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On the Keldys-Fichera boundary-value problem for degenerate quasilinear elliptic equations *

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Abstract

We prove existence and uniqueness theorems for the Keldys-Fichera boundary-value problem using pseudo-monotone operators. The equation studied here is quasilinear, elliptic, and its set of degenerate points may be of non-zero measure. We also obtain comparison and maximum principles for this problem.

1 Introduction

This article studies the Keldys-Fichera boundary-value problems (KFBVP) for degenerate quasilinear elliptic equations. For linear elliptic equations with non-negative characteristic form of second order, the KFBVP is well known and has been summarized in detail by Oleinik and Radkevich [7]. However little information is known about this problem for nonlinear equations. Ma and Yu [6] discussed the KFBVP for the degenerate quasilinear elliptic equation

$$Lu = D_i[a_{ij}(x,u)D_j u + b_i(x)u] - c(x,u) = f(x), \quad x \in \Omega. \tag{1.1}$$

In this article, the summation from 1 to n over repeated indices is understood. They obtained an existence theorem by using the acute angle principle for weakly continuous operators. In this paper we consider the more general degenerate quasilinear elliptic equation

$$Qu = -D_i a_i(x,u,Du) + a(x,u) = f(x), \quad x \in \Omega, \tag{1.2}$$

in a bounded domain $\Omega \subset R^n$, $n \geq 2$, with piecewise C^1-smooth boundary $\partial\Omega$. We also obtain existence results, different from [6], using the pseudo-monotone operator method. Moreover, in this paper the set of degenerate points may be of non-zero measure, which is different from [6]. Also, the comparison principle, the maximum principle and a uniqueness theorem of the solution to the KFBVP for (1.2) are discussed.

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The article is organized as follows. Section 2 formulates the existence theorem, the main result of this paper, and the preliminaries. Section 3 contains the proof of the main result. In section 4, we prove the comparison principle, the maximum principle and the uniqueness theorem.

2 Main Result

Let Ω be a bounded domain in R^n $(n \geq 2)$ with piecewise C^1-boundary $\partial\Omega$ and the Sobolev imbedding theorems are valid for this domain. Assume the following hypotheses:

(A1) $a_i(x, z, p)$, $i = 1, 2, \dots, n$, satisfy Carathéodory conditions, i.e., they are continuous in $(z, p) \in R \times R^n$ for x a.e. in $\bar{\Omega}$ and measurable in $x \in \bar{\Omega}$ for every $(z, p) \in R \times R^n$. Moreover, $a_i(x, z, p)$ and $a(x, z)$ possess integrable continuous derivatives in p_i and z

(A2) $a_i(x, 0, 0) \in W^{1/m,m'}(\partial\Omega)$ for $m \geq 2$,

$$|a_i(x, z, p)| \leq c(|p|^{m-1} + |z|^{l/m'} + \varphi_1(x)), \quad \varphi_1(x) \in W^{1/m,m'}(\partial\Omega),$$
$$|a(x, z)| \leq c(|z|^{l-1} + \varphi_2(x)), \quad \varphi_2(x) \in L_{l'}(\Omega),$$

for $(x, z, p) \in \bar{\Omega} \times R \times R^n$, where $c > 0$, $2 \leq l < \bar{m} = m(n-1)/(n-m)$ if $m < n$ and $2 \leq l < \infty$ if $m = n$

(A3) There exist constants $\alpha > 0$, $\beta > 0$ and a continuous function $\lambda(x) \geq 0$ such that for all $(x, z, p) \in \bar{\Omega} \times R \times R^n$ and any $\xi \in R^n$ it holds that

$$\alpha^{-1} \frac{\partial a_i}{\partial p_j}(x, 0, 0)\xi_i\xi_j \leq \frac{\partial a_i}{\partial p_j}(x, z, p)\xi_i\xi_j \leq \alpha \frac{\partial a_i}{\partial p_j}(x, 0, 0)\xi_i\xi_j, \tag{2.1}$$
$$\beta^{-1} \frac{\partial a_i}{\partial z}(x, 0, 0)\xi_i \leq \frac{\partial a_i}{\partial z}(x, z, p)\xi_i \leq \beta \frac{\partial a_i}{\partial z}(x, 0, 0)\xi_i, \tag{2.2}$$
$$\frac{\partial a_i}{\partial p_j}(x, 0, 0)\xi_i\xi_j \geq \lambda(x)|\xi|^2; \tag{2.3}$$

(A4) There exists a positive function $\varphi_3(x) \in L^1(\Omega)$ such that

$$a_i(x, z, p)p_i + a(x, z)z \geq h(|p|)|p|^m - \varphi_3(x) \tag{2.4}$$

for $(x, z, p) \in \bar{\Omega} \times R \times R^n$, where $h(t)$ with $h(0) = 0$ is a bounded, non-decreasing and continuous function on $[0, \infty)$. Without loss of generality, we assume that $\sup h(t) > 1$.

Remark 2.1 From (2.1) and (2.3) we know that (1.2) may degenerate at the set $\{x \mid \lambda(x) = 0\}$ which may have positive measure, while in [6] the set is of zero measure.

