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Travelling waves for a neural network *

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Abstract

In this note, we give another proof of existence and uniqueness of travelling waves for a neural network equations and prove that all travelling waves are monotonic.

1 Introduction

The following single-layer neural network over the real line was introduced by Ermentrout and Mcleod [6]:

$$u(x,t)=\int_{-\infty}^{t}ds\int_{-\infty}^{\infty}dyh(t-s)k(x-y)S(u(y,s)) \tag{1.1}$$

where $x\in\mathbb{R}$ and $t\in\mathbb{R}$; $u(x,t)$ is the mean membrane potential of a patch of tissue at position x and at time t; $S(u)$ is a nonlinear function and $S(u(x,t))$ is the firing rate; h and k are nonnegative functions defined $[0,\infty)$ and $\mathbb{R}$ respectively. When $h(t)=e^{-t}$ for $t>0$, then equation (1.1) is equivalent to the following differential equation:

$$\partial u(x,t)/\partial t+u(x,t)=k*S(u)(x,t), \tag{1.2}$$

where $k*S(u)$ denotes the convolution of k with $S(u)$, i.e., $k*S(u)(x,t)=\int_{-\infty}^{\infty}k(x-y)S(u(y,t))dy$.

The existence and uniqueness of travelling waves of (1.1) of the form $u(x,t)=\phi(x-ct)$ satisfying $\phi(-\infty)=0$ and $\phi(\infty)=1$ are established in [6], where ϕ is a smooth function, called the wave profile, and c is a constant, called the wave speed. A homotopy argument is employed to prove the existence, which has fostered other studies in similar topics (see [2, 3, 4, 5, 7, 8], for example). This note serves to supplement the results obtained in [6], by applying results in [7], where a comparison argument, together with constructions of appropriate super- and sub- solutions, is used to study travelling waves for (1.2).

First we state the conditions on h, k, and S. We assume that

(A1) $h\in C^1[0,\infty)$ is a positive function on $[0,\infty)$ with $\int_0^\infty h(t)dt=1$ and $\int_0^\infty th(t)dt<\infty$.

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(**A2**) k is a nonnegative, continuous function on $\mathbb{R}$ with $\int_{\mathbb{R}} k(x)dx = 1$, $k' \in L^1(\mathbb{R})$ and $\operatorname{supp} J \bigcap (0,\infty) \neq \emptyset \neq \operatorname{supp} J \bigcap (-\infty, 0)$.

(**A3**) $S \in C^1([0,1])$ satisfies that $S'(u) > 0$, for $u \in [0,1]$, and that $f(u) = -u + S(u)$ has precisely three zeros at $u = 0, a, 1$ satisfying $f'(0) < 0$ and $f'(1) < 0$, where $0 < a < 1$.

Under the above assumptions, we can improve the results in [6]:

Theorem 1.1. *Under the above assumptions on h, k and S, we have*

(a) There exists a travelling wave solution $u = \phi(x - ct)$ to (1.1) satisfying $\phi \in C^1$, $\phi(-\infty) = 0$ and $\phi(\infty) = 1$.

(b) Any travelling wave solution to (1.1) satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = 1$ is strictly increasing.

(c) Traveling wave solution to (1.1) is unique module spatial translation.

Remark 1.2. (a) The monotonicity of travelling wave solutions to (1.1) is established in [6] for special kernels h and k and is conjectured for general case. Our result gives a positive answer.

(b) For the existence and uniqueness in [6], that k is even and h is monotonically decreasing is assumed. While it is natural, we can relax these restrictions.

2 Proof of Theorem 1.1

First we need the following result:

Lemma 2.1. *[7] For any k and S satisfying (**A2**) and (**A3**) respectively, there exists one and only one (modulo spatial translation) travelling wave solution $u(x,t) = \phi(x - ct)$ to (1.2) satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. Moreover, $\phi' > 0$ for all $x \in \mathbb{R}$.*

For any $c \in \mathbb{R}$, let $J_c(\cdot) = \int_0^\infty h(s)k(\cdot + cs)ds$. Then J_c satisfies (**A2**). For each $c \in \mathbb{R}$, by Lemma 2.1, there is a travelling wave solution $\phi_c(x - \alpha(c)t)$ to the equation (1.2) with k replaced by J_c, where ϕ_c is the profile and $\alpha(c)$ is the wave speed, depending on c. Let $\xi = x - ct$. Then the pair $(\phi_c, \alpha(c))$ satisfies the following equations:

$$-\alpha(c)\phi_c'(\xi) + \phi_c(\xi) - J_c * S(\phi_c)(\xi) = 0, \tag{2.1}$$

$$\phi(-\infty) = 0, \text{ and } \phi(\infty) = 1. \tag{2.2}$$

On the other hand, a travelling wave solution $u = u(x - ct)$ to (1.1) satisfies

$$u(\xi) = J_c * S(u)(\xi). \tag{2.3}$$

Therefore, if (u, c) is a travelling wave solution to (1.1), $(u, 0)$ is a travelling wave solution to (1.2) corresponding to $k(x) = J_c(x)$. Similarly, if $(\phi_c, 0)$ is a travelling wave solution to (1.2) with $k(x) = J_c(x)$, then (ϕ_c, c) is a travelling wave solution to (1.1). Therefore to prove the existence of a travelling wave, we only need to prove that there is a $c \in \mathbb{R}$ such that $\alpha(c) = 0$. To that end, we need:

