Electronic Journal of Differential Equations, Vol. **2003**(2003), No. 14, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

A new theorem on exponential stability of periodic evolution families on Banach spaces *

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Abstract

We consider a mild solution $v_f(\cdot,0)$ of a well-posed inhomogeneous Cauchy problem $\dot{v}(t) = A(t)v(t) + f(t)$, $v(0) = 0$ on a complex Banach space X, where $A(\cdot)$ is a 1-periodic operator-valued function. We prove that if $v_f(\cdot,0)$ belongs to $AP_0(\mathbb{R}_+, X)$ for each $f \in AP_0(\mathbb{R}_+, X)$ then for each $x \in X$ the solution of the well-posed Cauchy problem $\dot{u}(t) = A(t)v(t)$, $u(0) = x$ is uniformly exponentially stable. The converse statement is also true. Details about the space $AP_0(\mathbb{R}_+, X)$ are given in the section 1, below. Our approach is based on the spectral theory of evolution semigroups.

1 Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms of vectors in X and of operators in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. Let $\mathbb{R}_+$ the set of all non-negative real numbers and let $\mathbb{J}$ be either $\mathbb{R}$ or $\mathbb{R}_+$. The Banach space of all X-valued, bounded and uniformly continuous functions on $\mathbb{J}$ will be denoted by $BUC(\mathbb{J}, X)$, and the Banach space of all X-valued, almost periodic functions on $\mathbb{J}$ will be denoted by $AP(\mathbb{J}, X)$. It is known that $AP(\mathbb{J}, X)$ is the smallest closed subspace of $BUC(\mathbb{J}, X)$ containing functions of the form

$$t \mapsto f_{\mu,x}(t) := e^{i\mu t}x : J \to X, \quad \mu \in \mathbb{R}, \quad x \in X,$$

see e.g. [14]. The set of all X-valued functions on $\mathbb{R}_+$ for which there exist $t_f \geq 0$ and $F_f \in AP(\mathbb{R}_+, X)$ such that $f(t) = 0$ if $t \in [0, t_f]$ and $f(t) = F_f(t)$ if $t \geq t_f$ will be denoted by $\mathcal{A}_0(\mathbb{R}_+, X)$. Let $AP_0(\mathbb{R}_+, X)$ the smallest closed subspace of $BUC(\mathbb{R}_+, X)$ which contains $\mathcal{A}_0(\mathbb{R}_+, X)$. The subspace of $BUC(\mathbb{J}, X)$ consisting of all X-valued, continuous, 1-periodic functions such that $f(0) = 0$ will be denoted by $P_1^0(\mathbb{J}, X)$. An X-valued, trigonometric polynomial function is given by

$$t \mapsto p(t) := \sum_{k=-n}^{n} c_k e^{i\mu_k t} x_k : \mathbb{R} \to X, \quad c_k \in \mathbb{C}, \mu_k \in \mathbb{R}, x_k \in X.$$

* *Mathematics Subject Classifications:* 26D10, 34A35, 34D05, 34B15, 45M10, 47A06.
Key words: Almost periodic functions, exponential stability, periodic evolution families of operators, integral inequality, differential inequality on Banach spaces.

Submitted November 13, 2002. Published February 11, 2003.

The set of all functions f on $\mathbb{R}_+$ for which there exist $t_f \geq 0$ and a X-valued, trigonometric polynomial function p_f such that $f(t) = 0$ if $t \in [0, t_f]$ and $f(t) = p_f(t)$ if $t \geq t_f$ will be denoted by $TP_0(\mathbb{R}_+, X)$. It is clear that $TP_0(\mathbb{R}_+, X)$ is a subset of $\mathcal{A}_0(\mathbb{R}_+, X)$ and $P_1^0(\mathbb{R}_+, X)$ is the closure in $BUC(\mathbb{R}_+, X)$ of a part of $TP_0(\mathbb{R}_+, X)$. Let $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a strongly continuous semigroup on X and $A : D(A) \subset X \to X$ its infinitesimal generator. It is well known that the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad t \geq 0 \\ u(0) &= x, \quad x \in X \end{aligned} \tag{1.1}$$

is well-posed (see [22, 23, 15] and the references therein for the well-posednness of abstract differential equations) and the mild solution of (1.1) is given by $u(t) = T(t)x, (t \geq 0)$. Moreover, for a locally Bochner integrable function $f : \mathbb{R}_+ \to X$, the mild solution of the inhomogeneous Cauchy Problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t), \quad t \geq 0, \\ u(0) &= y, \quad y \in X \end{aligned}$$