Remark 2.2 Condition (A2) shows the growth orders of $a_i(x,z,p)$ and $a(x,z)$ in $|z|$, $|p|$ and $|z|$ respectively which often used in [2, 3] and other related papers. Condition (A3) is an extension of the one in [6]. Condition (A4) is a version of the one in [1].

We divide $\partial\Omega$ into the parts of the Keldys-Fichera type as follows

$$\begin{gathered}\Sigma^0 = \big\{x \in \partial\Omega : \frac{\partial a_i}{\partial p_j}(x,0,0)\nu_i\nu_j = 0\big\},\\ \Sigma_1 = \big\{x \in \Sigma^0 : \frac{\partial a_i}{\partial z}(x,0,0)\nu_i \le 0\big\},\\ \Sigma_2 = \Sigma^0\backslash\Sigma_1, \quad \Sigma_3 = \partial\Omega\backslash\Sigma^0,\end{gathered} \tag{2.5}$$

where $\vec{\nu} = (\nu_1,\nu_2,\cdots,\nu_n)$ is the unit outward normal to $\partial\Omega$. For $m \ge 2$, we denote by $W^{1,m}(\Omega)$ the Sobolev space with the norm

$$\|u\| = (\|u\|_m^m + \|Du\|_m^m)^{1/m},$$

where $\|\cdot\|_m$ is the $L^m(\Omega)$ norm. For $2 \le m < n$, we define that $\tilde{W}^{1,m}(\Omega)$, the subspace of $W^{1,m}(\Omega)$, is the closure of the set $\{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Sigma_3\}$ in the norm $\|u\|$. Attaching the Keldys-Fichera boundary condition that $u(x) = 0$ on $\Sigma_2 \cup \Sigma_3$ to (1.2) we then get the Keldys-Fichera boundary-value problem

$$\begin{gathered}-D_i a_i(x,u,Du) + a(x,u) = f(x), \quad x \in \Omega\\ u = 0, \quad x \in \Sigma_2 \cup \Sigma_3.\end{gathered} \tag{2.6}$$

Definition 2.3 A weak solution of (2.6) is defined to be an element $u \in \tilde{W}^{1,m}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega [a_i(x,u,Du)D_i v + a(x,u)v]dx - \int_{\Sigma_1} a_i(x,u,0)\nu_i v ds\\ &= \int_{\Sigma_2} a_i(x,0,0)\nu_i v ds + \int_\Omega f(x)v dx,\end{aligned} \tag{2.7}$$

for any $v \in C^1(\bar{\Omega})$ vanishing on Σ_3.

Remark 2.4 One can find out that this definition is a proper counterpart of the weak solution of the KFBVP for linear equations in [7].

Remark 2.5 (2.1) and (2.2) mean that in the definitions of Σ^0 and Σ_1 we can replace $\frac{\partial a_i}{\partial p_j}(x,0,0)$ and $\frac{\partial a_i}{\partial z}(x,0,0)$ by $\frac{\partial a_i}{\partial p_j}(x,z,p)$ and $\frac{\partial a_i}{\partial z}(x,z,p)$ respectively.

Remark 2.6 Multiplying (2.6) by $v \in C^1(\bar{\Omega})$ vanishing on Σ_3 and then integrating by parts, we can obtain (2.7). Hence, the classical solution of (2.6) must be a weak solution.

Theorem 2.7 (Existence theorem) *Assume that (A1)-(A4) hold and that* $f(x) \in W^{-1,m'}(\Omega)$. *Then (2.6) has a weak solution in* $\tilde{W}^{1,m}(\Omega)$.

3 Proof of Theorem 2.7

We confine ourselves to the case of $m < n$, and refer the reader to Remark 3.3 for the case of $m \geq n$. For $\delta \in (0,\ 1]$ and $i = 1, 2, \cdots, n$, define

$$a_{i\delta}(x, z, p) = a_i(x, z, p) + \delta p_i |p_i|^{m-2}\,.$$

Lemma 3.1 *For any $\delta \in (0,\ 1]$ the integral equation*

$$\begin{aligned}&\int_\Omega [a_{i\delta}(x, u, Du)D_i v + a(x, u)v]dx - \int_{\Sigma_1} a_i(x, u, 0)\nu_i v ds \\ &= \int_\Omega f(x)v dx + \int_{\Sigma_2} a_i(x, 0, 0)\nu_i v ds, \forall v \in \tilde{W}^{1,m}(\Omega) \qquad (3.1)\end{aligned}$$

has a solution in $\tilde{W}^{1,m}(\Omega)$. This solution is denoted , u_δ.

