Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 80, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

A note on the singular Sturm-Liouville problem with infinitely many solutions *

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Abstract

We consider the Sturm-Liouville nonlinear boundary-value problem

$$-u''(t) = a(t)f(u(t)), \quad 0 < t < 1,$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0,$$

where α, β, γ, $\delta \geq 0$, $\alpha\gamma + \alpha\delta + \beta\gamma > 0$ and $a(t)$ is in a class of singular functions. Using a fixed point theorem we show that under certain growth conditions imposed on $f(u)$ the problem admits infinitely many solutions.

1 Introduction

In this paper we are interested in the Sturm-Liouville nonlinear boundary-value problem

$$-u''(t) = a(t)f(u(t)), \quad 0 < t < 1, \tag{1}$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \tag{2}$$

The paper is organized in the following manner. In the introduction we briefly discuss the background of the problem, make standing assumptions on the right side of (1) and state the theorems that will be used to obtain our main results presented in Section 3. The approach is based on the properties of the Green's function of the homogeneous (1)-(2). They will be presented in Section 2.

Fixed point theorems have been applied to various boundary value problems to establish the existence of multiple positive solutions. Just recently there have been obtained several results concerning the existence of countably infinitely many positive solutions, e.g., Ehme [3], Eloe, Henderson and Kosmatov [4], Kaufmann and Kosmatov [5] and Kosmatov [6]. They cover the cases of $(k,\ n-k)$ and second order conjugate type boundary value problems. In addition, [5, 6] deal with a singular BVP. Singular boundary value problems have been

* *Mathematics Subject Classifications:* 34B16, 34B18.
Key words: Sturm-Liouville problem, Green's function, fixed point theorem,
Hölder's inequality, multiple solutions.

Submitted November 13, 2001. Published September 27, 2002.

considered by many authors, e.g., Agarwal, O'Regan and Wong [1] and Baxley and Thompson [2]. This paper complements the results of [6] in which we considered the conjugate type boundary value problem

$$\begin{gathered} -u''(t) = a(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = 0. \end{gathered}$$

It needs to be mentioned that [6] only treats the class of symmetric about $t = \frac{1}{2}$ solutions. The assumption of symmetry imposes a constraint on $a(t)$: it must also be chosen to be symmetric about $t = \frac{1}{2}$. In our situation we can more generally locate the point of singularity anywhere in $[0, 1]$.

We will analyze a family of singular functions

$$a(t) = |t - t'|^{-\epsilon}, \tag{3}$$

where $t' \in (0, 1)$ and $0 < \epsilon < 1$. We will show that it is possible to construct $f(u)$ in such a fashion that it can be uniformly used for a range of value of the parameter ϵ. To achieve a result that does not involve ϵ in growth conditions Hölder's inequality is to our advantage. It is utilized to yield norm-estimates from "above" of Theorem 1.2 presented below.

The Green's function of

$$-u'' = 0$$

satisfying (2) is

$$G(t, s) = \begin{cases} \sigma(\alpha t + \beta)(\gamma + \delta - \gamma s), & t \le s \le 1, \\ \sigma(\alpha s + \beta)(\gamma + \delta - \gamma t), & 0 \le s \le t, \end{cases} \tag{4}$$

where $\sigma = 1/(\alpha\gamma + \alpha\delta + \beta\gamma)$ (note that $\alpha\gamma + \alpha\delta + \beta\gamma > 0$).

At this point we assume that $f(u)$ is a continuous nonnegative function and introduce an integral operator, T, associated with the BVP (1), (2) as follows

$$Tu(t) = \int_0^1 G(t, s)a(s)f(u(s))ds, \quad 0 \le t \le 1. \tag{5}$$

Fixed points of (5) are in fact (positive) solutions of (1), (2).

The main tools used in this paper are the Krasnosel'skiĭ's fixed point theorem [7] and Hölder's inequality stated below. Now we define a cone in a Banach space.

Definition 1.1 Let $\mathcal{B}$ be a real Banach space. A nonempty, closed set $\mathcal{C} \subset \mathcal{B}$ is said to be a cone provided:

(i) $\alpha u + \beta v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$ and $\alpha, \beta \ge 0$,

(ii) $u, -u \in \mathcal{C}$ implies $u = 0$.