is given by

$$u_f(t, y) = T(t)y + \int_0^t T(t - \xi) f(\xi) d\xi, \quad t \geq 0.$$

In particular the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + e^{i\mu t}x \quad t \geq 0, \\ u(0) &= 0\,, \end{aligned}$$

where $\mu \in \mathbb{R}$ and $x \in X$, has the solution

$$u_f(t, 0) = u_{\mu,x}(t) = \int_0^t T(t - \xi) e^{i\mu\xi} x d\xi, \quad t \geq 0.$$

The Datko-Neerven's theorem ([8, 18]) states that *a strongly continuous semigroup* $\mathbf{T} = \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ *is exponentially stable*, that is, there exist the constants $N > 0$ and $\nu > 0$ such that

$$\|T(t)\| \leq Ne^{-\nu t} \quad \text{for all } t \geq 0,$$

if and only if it acts boundedly on one of the spaces $L^p(\mathbb{R}_+, X)$ *or* $C_0(\mathbb{R}_+, X)$ *by convolution.* With other words if $\mathcal{X}$ is one of the spaces $L^p(\mathbb{R}_+, X)$ or $C_0(\mathbb{R}_+, X)$ then the strongly continuous semigroup $\mathbf{T}$ is exponentially stable if and only if for each function $f \in \mathcal{X}$ the solution $u_f(\cdot, 0)$ belongs to $\mathcal{X}$. Here $C_0(\mathbb{R}_+, X)$ is the space consisting of all X-valued, continuous functions vanishing at infinity, endowed with the sup-norm, and $L^p(\mathbb{R}_+, X)$, $1 \leq p < \infty$, denotes the usual Lebesgue-Bochner space of all measurable functions $f : \mathbb{R}_+ \to X$ identifying functions which are equal almost everywhere and such that

$$\|f\|_p := \Big(\int_0^\infty \|f(s)\|^p ds \Big)^{1/p} < \infty.$$

When X is a complex Hilbert space the Neerven-Vu's theorem ([19, 20, 24]) states that *the strongly continuous semigroup* $\mathbf{T}$ *on* X *is exponentially stable if and only if*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\|u_{\mu,x}(t)\| = M(x) < \infty \quad \text{for each } x\in X. \tag{1.2}$$

In fact Neerven and Vu showed that if (1.2) holds then the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$ exists and is uniformly bounded in $\{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. This result is valid for semigroups defined on Banach spaces. The Gearhart-Prüss-Herbst-Howland's theorem (see [10, 11, 12, 13, 21, 25]) states that for semigroups on Hilbert spaces the uniform boundedness of the resolvent in $\{\mathrm{Re}(\lambda) > 0\}$ implies the exponential stability. A short history of these results and many more details about their relationships with abstract differential equations can be found in [2, 4, 24].

For a well-posed, non-autonomous Cauchy problem

$$\begin{aligned} \dot u(t) &= A(t)u(t), \quad t\geq 0,\\ u(0) &= x, \quad x\in X \end{aligned} \tag{1.3}$$

with (possibly unbounded) linear operators $A(t)$, the mild solutions lead to an evolution family on $\mathbb{R}_+$, $\mathcal{U} = \{U(t,s) : \ t\geq s\geq 0\}\subset\mathcal{L}(X)$, that is:

(e_1) $U(t,r) = U(t,s)U(s,r)$ for all $t\geq s\geq r\geq 0$ and $U(t,t) = I$ for any $t\geq 0$, (I is the identity operator in $\mathcal{L}(X)$);

(e_2) the maps $(t,s)\mapsto U(t,s)x : \{(t,s) : t\geq s\geq 0\}\to X$ are continuous for each $x\in X$.

An evolution family is *exponentially bounded* if there exist $\omega\in\mathbb{R}$ and $M_\omega > 0$ such that

$$\|U(t,s)\| \leq M_\omega e^{\omega(t-s)}, \text{ forall } t\geq s\geq 0, \tag{1.4}$$

and *exponentially stable* if (1.4) holds with some negative ω. If the evolution family $\mathcal{U}$ verifies the condition

(e_3) $U(t,s) = U(t-s,0)$ for all $t\geq s\geq 0$,

then the family $\mathbf{T} = \{U(t,0) : t\geq 0\}\subset\mathcal{L}(X)$ is a strongly continuous semigroup on X. In this case the estimate (1.4) holds automatically. If the Cauchy problem (1.3) is 1-periodic, that is, $A(t{+}1) = A(t)$ for every $t\geq 0$, then the corresponding evolution family $\mathcal{U}$ is 1-periodic, that is,

(e_4) $U(t+1, s+1) = U(t,s)$ for all $t\geq s\geq 0$.