Proof Denote

$$\begin{aligned}&B_\delta(u, v)\\ &= \int_\Omega [a_{i\delta}(x, u, Du)D_i v + a(x, u)v]dx - \int_{\Sigma_1} a_i(x, u, 0)\nu_i v ds \quad \forall v \in \tilde{W}^{1,m}(\Omega).\end{aligned}$$

Then, by (A2), the Hölder's inequality and the Sobolev imbedding theorem, $W^{1,m}(\Omega) \hookrightarrow L^l(\Omega)$, we obtain

$$\begin{aligned}&\Big|\int_\Omega [a_{i\delta}(x, u, Du)D_i v + a(x, u)v]dx\Big|\\ &\leq c(\|Du\|_m^{m/m'} + \|u\|_l^{l/m'} + n\|Du\|_m^{m/m'} + \|\varphi_1\|_{m'})\|v\|\\ &\quad +(\|u\|_l^{l/l'} + \|\varphi_2\|_{l'})\|v\| \qquad (3.2)\\ &\leq c\|v\|,\end{aligned}$$

where $c = c(\|u\|, \|\varphi_1\|_{m'}, \|\varphi_2\|_{l'}, l, l', m, m', n)$. Here and in the sequel, the constant c may vary in the context, and if necessary we will then indicate its dependence on other known quantities.

Because Σ_1 and Σ_2 are of C^1, the trace imbedding $W^{1,m}(\Omega) \hookrightarrow L^{\bar{m}}(\Sigma_i), i = 1, 2$, is valid (see Σ_1 of Chapter 3 in [4]). Notice that $l < \bar{m},\ m < \bar{m}$, so the following integrals are well defined, and then by (A2) and the Hölder's inequality, it follows that

$$\begin{aligned}\Big|\int_{\Sigma_1} a_i(x, u, 0)\nu_i v ds| &\leq c\int_{\Sigma_1} (|u|^{l/m'} + |\varphi_1|)|v| ds\\ &\leq c(\|u\|_{L^l(\Sigma_1)}^{l/m'} + \|\varphi_1\|_{m'})\|v\|_{L^m(\Sigma_1)} \leq c\|v\|,\end{aligned}$$

where $c = c(\|u\|, \|\varphi_1\|_{m'}, l, m, m', |\Omega|)$, and the last inequality is based on the trace imbedding theorem mentioned above. From (3.2) and (3.3) we then know that there exists a continuous mapping $B(u) : \tilde{W}^{1,m}(\Omega) \to W^{-1,m'}(\Omega)$ such that $(B(u), v) = B_\delta(u, v)$. Hence, for solving (3.1) it is sufficient to find an element, say u_δ, in $\tilde{W}^{1,m}(\Omega)$ such that

$$B(u_\delta) = F, \tag{3.4}$$

where F defined by

$$(F, v) = \int_\Omega f(x) v dx + \int_{\Sigma_2} a_i(x, 0, 0)\nu_i v ds. \ \forall v \in \tilde{W}^{1,m}(\Omega).$$

It is obvious that $F \in W^{-1,m'}$. In the following we employ the pseudo-monotone operator method to solve (3.4). By (A2), it follows that

$$|a_{i\delta}(x, z, p)| \le (c + n\delta)\big(|p|^{m-1} + |z|^{l/m'} + \varphi_1(x)\big). \tag{3.5}$$

Let $\xi, \zeta \in R^n$, one can deduce, by (A3), that

$$\begin{aligned}
&[a_{i\delta}(x, z, \xi) - a_{i\delta}(x, z, \zeta)](\xi_i - \zeta_i)\\
&= [a_i(x, z, \xi) - a_i(x, z, \zeta)](\xi_i - \zeta_i) + \delta(\xi_i|\xi_i|^{m-2} - \zeta_i|\zeta_i|^{m-2})(\xi_i - \zeta_i)\\
&= \int_0^1 \frac{\partial a_i}{\partial p_j}(x, z, \zeta + t(\xi - \zeta))(\xi_i - \zeta_i)(\xi_j - \zeta_j)dt \\
&\quad +\delta(\xi_i|\xi_i|^{m-2} - \zeta_i|\zeta_i|^{m-2})(\xi_i - \zeta_i)\\
&\ge \delta(|\xi_i|^{m-1} - |\zeta|^{m-1})(|\xi_i| - |\zeta_i|) > 0, \text{ for } \xi \ne \zeta.
\end{aligned} \tag{3.6}$$

Using (A4), we have

$$a_{i\delta}(x, z, p)p_i + a(x, z)z \ge \delta \sum_{i=1}^n |p_i|^m - \varphi_3(x) \ge \delta 2^{(1-m)n}|p|^m - \varphi_3(x). \tag{3.7}$$

Inequalities (3.5)-(3.7) show that the operator B in $B(u) = F$ is pseudo-monotone(see Theorem 1 in [1]). Moreover, B is coercive. In fact, by (3.7) and the Poincaré inequality, we obtain

$$\begin{aligned}
\int_\Omega [a_{i\delta}(x, u, Du)D_i u + a(x, u)u]dx &\ge \delta 2^{(1-m)n} \int_\Omega |Du|^m dx - \int_\Omega \varphi_3(x)dx\\
&\ge c\|u\|^m - \int_\Omega \varphi_3(x)dx,
\end{aligned}$$

for $u \in \tilde{W}^{1,m}(\Omega)$, where $c = c(m, n, |\Omega|, \delta)$. Hence,

$$(B(u), u)\|u\|^{-1} \ge c\|u\|^{m-1} - \|u\|^{-1} \int_\Omega \varphi_3(x)dx \to +\infty, \ as \ \|u\| \to +\infty,$$

which says that the pseudo-monotone operator B is coercive. Therefore, the equation $B(u) = F$ has a solution, say u_δ, in $\tilde{W}^{1,m}(\Omega)$. This completes the proof of Lemma 3.1. □