Theorem 1.2 *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open bounded subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{C} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{C} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Under appropriate growth assumptions on f, Theorem 1.2 guarantees the existence of fixed points of (5). With $a(t)$ specified by (3), (5) becomes

$$Tu(t) = \int_0^1 G(t,s)|t' - s|^{-\epsilon} f(u(s))ds, \quad 0 \leq t \leq 1. \tag{6}$$

Evidently, T is a completely continuous operator.

We say that f is in the Lebesgue space of (real valued) functions, $L^p[a,b]$, if

$$\int_a^b |f|^p dx < \infty.$$

The norm on $L^p[a,b]$ is

$$\|f\|_p = \Big(\int_a^b |f|^p dx \Big)^{1/p}.$$

Now we state Hölder's inequality.

Theorem 1.3 *Let $f \in L^p$ and $g \in L^q$, where $p > 1$ and $q = \frac{p}{p-1}$. Then $fg \in L^1$ and we have*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{7}$$

2 Technical Results

For $\tau \in [0, \frac{1}{2})$, denote the interval $[\tau, 1-\tau]$ by I_τ. Note that, for each $\tau \in (0, \frac{1}{2})$, (4) satisfies

$$\min_{t \in I_\tau} G(t,s) \geq L_\tau G(t', s),$$

where $L_\tau = \min\{\frac{\delta+\gamma\tau}{\delta+\gamma}, \frac{\beta+\alpha\tau}{\beta+\alpha}\}$ for all $t', s \in [0,1]$. Let $\mathcal{B} = C[0,1]$ endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and define $\mathcal{C}_\tau \subset \mathcal{B}$ by

$$\mathcal{C}_\tau = \{u(t) \in \mathcal{B} \,|\, u(t) \geq 0 \quad \text{on}[0,1],\ \min_{t \in I_\tau} u(t) \geq L_\tau \|u\|\}.$$

Clearly, $\mathcal{C}_\tau$ is a cone and it can be shown that (6) preserves $\mathcal{C}_\tau$.

At this point we would like to establish the L^p-norm estimates on (3) and (4) which will be used in Section 3. It is easy to see that $a \in L^p$ for all $p < \frac{1}{\epsilon}$ with

$$\|a\|_p = \frac{1}{(1-\epsilon p)^{1/p}}({t'}^{1-\epsilon p} + (1-t')^{1-\epsilon p})^{1/p} \le \frac{2^\epsilon}{(1-\epsilon p)^{1/p}}. \tag{8}$$

To obtain the required estimates on (4) we consider the following three cases: (1) $\alpha = 0$, $\gamma > 0$, (2) $\gamma = 0$, $\alpha > 0$, and (3) $\alpha\gamma > 0$.

If $\alpha = 0$ and $\gamma > 0$, then (4) becomes

$$G(t,s) = \begin{cases} 1 + \frac{\delta}{\gamma} - s, & t \le s \le 1, \\ 1 + \frac{\delta}{\gamma} - t, & 0 \le s \le t. \end{cases} \tag{9}$$

Now,

$$\int_\tau^{1-\tau} G(t,s)ds = (1 + \frac{\delta}{\gamma})(1-2\tau) + \frac{1}{2}\tau^2 - t(1-\tau) - \frac{1}{2}t^2$$

attains its maximum at $t = \tau$ and

$$\max_{t\in[0,1]} \int_\tau^{1-\tau} G(t,s)ds = 2(1 - \tau + \frac{\delta}{\gamma})(\frac{1}{2} - \tau) \ge \frac{1}{2} - \tau. \tag{10}$$

If $\gamma = 0$ and $\alpha > 0$, then (4) becomes

$$G(t,s) = \begin{cases} \frac{\beta}{\alpha} + t, & t \le s \le 1, \\ \frac{\beta}{\alpha} + s, & 0 \le s \le t. \end{cases} \tag{11}$$