Every 1-periodic evolution family is exponentially bounded, see for example ([5], Lemma 4.1). For a locally Bochner integrable function $f:\mathbb{R}_+\to X$, the mild solution of the well-posed, inhomogeneous Cauchy problem

$$\begin{aligned} \dot v(t) &= A(t)v(t) + f(t), \quad t\geq 0,\\ u(0) &= x \end{aligned}$$

is given by

$$v_f(t,x) := U(t,0)x + \int_0^t U(t,\tau)f(\tau)d\tau, \quad (t \geq 0).$$

We also consider evolution families on the line. We shall use the same notations as in the case of evolution families on $\mathbb{R}_+$ except that the variables s and t can take any value in $\mathbb{R}$. For more details about the strongly continuous semigroups and evolution families we refer to [9]. The Datko-Neerven's theorem can be extended for evolution families in the both cases, on the line and on the half-line, see the papers ([8, Theorem 6], [17, Theorem 2.2], [7], or the monograph [6]. It seems that the Neerven-Vu's theorem cannot be extended for periodic evolution families, but some weaker results, which will be described as follows, hold.

We recall the notion of evolution semigroup. For more details we refer to [6, 7] and references therein. Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be a 1-periodic evolution family, $t \geq 0$, and $G \in AP(\mathbb{R}, X)$. The function given by

$$s \mapsto (\mathcal{S}(t)G)(s) := U(s, s-t)G(s-t) : \mathbb{R} \to X, \tag{1.5}$$

belongs to $AP(\mathbb{R}, X)$ and the one-parameter family $\mathcal{S} = \{\mathcal{S}(t) : t \geq 0\}$ is a strongly continuous semigroup on $AP(\mathbb{R}, X)$, see for example [16]. $\mathcal{S}$ is called an evolution semigroup on $AP(\mathbb{R}, X)$.

2 Results

Lemma 2.1 *Let $f \in AP_0(\mathbb{R}_+, X)$, $\tau \geq 0$ and $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\} \subset \mathcal{L}(X)$ be a 1-periodic evolution family of bounded linear operators on X. Then the function $S(\tau)f$ given by*

$$[S(\tau)f](s) := \begin{cases} U(s, s-\tau)f(s-\tau), & \text{if } s \geq \tau \\ 0, & \text{if } 0 \leq s < \tau \end{cases} \tag{2.1}$$

belongs to $AP_0(\mathbb{R}_+, X)$.

Proof First we prove that $S(\tau)g \in \mathcal{A}_0(\mathbb{R}_+, X)$ for any g in $\mathcal{A}_0(\mathbb{R}_+, X)$. Let t_g and F_g (as in the definition of the set $\mathcal{A}_0(\mathbb{R}_+, X)$). Let $t_{S(\tau)g} := \tau + t_g$ and $F_{S(\tau)g} := U(\cdot, \cdot - \tau)F_g(\cdot - \tau)$. If $\tau \leq s \leq \tau + t_g$ then $g(s-\tau) = 0$ and $[S(\tau)g](s) = 0$. Moreover, if $s \geq \tau + t_g$ then $g(s-\tau) = F_g(s-\tau)$ and therefore $[S(\tau)g](s) = F_{S(\tau)g}(s)$. Thus $S(\tau)g \in \mathcal{A}_0(\mathbb{R}_+, X)$. Let now $\varepsilon > 0$ and $g \in \mathcal{A}_0(\mathbb{R}_+, X)$ such that $\|f - g\|_{BUC(\mathbb{R}_+, X)} < \varepsilon$. Then $S(\tau)g$ belongs to $\mathcal{A}_0(\mathbb{R}_+, X)$, and

$$\begin{aligned}\|S(\tau)f - S(\tau)g\|_{BUC(\mathbb{R}_+, X)} &= \sup_{s \geq \tau} \|U(s, s-\tau)[f(s-\tau) - g(s-\tau)]\| \\ &\leq Me^{\omega\tau} \sup_{s \geq \tau} \|f(s-\tau) - g(s-\tau)\| \leq Me^{\omega\tau}\varepsilon.\end{aligned}$$

This completes the proof. □

Now, it is easy to see, that the family $\{S(\tau) : \tau \geq 0\}$ is a semigroup of linear and bounded operators on $AP_0(\mathbb{R}_+, X)$.