Let $A_\alpha(x,\eta,\xi)$, $|\alpha| \le m$, be the functions defined in $\Omega \times R^{N_1} \times R^{N_2}$ and satisfy the following conditions: they are continuous in (η,ξ) for a.e. $x \in \Omega$ and measurable in x for every $(\eta,\xi) \in R^{N_1} \times R^{N_2}$ where N_1 is the number of multi-index α with $|\alpha| \le m-1$ and N_2 is that of α with $|\alpha| = m$. Moreover,

$$|A_\alpha(x,\eta,\xi)| \le C(|\eta|^{p-1} + |\xi|^{p-1} + k(x)),\ k(x) \in L^{p'}(\Omega), 1 < p < \infty.$$

We recall the following lemma.

Lemma 3.2 (Lemma 2.1 of Chapter 2 in [5]) *Assume that $u_\mu \to u$ with $u \in W^{m-1,p}(\Omega)$, and $v \in W^{m,p}(\Omega)$. Then,*

$$A_\alpha(x,\delta u_\mu, D^m v) \to A_\alpha(x,\delta u, D^m v) \text{ strongly in } L^{p'}(\Omega),$$

where $A_\alpha \in L^{p'}(\Omega)$ and $\delta u = \{u, Du, \cdots, D^{m-1}u\}$.

Proof of Theorem 2.7 First we estimate a uniform bound for $u_\delta, 0 < \delta \le 1$. Let u_δ take the place of u and v in (3.1), then (3.1) reads

$$\begin{aligned}\int_\Omega [a_{i\delta}(x,u_\delta,Du_\delta)D_i u_\delta + a(x,u_\delta)u_\delta]dx - \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i u_\delta ds \\ = \int_\Omega f(x)u_\delta dx + \int_{\Sigma_2} a_i(x,0,0)\nu_i u_\delta ds. \quad (3.8)\end{aligned}$$

By condition (A4), it follows that

$$a_{i\delta}(x,u_\delta,Du_\delta)D_i u_\delta + a(x,u_\delta)u_\delta \ge \delta 2^{(1-m)n}|Du_\delta|^m + h(|Du_\delta|)|Du_\delta|^m - \varphi_3(x),$$

and then combining this with (3.8) yields

$$\begin{aligned}\int_\Omega h(|Du_\delta|)|Du_\delta|^m dx \le & \int_\Omega |Du_\delta|^m dx + \int_\Omega \varphi_3(x)dx + \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i u_\delta ds \\ & + \int_{\Sigma_2} a_i(x,0,0)\nu_i u_\delta ds + \int_\Omega f(x)u_\delta dx. \quad (3.9)\end{aligned}$$

For the right-hand side of (3.9), we have the following estimates

$$\begin{aligned}&\Big|\int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i u_\delta ds\Big| \\ &\le c\int_{\Sigma_1}(|u_\delta|^{l/m'} + |\varphi_1|)|u_\delta|ds \quad \text{by condition (A2)} \\ &\le c(\|u_\delta\|^{l/m'}_{L^l(\Sigma_1)} + \|\varphi_1\|_{m'})\|u_\delta\|_{L^m(\Sigma_1)} \quad \text{by Hölder's inequality} \\ &\le c(\|Du_\delta\|^{l/m'}_m + \|\varphi_1\|_{m'})\|Du_\delta\|_m \\ &\qquad \text{by the trace imbedding and Poincaré inequality} \\ &\le c(\frac{1}{m'} + \frac{2}{m})\|Du_\delta\|^m_m + \frac{1}{m'}\|\varphi_1\|^{m'}_{m'} \\ &\qquad \text{by Young's inequality and } L^m(\Omega) \hookrightarrow L^l(\Omega) \\ &\le \varepsilon\|Du_\delta\|^m_m + M_1. \quad \text{by Young's inequality}\end{aligned}$$

where $M_1 = M_1(m, m', n, |\Omega|, \|\varphi_1\|_{m'}, \varepsilon)$ and $\varepsilon > 0$ is any real number. Similarly,

$$\Big| \int_\Omega |Du_\delta|^m dx + \int_{\Sigma_2} a_i(x,0,0)\nu_i u_\delta ds + \int_\Omega f(x) u_\delta dx \Big| \le \varepsilon \|u_\delta\|_m^m + M_2,$$

where $M_2 = M_2(\|f\|_{m'}, \|\varphi_1\|_{m'}, m', |\Omega|, n, \varepsilon)$. Then, from (3.9) we get

$$\int_\Omega h(|Du_\delta|)|Du_\delta|^m dx \le 2\varepsilon \|Du_\delta\|_m^m + M, \tag{3.10}$$

where $M = M_1 + M_2 + \|\varphi_3\|_{L^1(\Omega)}$. Let

$$\Omega(\delta^*) = \{x \in \Omega \mid h(|Du_\delta|) \le \delta^*\}, \ \alpha(\delta^*) = \sup_{h(\eta)\le\delta^*} \eta, \ 0 < \delta^* \le 1.$$

