Electronic Journal of Differential Equations, Vol. **2003**(2003), No. 18, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Remarks on semilinear problems with nonlinearities depending on the derivative *

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Abstract

In this paper, we continue some work by Cañada and Drábek [1] and Mawhin [6] on the range of the Neumann and Periodic boundary value problems:

$$\begin{gathered}\mathbf{u}''(t)+\mathbf{g}(t,\mathbf{u}'(t))=\overline{\mathbf{f}}+\widetilde{\mathbf{f}}(t),\quad t\in(a,b)\\ \mathbf{u}'(a)=\mathbf{u}'(b)=0\\ \text{or}\quad \mathbf{u}(a)=\mathbf{u}(b),\quad \mathbf{u}'(a)=\mathbf{u}'(b)\end{gathered}$$

where $\mathbf{g}\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, $\overline{\mathbf{f}}\in\mathbb{R}^n$, and $\widetilde{\mathbf{f}}$ has mean value zero. For the Neumann problem with $n>1$, we prove that for a fixed $\widetilde{\mathbf{f}}$ the range can contain an infinity continuum. For the one dimensional case, we study the asymptotic behavior of the range in both problems.

1 Introduction

Let us consider the resonance problem

$$\begin{gathered}u''(t)+g(u'(t))=f(t),\quad t\in(a,b)\\ u'(a)=u'(b)=0\end{gathered}\tag{1.1}$$

where $f\in C[a,b]$ and $g:\mathbb{R}\to\mathbb{R}$ is continuous. The linearized part of (1.1) is the resonance system

$$\begin{gathered}u''(t)=f(t),\quad t\in(a,b)\\ u'(a)=u'(b)=0\end{gathered}\tag{1.2}$$

and the corresponding eigenfunction is $u_1(t)=1$. The change of variable $v=u'$ transforms (1.2) into the problem

$$\begin{gathered}v'(t)=f(t),\quad t\in(a,b)\\ v(a)=v(b)=0\end{gathered}\tag{1.3}$$

*Mathematics Subject Classifications: 34B15, 34L30.
Key words: Nonlinear boundary-value problem, Neumann and Periodic problems.

Submitted December 5, 2002. Published February 20, 2003.
Partially supported by grant FQM-0178 from the Junta de Andalucía,
Grupo de Investigación "Aproximación y Métodos Numéricos" .

which obviously is solvable if and only if $\int_a^b f(t)dt = 0$. Moreover, its solution is given by $v(t) = \int_a^t f(s)ds$. Hence (1.2) is solvable if and only if $f = \widetilde{f} \in \widetilde{C}[a,b] := \{\widetilde{f} \in C[a,b] : \int_a^b \widetilde{f}(t)dt = 0\}$ and its set of solutions is

$$u_c(t) = c + \int_a^t v(s)ds$$

where $c \in \mathbb{R}$ and $v(t) = \int_a^t f(s)ds$. Let us now consider problem (1.1). When we decompose

$$f(t) = s + \widetilde{f}(t) \tag{1.4}$$

where $s \in \mathbb{R}$ and $\widetilde{f} \in \widetilde{C}[a,b]$, it is quite natural to ask for which values $s \in \mathbb{R}$ the problem (1.1) is solvable. This question has been studied by several authors. In particular, Cañada and Drábek (see [1]) proved that if $g \in C^1(\mathbb{R})$ and is bounded, then for each $\widetilde{f}$ there is a unique value $s = s(\widetilde{f}) \in \mathbb{R}$ such that (1.1) is solvable. Moreover, in such a case they also proved that the map $s(\cdot) : \widetilde{C}[a,b] \to \mathbb{R}$, $\widetilde{f} \to s(\widetilde{f})$ is continuously differentiable and satisfies $|s(\widetilde{f})| \leq \|g\|$ for all $\widetilde{f} \in \widetilde{C}[a,b]$, where $\|g\| = \sup_{t\in\mathbb{R}} |g(t)|$. In the same paper the authors noted that their proofs are also applicable to the more general problem

$$\begin{gathered} u''(t) + g(t, u'(t)) = s + \widetilde{f}(t), \quad t \in (a,b) \\ u'(a) = u'(b) = 0 \end{gathered} \tag{1.5}$$