Lemma 2.2 *Let $\mathcal{U}$ be an 1-periodic evolution family of bounded linear operators on X. The semigroup $\mathbf{S} = \{S(t) : t \geq 0\}$ associated to $\mathcal{U}$ on $AP_0(\mathbb{R}_+, X)$), defined in* (2.1) *is strongly continuous.*

Proof For each $f \in AP_0(\mathbb{R}_+, X)$ and any $\tau \geq 0$, we have

$$\begin{aligned} \|S(\tau)f - f\|_{AP_0(\mathbb{R}_+,X)} &\leq \sup_{s\in[t_f,\tau+t_f]} \|f(s)\| + \sup_{s\in[t+t_f,\infty)} \|(S(\tau)f)(s) - F_f(s)\| \\ &\leq \|\mathcal{S}(\tau)F_f - F_f\|_{AP(\mathbb{R},X)} + \sup_{s\in[t_f,\tau+t_f]} \|f(s)\|. \end{aligned}$$

The last term tends to 0 when $\tau \to 0$, because the semigroup $\mathcal{S}$ (which is defined in (1.5)) is strongly continuous, the function f is uniformly continuous on $\mathbb{R}_+$, and $f(t_f) = 0$. □

The semigroup $\mathbf{S}$ is called an *evolution semigroup associated to $\mathcal{U}$ on the space* $AP_0(\mathbb{R}_+, X)$. The main result of the our paper is the following theorem.

Theorem 2.3 *Let $\mathcal{U} = \{U(t, s) : t \geq s \in \mathbb{R}\}$ be an 1-periodic evolution family of bounded linear operators acting on X. The following two statements are equivalent:*

(i) $\mathcal{U}$ is exponentially stable;

(ii) $v_f(\cdot, 0)$ belongs to $AP_0(\mathbb{R}_+, X)$ for each $f \in AP_0(\mathbb{R}_+, X)$.

The following Lemma is the key tool in our proof of (i) implies (ii) from Theorem 2.3.

Lemma 2.4 *Let $\mathcal{U} = \{U(t, s) : t \geq s \in \mathbb{R}\}$ be a* 1*-periodic evolution family of bounded linear operators on X, $\mathbf{S} = \{S(t) : t \geq 0\}$ the evolution semigroup associated to $\mathcal{U}$ on the space $AP_0(\mathbb{R}_+, X)$, defined in* (2.1)*, $(G, D(G))$ the infinitesimal generator of $\mathbf{S}$ and u and $f \in AP_0(\mathbb{R}_+, X)$. The following two statements are equivalent:*

(j) $u \in D(G)$ and $Gu = -f$;

(jj) $u(t) = \int_0^t U(t, s)f(s)ds$ for all $t \geq 0$.

Proof This Lemma can be shown as in [17, Lemma 1.1]. For sake of completness we present the details.

(j) implies (jj). For each $t \geq 0$, $S(t)u - u = \int_0^t S(\xi)Gud\xi$; therefore,

$$\begin{aligned}
u(t) &= (S(t)u)(t) - (\int_0^t S(\xi)Gu\,d\xi)(t) \\
&= U(t,0)u(0) - \int_0^t U(t,t-\xi)(Gu)(t-\xi)\,d\xi \\
&= \int_0^t U(t,t-\xi)f(t-\xi)\,d\xi = \int_0^t U(t,\tau)f(\tau)\,d\tau.
\end{aligned}$$

(jj) implies (j). Let $t > 0$ be fixed. We prove that

$$\frac{1}{t}(-S(t)u + u) = \frac{1}{t}\int_0^t S(r)f dr. \tag{2.2}$$

If $s \geq t$, we have:

$$\begin{aligned}
\frac{1}{t}(-S(t)u+u)(s) &= \frac{1}{t}[-U(s,s-t)u(s-t)+u(s)] \\
&= \frac{1}{t}[\int_0^s U(s,\tau)f(\tau)d\tau - \int_0^{s-t} U(s,\tau)f(\tau)d\tau] \\
&= \frac{1}{t}\int_0^t U(s,s-r)f(s-r)dr \\
&= \frac{1}{t}(\int_0^t S(r)f dr)(s).
\end{aligned}$$