Obviously, $|Du_\delta| \le \alpha(\delta^*)$ when $x \in \Omega(\delta^*)$, then

$$\int_{\Omega(\delta^*)} |Du_\delta|^m dx \le \alpha^m(\delta^*)|\Omega|.$$

With the aid of (3.10), we have

$$\delta^* \int_{\Omega\setminus\Omega(\delta^*)} |Du_\delta|^m dx \le 2\varepsilon \|Du_\delta\|_m^m + M.$$

Therefore,

$$\delta^* \int_\Omega |Du_\delta|^m dx \le 2\varepsilon \|Du_\delta\|_m^m + \delta^* \alpha^m(\delta^*)|\Omega| + M.$$

Choosing $\delta^* = 1$, $\varepsilon = 1/4$ in above inequality yields $\|Du_\delta\|_m^m \le 2\alpha^m(1)|\Omega| + 2M$. Then, by Poincaré inequality, we finally obtain the uniform bound of $\{u_\delta\}$ that

$$\|u_\delta\| \le c, \tag{3.11}$$

where c is independent of δ. Hence, there is a subsequence of $\{u_\delta\}$, denoted still by $\{u_\delta\}$, converging weakly to an element $u \in \tilde{W}^{1,m}(\Omega)$. Replacing u and v in (3.1) by u_δ and $u_\delta - v$ respectively, then (3.1) reads

$$\begin{aligned} \int_\Omega [a_{i\delta}(x,u_\delta,Du_\delta)D_i(u_\delta - v) + a(x,u_\delta)(u_\delta - v)]dx - \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - v)ds \\ = \int_\Omega f(x)(u_\delta - v)dx + \int_{\Sigma_2} a_i(x,0,0)\nu_i(u_\delta - v)ds. \end{aligned}$$

Substituting (3.6) on the above equality yields

$$\begin{aligned} \int_\Omega f(x)(u_\delta - v)dx + \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - v)ds + \int_{\Sigma_2} a_i(x,0,0)\nu_i(u_\delta - v)ds \\ - \int_\Omega a_{i\delta}(x,u_\delta,Dv)D_i(u_\delta - v)dx - \int_\Omega a(x,u_\delta)(u_\delta - v)dx \\ \ge \delta \int_\Omega (|D_i u_\delta|^{m-1} - |D_i v|^{m-1})(|D_i u_\delta| - |D_i v|)dx. \end{aligned} \tag{3.12}$$

Next, we consider the convergence of the right hand side of (3.12) as $\delta \to 0$, in three steps.

step 1: Since $u_\delta \to u$ weakly in $\tilde{W}^{1,m}(\Omega)$ and because of the trace imbedding $W^{1,m}(\Omega) \hookrightarrow L^m(\Sigma_1)$, we can assume, choose a subsequence if necessary, that $u_\delta \to u$ weakly in $L^m(\Sigma_1)$. Therefore, noticing (3.11), we have

$$\int_\Omega f(x)(u_\delta - v)dx \to \int_\Omega f(x)(u - v)dx,$$
$$\int_{\Sigma_2} a_{i\delta}(x,0,0)\nu_i(u_\delta - v)ds \to \int_{\Sigma_2} a_i(x,0,0)\nu_i(u - v)ds.$$

step 2:

$$\begin{aligned}
&\int_\Omega a_{i\delta}(x,u_\delta,Dv)D_i(u_\delta - v)dx \\
&= \int_\Omega [a_{i\delta}(x,u_\delta,Dv) - a_{i\delta}(x,u,Dv)]D_i(u_\delta - v)dx \\
&\quad + \int_\Omega a_{i\delta}(x,u,Dv)D_i(u_\delta - v)dx = I_1 + I_2. \qquad (3.13)
\end{aligned}$$

Because $W^{1,m}(\Omega) \hookrightarrow L^m(\Omega)$ is compact, then $u_\delta \to u$ strongly in $L^m(\Omega)$. By lemma 3.2 and (3.11) we know that $a_{i\delta}(x,u_\delta,Dv)$ tends to $a_i(x,u,Dv)$ strongly in $L^{m'}(\Omega)$. This and (3.11) show that

$$|I_1| \le \|a_{i\delta}(x,u_\delta,Dv) - a_{i\delta}(x,u,Dv)\|_{m'}\|D_i(u_\delta - v)\|_m \to 0,$$

then $I_1 \to 0$. It is obvious that $I_2 \to \int_\Omega a_i(x,u,Dv)D_i(u - v)dx$ since $u_\delta \to u$ weakly in $\tilde{W}^{1,m}(\Omega)$ and (3.11). Therefore, (3.13) yields

$$\int_\Omega a_{i\delta}(x,u_\delta,Dv)D_i(u_\delta - v)dx \to \int_\Omega a_i(x,u,Dv)D_i(u - v)dx.$$

