u_0)| = M_0, \tag{4.57}$$

*uniformly with respect to $\varepsilon$, stabilizes exponentially to the corresponding equilibrium $(\bar\phi, \bar u) \in \mathbb{R}^2$, which is the unique solution of*

$$\bar u = f(\bar\phi), \quad \varepsilon \bar u + \bar\phi = I_0(\phi_0, u_0). \tag{4.58}$$

*This suggests that it is possible to construct the family of exponential attractors $\mathcal{M}_\varepsilon^M$ for* (4.3) *such that all the constants in estimates* (4.50)-(4.52) *are independent of $M$. We will come back to this problem in a forthcoming article.*

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