If $0 \leq s < t$, we have

$$\begin{aligned}
\frac{1}{t}(-S(t)u+u)(s) = \frac{1}{t}u(s) &= \frac{1}{t}\int_0^s U(s,\tau)f(\tau)d\tau \\
&= \frac{1}{t}\int_0^s U(s,s-r)f(s-r)dr \\
&= \frac{1}{t}(\int_0^s S(r)f dr)(s) \\
&= \frac{1}{t}(\int_0^t S(r)f dr)(s).
\end{aligned}$$

Passing to the limit as $t \to 0$ in (2.2) we get the conclusion (j). □

Recall that $\sigma(L)$ denotes the *spectrum* of the bounded linear operator L acting on X, and $\rho(L) := \mathbb{C} \setminus \sigma(L)$ is the resolvent set of L. The *spectral radius* of L is $r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\}$ and the *spectral bound* is $s(L) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L)\}$. For the proof of the following result see for example ([1], Proof of Theorem 4 and Lemma 3).

Theorem 2.5 *Let $\mathcal{U} = \{U(t,s) : t \geq s\}$ be a 1-periodic evolution family on the Banach space X, $V := U(1,0)$ the monodromy operator and $\mathcal{S}$ the evolution semigroup associated to $\mathcal{U}$ on the space $AP(\mathbb{R},X)$, given in* (1.5). *The following four statements are equivalent:*

(i) $\mathcal{U}$ *is exponentially stable;*

(ii) $r(V) < 1$*;*

(iii) $\sup_{n\in\mathbb{N}} \|\sum_{k=0}^{n} e^{i\mu k}V^k\| = M_\mu < \infty$*, for all* $\mu \in \mathbb{R}$*;*

(iv) for each $f \in P_1^0(\mathbb{R}_+, X)$ *and each* $\mu \in \mathbb{R}$ *the function* $t \mapsto \int_0^t U(t,s)e^{-i\mu s}f(s)ds$ *is bounded on* $\mathbb{R}_+$*.*

Proof of Theorem 2.3 (i) implies (ii). Let $\mathbf{S}$ denote the evolution semigroup associated to $\mathcal{U}$ on the space $AP_0(\mathbb{R}_+, X)$, defined in (9) and $(G, D(G))$ its infinitesimal generator. $\mathcal{U}$ is exponentially stable, that is, (2.1) holds with some negative ω for every pairs (t,s) with $t \geq s$, so $\omega_0(\mathbf{S})$ is negative and $0 \in \rho(G)$. Then G is an invertible operator. It follows that for every $f \in AP_0(\mathbb{R}_+, X)$ there is $u \in D(G)$ such that $Gu = -f$. Using Lemma 2.4 it results that $u = v_f(\cdot, 0)$, so $v_f(\cdot, 0)$ belongs to $AP_0(\mathbb{R}_+, X)$.
(ii) implies (i). Let $\mu \in \mathbb{R}$ and $f \in P_1^0(\mathbb{R}_+, X)$. The function $t \mapsto e^{-i\mu t}f(t)$ belongs to the space $AP_0(\mathbb{R}_+, X)$. Thus the function $t \mapsto \int_0^t U(t,s)e^{-i\mu s}f(s)ds$ is bounded on $\mathbb{R}_+$ because it belongs to the space $AP_0(\mathbb{R}_+, X)$, too. Using Theorem 2.5 ((iv) implies (i)), it follows that $\mathcal{U}$ is exponentially stable. $\square$

Remark 2.6 Combining the equivalence between (i) and (iv) from Theorem 2.5 with the result from Theorem 2.3 it is easy to see that an evolution family $\mathcal{U}$, as in Theorem 2.3, is exponentially stable if and only if for each $f \in AP_0(\mathbb{R}_+, X)$, the solution $v_f(\cdot, 0)$ is bounded on $\mathbb{R}_+$.