For the integral

$$\int_\Omega a(x,u_\delta)(u_\delta - v)dx = \int_\Omega a(x,u_\delta)(u_\delta - u)dx + \int_\Omega a(x,u_\delta)(u - v)dx = I_3 + I_4,$$

by the compact imbedding $W^{1,m}(\Omega) \hookrightarrow L^l(\Omega)$ it holds that $u_\delta \to u$ strongly in $L^l(\Omega)$, then, noticing (3.11), we get

$$\begin{aligned}
|I_3| &\le \|a(x,u_\delta)\|_{l'}\|u_\delta - u\|_l \le (\|u_\delta\|_l^{l/l'} + \|\varphi_2\|_{l'})\|u_\delta - u\|_l \\
&\le (\|u_\delta\|_l^{l/l'} + \|\varphi_2\|_{l'})\|u_\delta - u\|_l \le c\|u_\delta - u\|_l \to 0. \qquad (3.14)
\end{aligned}$$

Because $u_\delta \to u$ strongly in $L^l(\Omega)$, then $u_\delta \to u$ almost everywhere in Ω. On the other hand, we have already know that $\|a(x,u_\delta)\|_{l'} \le c$, and hence $a(x,u_\delta) \to A(x)$ weakly in $L^{l'}(\Omega)$. Then, Based on Lemma 1.3 of Chapter 1 in [5], we have $A(x) = a(x,u)$, and hence

$$\int_\Omega a(x,u_\delta)(u - v)dx \to \int_\Omega a(x,u)(u - v)dx.$$

Combining this with (3.14) yields

$$\int_\Omega a(x,u_\delta)(u_\delta - v)dx \to \int_\Omega a(x,u)(u-v)dx.$$

step 3:

$$\begin{aligned}
&\int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - v)ds \\
&= \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u-v)ds + \int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - u)ds. \quad (3.15)
\end{aligned}$$

For the first integral in the above equation, based on the compact trace imbedding $W^{1,m}(\Omega) \hookrightarrow L^m(\Sigma_1)$ and $u_\delta \to u$ weakly in $W^{1,m}(\Omega)$ we know that $u_\delta \to u$ strongly in $L^m(\Sigma_1)$. Then, by Lemma 3.2 and noticing that $a_i(x,u_\delta,0) \in L^{m'}(\Sigma_1)$, it yields

$$\int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u-v)ds \to \int_{\Sigma_1} a_i(x,u,0)\nu_i(u-v)ds. \quad (3.16)$$

Now, consider the second integral of (3.15). Because the trace imbedding $W^{1,m}(\Omega) \hookrightarrow L^m(\Sigma_1)$ is compact, hence $\|u_\delta - u\|_{L^m(\Sigma_1)} \to 0$. Noticing that $a_i(x,u_\delta,0) \in L^{m'}(\Sigma_1)$ and (3.11), using Hölder's inequality and the compact trace imbedding $W^{1,m}(\Omega) \hookrightarrow L^l(\Sigma_1)$, it holds that

$$\begin{aligned}
&\Big|\int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - u)ds\Big| \\
&\le \|a_i(x,u_\delta,0)\|_{L^{m'}(\Sigma_1)}\|u_\delta - u\|_{L^m(\Sigma_1)} \\
&\le c(\|u_\delta\|_{L^l(\Sigma_1)}^{l/m'} + \|\varphi\|_{m'})\|u_\delta - u\|_{L^m(\Sigma_1)} \quad (3.17) \\
&\le c(\|u_\delta\|^{l/m'} + \|\varphi_1\|_{m'})\|u_\delta - u\|_{L^m(\Sigma 1)} \to 0.
\end{aligned}$$

Returning to (3.15), by (3.16) and (3.17), we have

$$\int_{\Sigma_1} a_i(x,u_\delta,0)\nu_i(u_\delta - v)ds \to \int_{\Sigma_1} a_i(x,u,0)\nu_i(u-v)ds.$$

Now, let $\delta \to 0$ in (3.12), by *1), 2), 3)* and (3.11), we obtain that

$$\begin{aligned}
&\int_\Omega [a_i(x,u,Dv)D_i(u-v) + a(x,u)(u-v)]dx - \int_{\Sigma_1} a_i(x,u,0)\nu_i(u-v)ds \\
&\le \int_\Omega f(x)(u-v)dx + \int_{\Sigma_2} a_i(x,0,0)\nu_i(u-v)ds, \quad \forall v \in \tilde{W}^{1,m}(\Omega)\,. \quad (3.18)
\end{aligned}$$

For any real number $\alpha > 0$ and any $\zeta(x) \in \tilde{W}^{1,m}(\Omega)$, choosing $v = u - \alpha\zeta(x)$ in (3.18) and then let $\alpha \to 0$, it yields

$$\int_\Omega [a_i(x,u,Du)D_i\zeta + a(x,u)\zeta]dx - \int_{\Sigma_1} a_i(x,u,0)\nu_i\zeta(x)ds \\ \le \int_\Omega f(x)\zeta dx + \int_{\Sigma_2} a_i(x,0,0)\nu_i\zeta ds. \quad (3.19)$$

The inverse inequality of (3.19) holds if $\alpha < 0$. This completes the proof of Theorem 2.7. □

Remark 3.3 If $m > n$ or $n = 2$, Assumption (A2) is replaced by the following assumption (A2) ′ which allows us to use the Sobolev imbedding $W^{1,m}(\Omega)$ to bounded and continuous function space $C_B(\omega)$ (see Chapter 7.7 in [2]). So the proof is easier than that of case $m < n$.