Then

$$\int_\tau^{1-\tau} G(t,s)ds = \frac{\beta}{\alpha}(1-2\tau) - \frac{1}{2}\tau^2 + t(1-\tau) - \frac{1}{2}t^2$$

has its maximum at $t = 1 - \tau$ and

$$\max_{t\in[0,1]} \int_\tau^{1-\tau} G(t,s)ds = (1 + 2\frac{\beta}{\alpha})(\frac{1}{2} - \tau) \ge \frac{1}{2} - \tau. \tag{12}$$

If $\alpha\gamma > 0$, then (4) takes shape of

$$G(t,s) = \begin{cases} \sigma'(t+\beta')(1+\delta'-s), & t \le s \le 1, \\ \sigma'(s+\beta')(1+\delta'-t), & 0 \le s \le t, \end{cases} \tag{13}$$

where $\beta' = \beta/\alpha$, $\delta' = \delta/\gamma$, $\sigma' = 1/(\beta' + \delta' + 1)$.

A direct computation gives that for all $t \in [0,1]$,

$$\int_\tau^{1-\tau} G(t,s)ds = \sigma'(-2\beta'\delta'\tau - \frac{1}{2\sigma'} + \beta'(\frac{1}{2} - \tau) + \beta'\delta' + (\beta'\tau + \delta' - \delta'\tau + \frac{1}{2})t - \frac{1}{2\sigma'}t^2).$$

The right side of the equation above attains its max at $t_m = {\sigma'}^2(\beta'\tau + \delta' - \delta'\tau + \frac{1}{2}) \in (\tau, 1-\tau)$ and we obtain that

$$\begin{aligned}
\max_{t\in[0,1]} \int_\tau^{1-\tau} G(t,s)ds &= \sigma'[\beta'(1+2\delta')(\frac{1}{2}-\tau) + \frac{1}{2\sigma'}({t_m}^2 - \tau^2)] \\
&= \sigma'[\beta'(1+2\delta')(\frac{1}{2}-\tau) + \frac{1}{2}(2\tau + \sigma'(1+2\delta')(\frac{1}{2}-\tau))] \\
&\geq \sigma'[\beta'(1+2\delta')(\frac{1}{2}-\tau) + \frac{\sigma'}{4}(1+2\alpha')^2(\frac{1}{2}-\tau)] \\
&= \frac{{\sigma'}^2}{4}(1+2\beta')(2\beta'+1+2\delta')(2\delta'+1)(\frac{1}{2}-\tau) \qquad (14) \\
&\geq \frac{1}{4}(\frac{1}{2}-\tau)
\end{aligned}$$

Combining (10), (12), and (14) we get that (4) satisfies

$$\max_{t\in[0,1]} \int_\tau^{1-\tau} G(t,s)ds \geq \frac{1}{4}(\frac{1}{2}-\tau). \qquad (15)$$

Now we establish an estimate from above for (9). Let $q \geq 1$. It is easy to see that $G(t,s) \leq G(s,s)$ for all $t, s \in [0,1]$. If $\alpha = 0$, $\gamma > 0$, then

$$\begin{aligned}
\|G(t,\cdot)\|_q^q &= \int_0^1 G^q(t,s)ds \\
&\leq \int_0^1 G^q(s,s)ds \\
&= \int_0^1 (1+\frac{\delta}{\gamma}-s)^q ds \\
&\leq \frac{1}{q+1}(1+\frac{\delta}{\gamma})^{q+1} \\
&< (1+\frac{\delta}{\gamma})^{2q}
\end{aligned}$$

so that

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q < (1+\frac{\delta}{\gamma})^2. \qquad (16)$$

By the same argument applied to the case of $\gamma = 0$, $\alpha > 0$ we get for (11)

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q < (1+\frac{\beta}{\alpha})^2. \qquad (17)$$

If $\alpha\gamma > 0$, then we have for (13)

$$G(t,s) \leq G(s,s) = \sigma'(\beta'\delta' + \beta' + (\delta' + 1 - \beta')s - s^2).$$