(with $g \in C^1([a,b] \times \mathbb{R}, \mathbb{R})$ and bounded) and also to the periodic problem

$$\begin{gathered} u''(t) + g(t, u'(t)) = s + \widetilde{f}(t), \quad t \in (a,b) \\ u(a) = u(b), \quad u'(a) = u'(b); \end{gathered} \tag{1.6}$$

and proposed as an open question to study these kind of problems for systems of equations and for higher order equations. This was made by Mawhin in [6]. In particular, he studied the problems

$$\begin{gathered} \mathbf{u}''(t) + \mathbf{g}(t, \mathbf{u}'(t)) = \overline{\mathbf{f}} + \widetilde{\mathbf{f}}(t), \quad t \in (a,b) \\ \mathbf{u}'(a) = \mathbf{u}'(b) = \mathbf{0} \end{gathered} \tag{1.7}$$

and

$$\begin{gathered} \mathbf{u}''(t) + \mathbf{g}(t, \mathbf{u}'(t)) = \overline{\mathbf{f}} + \widetilde{\mathbf{f}}(t), \quad t \in (a,b) \\ \mathbf{u}(a) = \mathbf{u}(b), \quad \mathbf{u}'(a) = \mathbf{u}'(b), \end{gathered} \tag{1.8}$$

where $\mathbf{g} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory function, $\mathbf{u} : [a,b] \to \mathbb{R}^n$, $\overline{\mathbf{f}} \in \mathbb{R}^n$ and

$$\widetilde{\mathbf{f}} \in \widetilde{L^1}([a,b], \mathbb{R}^n) := \{\widetilde{\mathbf{f}} \in L^1([a,b], \mathbb{R}^n) : \int_a^b \widetilde{\mathbf{f}}(t)dt = 0\};$$

and proved that if

$$\lim_{\|\mathbf{v}\|_2 \to \infty} \big\| \mathbf{g}(t, \mathbf{v}) / \|\mathbf{v}\|_2 \big\|_2 = 0 \quad \text{uniformly a.e. in } t \in [a,b], \tag{1.9}$$

then for each $\widetilde{\mathbf{f}} \in \widetilde{L^1}([a,b],\mathbb{R}^n)$ the sets

$$\mathcal{J}_{\widetilde{\mathbf{f}}}^{(\mathcal{N})} = \{\overline{\mathbf{f}} \in \mathbb{R}^n : \text{the problem (1.7) is solvable}\}$$
$$\mathcal{J}_{\widetilde{\mathbf{f}}}^{(\mathcal{P})} = \{\overline{\mathbf{f}} \in \mathbb{R}^n : \text{the problem (1.8) is solvable}\}$$

are both nonempty, where $\|\cdot\|_2$ denotes the Euclidean norm of $\mathbb{R}^n$. Moreover, he also proved that for $n=1$ and $\widetilde{f} \in \widetilde{L^1}(a,b) := \widetilde{L^1}([a,b],\mathbb{R})$, $\#\mathcal{J}_{\widetilde{f}}^{(\mathcal{N})} = \#\mathcal{J}_{\widetilde{f}}^{(\mathcal{P})} = 1$ and stated the uniqueness problem for $n>1$ as an open question. In this note we solve this problem in the negative sense for the Neumann case (1.7).

For $n=1$ and $g \in C([a,b]\times\mathbb{R},\mathbb{R})$ satisfying (1.9), we denote by $s_{\mathcal{N}}(\widetilde{f})$ the unique element of $\mathcal{J}_{\widetilde{f}}^{(\mathcal{N})}$ and by $s_{\mathcal{P}}(\widetilde{f})$ the unique element of $\mathcal{J}_{\widetilde{f}}^{(\mathcal{P})}$. We study the asymptotic behavior of the functionals $s_{\mathcal{N}}(\widetilde{f})$ and $s_{\mathcal{P}}(\widetilde{f})$ for $\|\widetilde{f}\| \to \infty$ when the uniqueness results are applicable.

2 Uniqueness Problem

The first contribution of this note to the subject is that we solve for the Neumann problem (1.7) the uniqueness question in the negative sense for all $n>1$. With this objective in mind, we take $h:\mathbb{R}\to\mathbb{R}$ a C^∞ function such that it is bounded and satisfies $h(x)=x$ for all $x\in[-2,2]$ and we set $\widetilde{\mathbf{f}}=\mathbf{0}$ and

$$\mathbf{g}(t,x_1,x_2,x_3,\dots,x_n) = -(-h(x_2),h(x_1),0,\dots,0).$$

Then $\mathbf{g}:[0,2\pi]\times\mathbb{R}^n\to\mathbb{R}^n$ belongs to $C^\infty([0,2\pi]\times\mathbb{R}^n,\mathbb{R}^n)$ and it is bounded. Let us now consider the problem

$$\begin{gathered}\mathbf{u}''(t)+\mathbf{g}(t,\mathbf{u}'(t)) = \overline{\mathbf{f}}, \quad t\in(0,2\pi)\\ \mathbf{u}'(0)=\mathbf{u}'(2\pi)=\mathbf{0}\end{gathered} \tag{2.1}$$