(A2)' $|a_i(x,z,p)| \le h_1(z)(|p|^{m-1} + k_1(x))$, $k_1(x) \in L^{m'}(\omega)$, for $i = 1,2,\cdots,n$. $|a(x,z)| \le h_2(x)k_2(x)$, $k_2(x) \in L^1(\omega)$, where $h_i(z)(i = 1,2)$ is a continuous function.

4 Comparison Principle and Uniqueness Theorem

We first consider the linear operator

$$Lu = -D_i(a^{ij}(x)D_ju + b^i(x)u) + cu, \quad x \in \omega, \quad (a^{ij}(x)) \ge 0. \quad (4.1)$$

Denote

$$\Sigma^0 = \{x \in \partial\Omega : a^{ij}(x)\nu_i\nu_j = 0\},$$
$$\Sigma_1 = \{x \in \Sigma^0 : b^i(x)\nu_i \le 0\},$$
$$\Sigma_2 = \Sigma^0 \setminus \Sigma_1, \quad \Sigma_3 = \partial\Omega \setminus \Sigma^0,$$
$$C^* = \{\varphi \in C^1(\bar\Omega) : \varphi \ge 0, \varphi|_{\Sigma_3} = 0\},$$

For $u \in \tilde{W}^{1,m}(\Omega)$ and $v \in C^*$, let

$$(Lu,\ v) = \int_\Omega [(a^{ij}D_ju + b^iu)D_iv + cuv]dx - \int_{\Sigma_1} b^i\nu_iuvds, \quad (4.2)$$

Definition 4.1 By a weak subsolution (supersolution) $u \in W^{1,m}(\Omega)(m \ge 2)$ of $Lu = 0$ in Ω, we mean that $(Lu,\ v) \ge 0$ (≤ 0) holds for all $\varphi \in C^*$.

Lemma 4.2 (Maximum Principle) *Suppose that the coefficients of* L *satisfy*

$$a^{ij}, c \in C(\bar\Omega), \quad b^i \in C^1(\bar\Omega), \quad b^i\nu_i\big|_{\Sigma_1} \le 0, \quad b^i\nu_i\big|_{\Sigma_2} > 0,$$
$$D_ib^i \le \min\{c,\ 2c\}, \quad \Sigma_1 \in C^1, \quad \Sigma_2 \cup \Sigma_3 \ne \emptyset.$$

If the minimum of the weak subsolution of $Lu = 0$ *is nonpositive (or the maximum of the weak supersolution is nonnegative), then it must be achieved on* $\overline{\Sigma_2 \cup \Sigma_3}$.

Proof Suppose that u is a weak subsolution of $Lu=0$, by definition 4.1, we have

$$\int_\Omega [(a^{ij}D_ju+b^iu)D_iv+cuv]dx-\int_{\Sigma_1} b^i\nu_i uvds\geq 0,\quad \forall v\in C^*. \tag{4.3}$$

Let $l=\inf_{\Sigma_2\cup\Sigma_3} u\leq 0$,

$$w=(l-u)^+=\begin{cases} l-u, & \text{when } u<l,\\ 0 & \text{otherwise.}\end{cases}$$

Then w belongs to the closure of C^* in the norm of Sobolev space $W^{1,m}(\omega)$, so we can choose w as a test function in (4.3) and obtain

$$\int_\Omega [(a^{ij}D_ju+b^iu)D_iw+cuw]\,dx-\int_{\Sigma_1} b^i\nu_i uw\,ds\geq 0, \tag{4.4}$$

Noting that $\Sigma_1\in C^1$ and $W^{1,m}(\Omega)$ can be imbedded into $L^{mn/(n-m)}(\Omega)\subset L^2(\Omega)$, the left hand side of (4.4) is well defined. Integrating the term b^iuD_jw in (4.4) by parts yields

$$\begin{aligned}\int_{\Omega_+} &[-a^{ij}D_iwD_jw+\frac{1}{2}(D_ib^i-2c)w^2-(D_ib^i-c)lw]dx\\ \geq&\ -\frac{1}{2}\int_{\Sigma_1} b^i\nu_i w^2ds+\int_{\Sigma_2}(\frac{1}{2}w^2-lw)b^i\nu_i\,ds \\ \geq&\ -\frac{1}{2}\int_{\Sigma_1} b^i\nu_i w^2ds\geq 0,\end{aligned} \tag{4.5}$$