Consider the function $g(s) = \beta'\delta' + \beta' + (\delta' + 1 - \beta')s - s^2$ on $[0,1]$. There are three cases: (i) $\delta' - \beta' \leq -1$, (ii) $\delta' - \beta' \geq 1$ and (iii) $|\beta' - \delta'| < 1$. In cases (i) and (ii), the maximum of $g(s)$ occurs at $s = 0$ and $s = 1$, respectively. So that $\max_{s\in[0,1]} g(s) = g(0) = \beta'(1+\delta') < \frac{1}{\sigma'^2}$ and $\max_{s\in[0,1]} g(s) = g(1) = \delta'(1+\beta') < \frac{1}{\sigma'^2}$ as corresponding to the above two cases. Combining cases (i) and (ii), we get that if $|\delta' - \beta'| \geq 1$, then

$$\max_{s\in[0,1]} g(s) < \frac{1}{\sigma'^2}.$$

In case (iii) the maximum is attained at $s = \frac{\delta'+1-\beta'}{2} \in (0,1)$ and $\max_{s\in[0,1]} g(s) = g(\frac{\delta'+1-\beta'}{2}) = \frac{1}{4}(\beta' + \delta' + 1)^2 = \frac{1}{\sigma'^2}$. Pasting all the cases together,

$$\max_{s\in[0,1]} g(s) < \frac{1}{\sigma'^2}$$

and so since $G(s,s) = \sigma' g(s)$,

$$G(t,s) < \frac{1}{\sigma'}, \ t,s \in [0,1].$$

In particular,

$$\begin{aligned} \|G(t,\cdot)\|_q^q &= \int_0^1 G^q(t,s)ds \\ &\leq \int_0^1 G^q(s,s)ds \\ &= \frac{1}{\sigma'^{2q}} \\ &= (1 + \frac{\beta}{\alpha} + \frac{\delta}{\gamma})^{2q} \end{aligned}$$

and hence

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q < (1 + \frac{\beta}{\alpha} + \frac{\delta}{\gamma})^2. \tag{18}$$

Finally, combining (16), (17), and (18) we get

$$\max_{t\in[0,1]} \|G(t,\cdot)\|_q < A, \tag{19}$$

where

$$A = \begin{cases} (1 + \frac{\delta}{\gamma})^2, & \alpha = 0, \gamma > 0 \\ (1 + \frac{\beta}{\alpha})^2, & \gamma = 0, \alpha > 0 \\ (1 + \frac{\beta}{\alpha} + \frac{\delta}{\gamma})^2, & \alpha\gamma > 0. \end{cases} \tag{20}$$

3 Main Results

Without loss of generality, let $0 < t' < 1/2$. First, we are going to consider the case of $0 < \epsilon < 1/2$.

Theorem 3.1 *Suppose the sequence $\{t_i\}_{i=1}^{\infty}$ satisfies $0 < t_i < t_{i+1}$, $i \in N$, and*

$$\lim_{i\to\infty} t_i = t' < \frac{1}{2}.$$

Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are the sequences satisfying

$$a_{i+1} < L_{t_i} b_i < b_i < \frac{16A}{\frac{1}{2} - t_i} b_i < a_i \quad i \in N,$$

where A is given by (20). Assume also that f satisfies the following conditions:

(H1) $f(z) \le \frac{1}{4A} a_i$ for all $z \in [0, a_i]$, $i \in N$,

(H2) $f(z) \ge \frac{4}{\frac{1}{2}-t'} b_i$ for all $z \in [L_{t_i} b_i, b_i]$, $i \in N$.

Then (1), (2) has infinitely many fixed points $\{u_i\}_{i=1}^{\infty}$ such that $b_i < \|u_i\| < a_i$, $i \in N$.

Proof. Consider the sequences $\{\Omega_{1,i}\}_{i=1}^{\infty}$ and $\{\Omega_{2,i}\}_{i=1}^{\infty}$ of open sets in $\mathcal{B}$ defined, for each $i \in N$, by

$$\Omega_{1,i} = \{u \in \mathcal{B}\colon \|u\| < a_i\},$$
$$\Omega_{2,i} = \{u \in \mathcal{B}\colon \|u\| < b_i\},$$

Consider also the sequence of cones $\{\mathcal{C}_i\}_{i=1}^{\infty}$ defined by

$$\mathcal{C}_i = \{u(t) \in \mathcal{B} | u(t) \ge 0 \text{ on } [0,1] \text{ with } \min_{t\in I_{t_i}} u(t) \ge L(t_i)\|u\|\}.$$