Lemma 2.2. *The wave speed $\alpha(\cdot)$ is a continuous function on $\mathbb{R}$.*

Proof. Let $c_0 \in \mathbb{R}$ and $(\phi_{c_0}, \alpha(c_0))$ be a travelling wave solution to (1.2) corresponding to $k = J_{c_0}$. Then, $\phi_c' > 0$ for all $x \in \mathbb{R}$ and $(\phi_c, \alpha(c))$ can be obtained as a solution to (2.1) by the Implicit Function Theorem, applying in the neighborhood of c_0 (see [6], for example). Therefore, $\phi(c)$ is indeed continuously differentiable. □

Lemma 2.3. *$\alpha(c) < 0$ for c positively sufficiently large and $\alpha(c) > 0$ for c negatively sufficiently large.*

Proof. We only prove the lemma when c is positive. The other case can be proved similarly. We can choose $z_0 \in (0, 1)$ such that $\epsilon_0 = S(z_0) - z_0 > 0$. For this ϵ_0, we can choose two positive constants $A = A(\epsilon_0)$ and $B = B(\epsilon_0)$ such that $(\int_0^A + \int_B^\infty) h(s) ds < \epsilon_0/8$ and $(\int_{-\infty}^{-B} + \int_B^\infty) k(s) ds < \epsilon_0/8$. Since $(\phi_c, \alpha(c))$ satisfies (2.1), we have

$$\begin{aligned} &- \alpha(c)\phi_c'(x) + \phi_c(x) - S(\phi_c)(x) \\ &= \int_0^\infty h(s) \int_{-\infty}^\infty k(x + cs - y)\{S(\phi_c(y)) - S(\phi_c(x))\} dy\, ds \\ &\geq \int_A^B h(s) \int_{x+cs-B}^{x+cs+B} k(x + cs - y)\{S(\phi_c(y)) - S(\phi_c(x))\} dy\, ds - \epsilon_0/2 \end{aligned} \tag{2.4}$$

where we have used the fact that $S(u(x)) \leq 1$. If $c \geq A^{-1}B$, then $y > x$ for y in the range of the integration on the right of (2.4). Therefore the integral on the right side of (2.4) is positive and

$$-\alpha(c)\phi_c'(x) + \phi_c(x) - S(\phi_c)(x) > -\epsilon_0/2. \tag{2.5}$$

Since $\phi_c(-\infty) = 0$, and $\phi_c(\infty) = 1$, we choose x_0 such that $\phi_c(x_0) = z_0$, Then we deduce from (2.5) that $\alpha(c)\phi_c'(x_0) < 0$. Therefore, $\alpha(c) < 0$ since $\phi_c'(x_0) > 0$. □

Proof of Theorem 1.1 By lemma 2.2 and 2.3, there is constant c such that $\alpha(c) = 0$. The pair (ϕ_c, c) is the travelling wave solution to (1.1). By lemma 2.1, $\phi_c' > 0$ for all x. The uniqueness is established in [6], where uniqueness for monotonic travelling waves is proved. □

References

[1] P. W. Bates and F. Chen, Periodic travelling wave solutions of an integrodifferential model for phase transition, *Electronic J. Differential Equations*, **26** (1999) 1-19.

[2] P. W. Bates and A. Chmaj, A discrete convolution model for phase transitions, *Arch. Rational Mech. Anal.*, **150** (1999), 281-305.

[3] P. W. Bates, P. C. Fife, X. Ren, and X. Wang, Traveling waves in a nonlocal model of phase transitions, *Arch. Rat. Mech. Anal.,* **138** (1997), 105-136.

[4] F. Chen, Almost Periodic Traveling Waves of Nonlocal Evolution Equations *J. Nonlinear Anal.*,**50** (2002), 807–838.

[5] Z. Chen, B. Ermentrout, B. McLeod, Traveling fronts for a class of nonlocal convolution differential equations. *Appl. Anal.* **64** (1997), no. 3-4, 235–253.

[6] B. Ermentrout, B. McLeod, Existence and uniqueness of travelling waves for a neural network. Proc. *Roy. Soc. Edinburgh Sect. A* **123** (1993), no. 3, 461–478.

[7] X. Chen, Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations, *Adv. Diff. Eqs.,* **2** (1997), 125-160.

[8] A. de Masi, E. Orlandi, E. Presutti, and L. Triolo, Uniqueness of the instanton profile and global stability in nonlocal evolution equations, *Rend. Math.,* **14** (1994), 693-723.

[9] M. Katsoulakis and P. S. Souganidis, Interacting particle systems and generalized mean curvature evolution, *Arch. Rat Mech. Anal.,* **127** (1994), 133-157.

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