where $\Omega_+=\{x\in\Omega: u<l\}$. On the other hand, from the assumption of Lemma 4.2, the integrand on Ω_+ is negative which implies $|\Omega_+|=0$, and hence

$$\inf_\Omega u\geq \inf_{\Sigma_2\cup\Sigma_3} u.$$

For the case of weak supersolution, let u is a weak supersolution of $Lu=0$, replacing u by $-u$ in the preceding arguments, we obtain

$$\sup_\Omega u\leq \sup_{\Sigma_2\cup\Sigma_3} u.$$

Thus the proof of Lemma 4.2 is completed. □

Remark 4.3 Replacing $\overline{\Sigma_2\cup\Sigma_3}$ by $\overline{\Sigma_3}$ in the proof of Lemma 4.2, we can deduce that $\inf_\Omega u\geq\inf_{\Sigma_3} u$ and $\sup_\Omega u\leq \sup_{\Sigma_3} u$ for the subsolution and the supersolution of $Lu=0$ respectively. Hence for the weak solution of $Lu=0$ with $u|_{\Sigma_3}=0$, it must be that $u|_{\Sigma_2}=0$.

Let $u,\ v\in \tilde{W}^{1,m}(\Omega)$, we say that $Qu\le Qv$ in the sense of distributions, if

$$\begin{aligned}&\int_\Omega [a_i(x,u,Du)D_i\varphi + a(x,u)\varphi]dx - \int_{\Sigma_1} a_i(x,u,0)\nu_i\varphi\, ds\\ \le\ &\int_\Omega [a_i(x,v,Dv)D_i\varphi + a(x,v)\varphi]dx - \int_{\Sigma_1} a_i(x,v,0)\nu_i\varphi\, ds\end{aligned}$$

holds for any $\varphi\in C^*$. Denote $u_t(x)=tu+(1-t)v(x), 0\le t\le 1$ and $a_{it}=a_i(x,\ u_t,\ Du_t)$ for $i=1,2,\cdots,n$. $a_t=a_i(x,u_t)$. Then, we have the following comparison principle.

Theorem 4.4 (Comparison Principle) *Suppose that* $a_i(x,z,p)\in C^1(\bar\Omega\times R\times R^n)\times C^2(\Omega\times R\times R^n)$, $a(x,z)\in C^1(\bar\Omega\times R)$, $\nu_i D_z a_{it}|_{\Sigma_1}\le 0$, $\nu_i D_z a_{it}|_{\Sigma_2}>0$, $D_{x_i z}a_{it}\le \min\{D_z a_t,\ 2D_z a_t\}$ *and (A3) hold. If* $u,v\in C^1(\bar\Omega)\cap \tilde{W}^{1,m}(\Omega)$ *satisfy* $Qu\le Qv$ *in* Ω *and* $u\le v$ *on* $\Sigma_2\cup\Sigma_3$, *then* $u\le v$ *in* Ω.

Proof By the condition $Qu\le Qv$, for any $\varphi\in C^*$, we have

$$\begin{aligned}0\ &\ge \int_0^1 dt\int_\Omega \{[D_{p_j}a_{it}D_j(u-v)+D_z a_{it}(u-v)]D_i\varphi + D_z a_t(u-v)]\varphi\}dx\\ &- \int_0^1 dt\int_{\Sigma_1} D_z a_i(x,u_t,0)(u-v)\nu_i\varphi ds\\ &= \int_\Omega\{[a^{ij}D_j(u-v)+b^i(u-v)]D_i\varphi + c(u-v)\varphi\}dx\\ &- \int_{\Sigma_1} b^i\nu_i(u-v)\varphi ds,\end{aligned}\tag{4.6}$$

where $a^{ij}=\int_0^1 D_{p_j}a_{it}dt,\ b^i=\int_0^1 D_z a_{it}dt,\ c=\int_0^1 D_z a_t dt$. Let $w=u-v$. From (4.6), we have

$$\int_\Omega [(a^{ij}D_j w + b^i w)D_i\varphi + cw\varphi]dx - \int_{\Sigma_1} b^i\nu_i w\varphi ds\ge 0,$$

i.e., w is a supersolution of the liner equation

$$Lu = -D_i(a^{ij}(x)D_j u + b^i(x)u) + cu = 0.$$

From the assumptions of this theorem, the assumptions of Lemma 4.2 are all satisfied, thus Lemma 4.2 implies that $\sup_\omega w\le \sup_{\Sigma_2\cup\Sigma_3} w\le 0$, hence $u\le v$ in ω which proves Theorem 4.4. □

Theorem 4.5 (Uniqueness Theorem) *Suppose that the coefficients of* Q *satisfy the conditions in Theorem 4.4, then the* $C^1(\bar\Omega)\cap\tilde{W}^{1,m}(\Omega)$*-weak solution of problem (2.6) is unique.*

Proof Suppose that u and v are two weak solutions of problem (2.6). By Remark 4.3, it holds that $u = v = 0$ on $\overline{\Sigma_2 \cup \Sigma_3}$. Using Theorem 4.4, we find that $u \leq v$ as well as $v \leq u$, hence $u = v$ in Ω. This completes the proof. □

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