Let $i \in N$ and $u \in \mathcal{C}_i \cap \partial\Omega_{1,i}$, then

$$u(s) \le \|u\| = a_i$$

for all $s \in [0,1]$. So, by (H1)

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \int_0^1 a(s)G(t,s)f(u(s))ds \\ &\le \max_{t\in[0,1]} \int_0^1 a(s)G(t,s))ds \frac{1}{4A} a_i \end{aligned} \tag{21}$$

Let $1 < p < 1/\epsilon$ and set $q = p/(p-1)$. Then applying Hölder's inequality (7) to (21) yields

$$\|Tu\| \le \max_{t\in[0,1]} \|G(t,\cdot)\|_q \|a\|_p \frac{1}{4A} a_i,$$

which by (8), (19) transforms into

$$\|Tu\| < A\frac{2^\epsilon}{(1-\epsilon p)^{1/p}}\ \frac{1}{4A}a_i = \frac{2^\epsilon}{(1-\epsilon p)^{1/p}}\frac{1}{4}a_i. \tag{22}$$

Now, for every $0 < \epsilon < 1/2$ there exists $1 < p < 1/\epsilon$ such that $0 < p\epsilon < 1/2$. So, $1 < \frac{1}{1-\epsilon p} < 2$ and hence

$$1 < (\frac{1}{1-\epsilon p})^{\frac{1}{p}} < 2^{\frac{1}{p}}.$$

Hence (22) takes shape of

$$\|Tu\| < 2^{\epsilon+\frac{1}{p}}\frac{1}{4}a_i < a_i$$

$(\epsilon + \frac{1}{p} < 2)$, that is,

$$\|Tu\| < \|u\| \tag{23}$$

for all $u \in \mathcal{C}_i \cap \partial\Omega_{1,i}$, $i \in N$.

Let now $u \in \mathcal{C}_i \cap \partial\Omega_{2,i}$, then

$$b_i = \|u\| \ge u(s) \ge \min_{t\in[t_i,1-t_i]} u(s) \ge L_{t_i}\|u\| = L_{t_i}b_i$$

for all $s \in I_{t_i}$. Then, by (H2)

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]}\int_0^1 a(s)G(t,s)f(u(s))ds \\ &\ge \max_{t\in[0,1]}\int_{t_i}^{1-t_i} a(s)G(t,s)f(u(s))ds \\ &\ge \max_{t\in[0,1]}\int_{t_i}^{1-t_i} a(s)G(t,s)ds\frac{4}{\frac{1}{2}-t'}b_i. \end{aligned}$$

Note that because $t_i < t' < 1/2$ for all $i \in N$, $a(s) \ge \frac{1}{(t'-t_i)^\epsilon} > \frac{1}{(\frac{1}{2}-t_i)^\epsilon}$, $s \in I_{t_i}$, the inequality above becomes

$$\|Tu\| \ge \max_{t\in[0,1]}\int_{t_i}^{1-t_i} G(t,s)ds\frac{1}{(\frac{1}{2}-t_i)^\epsilon}\ \frac{4}{\frac{1}{2}-t'}\ b_i$$

Now (15) applies and we get

$$\begin{aligned} \|Tu\| &= \frac{1}{4}(\frac{1}{2}-t_i)\ \frac{1}{(\frac{1}{2}-t_i)^\epsilon}\ \frac{4}{\frac{1}{2}-t'}\ b_i \\ &> (\frac{1}{2}-t_i)^{1-\epsilon}\frac{1}{\frac{1}{2}-t'}b_i > b_i; \end{aligned}$$

that is,

$$\|Tu\| > \|u\| \tag{24}$$

for all $u \in \mathcal{C}_i \cap \partial\Omega_{2,i}$, $i \in N$.

Note that since $0 \in \Omega_{2,i} \subset \bar{\Omega}_{2,i} \subset \Omega_{1,i}$ and (23) and (24) hold, we can apply Theorem 1.2 to conclude that the operator T has a fixed point $u_i \in \mathcal{C}_i \cap (\bar{\Omega}_{1,i} \setminus \Omega_{2,i})$ such that $b_i < \|u_i\| < a_i$, $i \in N$. The proof is complete. □

Now, let us deal with the case $\frac{1}{2} \le \epsilon < 1$.