3 Applications

An immediate consequence of Theorem 2.3 is the spectral mapping theorem for the evolution semigroup $\mathbf{S}$ on $AP_0(\mathbb{R}_+, X)$. Similar results can be found in ([17], Theorem 2.2) for evolution semigroups on $C_{00}(\mathbb{R}_+, X)$ and in [2, Theorem 5] for evolution semigroups on $AAP_0(\mathbb{R}_+, X)$. Here $C_{00}(\mathbb{R}_+, X)$ denotes the space of all X-valued continuous functions on $\mathbb{R}_+$ such that $f(0) = \lim_{t\to\infty} f(t) = 0$ and $AAP_0(\mathbb{R}_+, X)$ is the space of all X-valued functions h on $\mathbb{R}_+$ such that $h(0) = 0$ and there exist $f \in C_0(\mathbb{R}_+, X)$ and $g \in AP(\mathbb{R}_+, X)$ such that $h = f + g$.

Theorem 3.1 *Let* $\mathcal{U}$ *be a* 1*-periodic evolution family of bounded linear operators on* X*. The evolution semigroup* $\mathbf{S}$ *associated to* $\mathcal{U}$ *on* $AP_0(\mathbb{R}_+, X)$ *satisfies the spectral mapping theorem, as follows*

$$e^{t\sigma(G)} = \sigma(S(t)) \setminus \{0\}, \quad t \geq 0.$$

Moreover, $\sigma(G) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq s(G)\}$*, and*

$$\sigma(S(t)) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(S(t))\}, \quad \textit{for all } t > 0.$$

Another application of Theorem 2.3 is the following inequality of Landau-Kallman-Rota's type. For more details about the theorems of this form, see [3] and [2]. Let $\mathcal{Y}$ one of the following spaces: $C_{00}(\mathbb{R}_+, X)$, $AAP_0(\mathbb{R}_+, X)$, or $AP_0(\mathbb{R}_+, X)$.

Theorem 3.2 *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a 1-periodic evolution family of bounded linear operators acting on X and let $f \in \mathcal{Y}$. Suppose that the following conditions are fulfilled:*

(i) $v_f(\cdot, 0) = \int_0^\cdot U(\cdot, s)f(s)ds$ belongs to $\mathcal{Y}$;

(ii) $w_f(\cdot) := \int_0^\cdot (\cdot - s)U(\cdot, s)f(s)ds$ belongs to $\mathcal{Y}$.

If $\sup\{\|U(t,s)\| : t \geq s \geq 0\} = M < \infty$ *then*

$$\|v_f(\cdot, 0)\|_{\mathcal{Y}}^2 \leq 4M^2 \|f\|_{\mathcal{Y}} \cdot \|w_f(\cdot)\|_{\mathcal{Y}}.$$

For the proof of Theorem 3.2 in the cases $\mathcal{Y} = C_{00}(\mathbb{R}_+, X)$ or $\mathcal{Y} = AAP_0(\mathbb{R}_+, X)$ we refer the reder to ([3, 2]). The last case can be obtained in a similar way.

The hypothesis from the Neerven-Vu's theorem can be formulated as follows:

There exist a positive constant K such that

$$\sup_{t \geq 0} \| \int_0^t T(\xi)e^{-i\mu\xi}x d\xi\| \leq K\|e^{i\mu\cdot}x\|_{BUC(\mathbb{R}_+,X)},$$

for all $x \in X$.

Then the following result is natural.

Theorem 3.3 *Let $\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$ be a 1-periodic evolution family of bounded linear operators acting on X. The following two statements are equivalent:*

1. *$\mathcal{U}$ is exponentially stable;*
2. *for each $p \in TP_0(\mathbb{R}_+, X)$ the solution $v_p(\cdot, 0)$ belongs to $AP_0(\mathbb{R}_+, X)$ and there exists a positive constant K such that*

$$\|v_p(\cdot, 0)\|_{AP_0(\mathbb{R}_+,X)} \leq K\|p\|_{AP_0(\mathbb{R}_+,X)}. \tag{3.1}$$

Proof The proof of $1 \Rightarrow 2$ is obvious. We will prove that 2 implies 1. Let $f \in AP_0(\mathbb{R}_+, X)$ and $p_n \in TP_0(\mathbb{R}_+, X)$ be such that the sequence (p_n) converges to f in $AP_0(\mathbb{R}_+, X)$. From (3.1) it follows that $(v_{p_n}(\cdot, 0))$ converges in $AP_0(\mathbb{R}_+, X)$. On the other hand it is easy to see that $(v_{p_n}(\cdot, 0))$ converges pointwise to $v_f(\cdot, 0)$. Thus $v_f(\cdot, o)$ lies in $AP_0(\mathbb{R}_+, X)$ and the assertion follows from Theorem 2.3.

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