and let $\alpha\in[-1,1]$ be fixed. We set $\mathbf{u}_\alpha(t) = (\alpha\sin(t-\frac{\pi}{2}), \alpha t - \alpha\cos(t-\frac{\pi}{2}),0,\dots,0)$ with $\alpha\in[-1,1]$. Then $\mathbf{u}_\alpha\in C^2([0,2\pi],\mathbb{R}^n)$ and $\mathbf{u}'_\alpha(t) = (\alpha\cos(t-\frac{\pi}{2}), \alpha+\alpha\sin(t-\frac{\pi}{2}),0,\dots,0)$, so that $\mathbf{u}'_\alpha(0)=\mathbf{u}'_\alpha(2\pi)=\mathbf{0}$ and

$$\begin{aligned}\mathbf{u}''_\alpha(t) &= (-\alpha\sin(t-\frac{\pi}{2}),\alpha\cos(t-\frac{\pi}{2}),0,\dots,0)\\ &= (-(\alpha+\alpha\sin(t-\frac{\pi}{2})),\alpha\cos(t-\frac{\pi}{2}),0,\dots,0)+(\alpha,0,0,\dots,0)\\ &= -\mathbf{g}(t,\mathbf{u}'_\alpha(t))+(\alpha,0,0,\dots,0)\end{aligned}$$

Hence $\mathbf{u}_\alpha$ solves (2.1) with $\overline{\mathbf{f}}=(\alpha,0,\dots,0)$ and we have proved that there exists a continuum of vectors $\overline{\mathbf{f}}\in\mathbb{R}^n$ for which the problem (2.1) is solvable. Moreover, we have got such a result not only for $\mathbf{g}(t,\mathbf{x})$ continuous but also C^∞ and bounded, so that $\mathbf{g}(t,\mathbf{x})$ satisfies the hypothesis of the existence and uniqueness results in the papers by Mawhin (see [6, Theorems 1 and 3]) and Cañada and Drábek (see [1, Theorem 3.3]). This proves that the mentioned uniqueness result for $n=1$ is impossible to generalize to higher dimensions. Of course, the same problem is still open for the periodic case.

3 Asymptotic behavior

In this section we set $n = 1$ and we consider the problems (1.5) and (1.6). Moreover, in order to have existence of solutions, we assume that $g(t, u)$ satisfies that $\lim_{|u|\to\infty} \frac{g(t,u)}{|u|} = 0$ uniformly in $t \in [a, b]$. With these hypotheses at hands we know that for each $\widetilde{f} \in \widetilde{C}[a, b]$, $\mathcal{J}^{(\mathcal{N})}_{\widetilde{f}} = \{s_{\mathcal{N}}(\widetilde{f})\}$ and $\mathcal{J}^{(\mathcal{P})}_{\widetilde{f}} = \{s_{\mathcal{P}}(\widetilde{f})\}$, where $s_{\mathcal{N}} : \widetilde{C}[a, b] \to \mathbb{R}$ and $s_{\mathcal{P}} : \widetilde{C}[a, b] \to \mathbb{R}$ are certain functionals. Furthermore, the change of variables $v = u'$ transforms (1.5) and (1.6) into the problems

$$\begin{gathered} v'(t) + g(t, v(t)) = s + \widetilde{f}(t), \quad t \in (a, b) \\ v(a) = v(b) = 0 \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} v'(t) + g(t, v(t)) = s + \widetilde{f}(t), \quad t \in (a, b) \\ v(a) = v(b), \quad \int_a^b v(t)dt = 0. \end{gathered} \tag{3.2}$$

Thus, if $w(t)$ solves (3.1) and $\omega(t)$ solves (3.2) and we integrate between a and b both sides of the equation, we get

$$s_{\mathcal{N}}(\widetilde{f}) = \frac{1}{b-a}\int_a^b g(t, w(t))dt \quad \text{and} \quad s_{\mathcal{P}}(\widetilde{f}) = \frac{1}{b-a}\int_a^b g(t, \omega(t))dt.$$

We will use the formulas above in order to prove certain asymptotic results for the functionals $s_{\mathcal{N}}(\cdot)$ and $s_{\mathcal{P}}(\cdot)$.

Now we state and prove the main results of this section.

Theorem 3.1 *Let us set* $\Theta = \{\frac{1}{b-a}\int_a^b g(t, v_0)dt : v_0 \in \mathbb{R}\}$. *Then for each* $g_0 \in \overline{\Theta}$, *the closure of* Θ *in* $\mathbb{R}$, *there exists a sequence* $\{\widetilde{f}_n\}_{n=1}^{\infty} \subset \widetilde{C}[a, b]$ *such that* $\lim_{n\to\infty} \|\widetilde{f}_n\| = \infty$ *and* $\lim_{n\to\infty} s_{\mathcal{N}}(\widetilde{f}_n) = g_0$, *where* $\|\widetilde{f}_n\| = \sup_{t\in[a,b]} |\widetilde{f}_n(t)|$.