Theorem 3.2 *Suppose the sequence $\{t_i\}_{i=1}^{\infty}$ satisfies $0 < t_i < t_{i+1}$, $i \in N$, and*

$$\lim_{i\to\infty} t_i = t' < \frac{1}{2}.$$

Suppose $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are sequences satisfying

$$a_{i+1} < L_{t_i} b_i < b_i < \frac{16A}{(\frac{1}{2} - t_i)(1-\epsilon)} b_i < a_i \ for\ each\ i \in N.$$

where B is as in Theorem 3.1 and $C = \frac{1}{4}(1-\epsilon)$. Assume also that f satisfies $(H2)$ of Theorem 3.1 and

(H3) $f(z) \le \frac{1}{4A}(1-\epsilon)a_i$ for all $z \in [0, a_i]$ all $i \in N$.

Then (6) has infinitely many fixed points $\{u_i\}_{i=1}^{\infty}$ such that $b_i < \|u_i\| \le a_i$, $i \in N$.

Proof. Let $\{\Omega_{1,i}\}_{i=1}^{\infty}$, $\{\Omega_{2,i}\}_{i=1}^{\infty}$ and $\{\mathcal{C}_i\}_{i=1}^{\infty}$ be as in the proof of Theorem 3.1. Let $i \in N$. Let $u \in \mathcal{C}_i \cap \partial\Omega_{1,i}$, then for all $s \in [0,1]$

$$u(s) \le \|u\| = a_i$$

and by $(H3)$

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \int_0^1 a(s)G(t,s)f(u(s))ds \\ &\le \max_{t\in[0,1]} \int_0^1 a(s)G(t,s))ds \frac{1}{4A}(1-\epsilon)a_i \end{aligned} \tag{25}$$

Let now $1 < p < 1/\epsilon$ and set $q = \frac{p}{p-1}$. Applying (7) to (25), we get

$$\|Tu\| \le \max_{t\in[0,1]} \|G(t,\cdot)\|_q \|a\|_p \frac{1}{4A}(1-\epsilon)a_i.$$

As in the proof of Theorem 3.1, we obtain that

$$\|Tu\| \le A \frac{2^{\epsilon}}{(1-\epsilon p)^{1/p}} \frac{1}{4A}(1-\epsilon)a_i. \tag{26}$$

Now we make a suitable choice of p. To this end, set $p = \frac{\epsilon+1}{2\epsilon}$ and $q = \frac{\epsilon+1}{1-\epsilon}$. Observe that $1 < p < 1/\epsilon$ and $q = \frac{p}{p-1}$ and so (18) becomes

$$\begin{aligned} \|Tu\| &\le \frac{2^{\epsilon}}{(\frac{1-\epsilon}{2})^{\frac{3\epsilon-1}{\epsilon+1}}} \frac{1}{4}(1-\epsilon)a_i \\ &\le \frac{2}{\frac{1-\epsilon}{2}} \frac{1}{4}(1-\epsilon)a_i = a_i; \end{aligned}$$

that is,

$$\|Tu\| \le \|u\| \tag{27}$$

for all $u \in \mathcal{C}_i \cap \partial\Omega_{1,i}$, $i \in N$. Let $u \in \mathcal{C}_i \cap \partial\Omega_{2,i}$, then

$$b_i = \|u\| \ge u(s) \ge \min_{t\in[t_i,1-t_i]} u(s) \ge L_{t_i}\|u\| = L_{t_i} b_i$$

for all $s \in I_{t_i}$. Then, as in the proof Theorem 3.1, we obtain

$$\|Tu\| > \|u\| \tag{28}$$

for all $u \in \mathcal{C}_i \cap \partial\Omega_{2,i}$, $i \in N$.

Note that since $\Omega_{2,i} \subset \bar{\Omega}_{2,i} \subset \Omega_{1,i}$ and (27) and (28) hold, we can apply Theorem 1.2 to conclude that the operator T has a fixed point $u_i \in \mathcal{C}_i \cap (\bar{\Omega}_{1,i} \setminus \Omega_{2,i})$ such that $b_i < \|u_i\| \le a_i,\ i \in N$, which completes the proof. □

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