Proof Let $g_0 = \frac{1}{b-a}\int_a^b g(t, v_0)dt \in \Theta$ be arbitrarily chosen. We define for each $n > 2(b-a)^{-1}$ a function $w_n : [a, b] \to \mathbb{R}$ which satisfies the following conditions

a) $w_n \in C^1[a, b]$

b) $w_n(a) = w_n(b) = 0$

c) $w_n(a + \frac{1}{2n}) = w_n(b - \frac{1}{2n}) = 1$

d) $w_n(t) = v_0$ for all $t \in [a + \frac{1}{n}, b - \frac{1}{n}]$

e) $\|w_n\| \leq |v_0| + 2$

and we set

$$\widetilde{f}_n(t) := w_n'(t) + g(t, w_n(t)) - \frac{1}{b-a}\int_a^b g(t, w_n(t))dt.$$

It is clear that a) implies that $\widetilde{f}_n \in C([a,b])$ for all $n \in \mathbb{N}$ and b) implies that $\int_a^b \widetilde{f}_n(t)dt = 0$. Moreover, using that $K = [a,b] \times [-|v_0| - 2, |v_0| + 2]$ is compact and $\{(t, w_n(t)) : t \in [a,b]\} \subset K$ for all $n \in \mathbb{N}$, we have that the functions $g(t, w_n(t))$ are uniformly bounded in $[a,b]$, so that the conditions b) and c) imply that $\lim_{n\to\infty} \|\widetilde{f}_n\| = \infty$.

Then $w = w_n$ solves the problem

$$w'(t) + g(t, w(t)) = s_{\mathcal{N}}(\widetilde{f}_n) + \widetilde{f}_n(t), \quad t \in (a,b)$$
$$w(a) = w(b) = 0$$

with $s_{\mathcal{N}}(\widetilde{f}_n) = \frac{1}{b-a}\int_a^b g(t, w_n(t))dt$. We will prove that $\lim_{n\to\infty} s_{\mathcal{N}}(\widetilde{f}_n) = g_0$. In fact, by d) we have that

$$\begin{aligned} s_{\mathcal{N}}(\widetilde{f}_n) &= \frac{1}{b-a}\int_a^b g(t, w_n(t))dt \\ &= \frac{1}{b-a}\Big(\int_a^{a+\frac{1}{n}} g(t, w_n(t))dt + \int_{a+\frac{1}{n}}^{b-\frac{1}{n}} g(t, v_0)dt + \int_{b-\frac{1}{n}}^{b} g(t, w_n(t))dt\Big). \end{aligned}$$

The uniform boundedness of $g(t, w_n(t))$ implies that

$$\lim_{n\to\infty}\int_a^{a+\frac{1}{n}} g(t, w_n(t))dt = \lim_{n\to\infty}\int_{b-\frac{1}{n}}^{b} g(t, w_n(t))dt = 0.$$

Hence

$$\lim_{n\to\infty} s_{\mathcal{N}}(\widetilde{f}_n) = \lim_{n\to\infty}\frac{1}{b-a}\int_{a+\frac{1}{n}}^{b-\frac{1}{n}} g(t, v_0)dt = g_0$$

which is what we wanted to prove.

Let us now take $g_0 \in \overline{\Theta} \setminus \Theta$. Then there exists a sequence of numbers $\{g_n\}_{n=1}^{\infty} \subset \Theta$ and a family of functions $\{\widetilde{f}_{n,k}\}_{n,k=1}^{\infty} \subset \widetilde{C}[a,b]$ such that $\|\widetilde{f}_{n,k}\| \geq k$ and $\big|s_{\mathcal{N}}(\widetilde{f}_{n,k}) - g_n\big| \leq \frac{1}{k}$ for all $k, n \geq 1$ and $\lim_{n\to\infty} g_n = g_0$. Thus the sequence $\{\widetilde{f}_{n,n}\}_{n=1}^{\infty}$ satisfies that $\lim_{n\to\infty}\|\widetilde{f}_{n,n}\| = \infty$ and $\lim_{n\to\infty} s_{\mathcal{N}}(\widetilde{f}_{n,n}) = g_0$. ♢

Corollary 3.2 *Let us assume that* $g = g(v) \in C(\mathbb{R})$ *and* $g_0 \in \overline{g(\mathbb{R})}$. *Then there exists a sequence* $\{\widetilde{f}_n\}_{n=1}^{\infty} \subset \widetilde{C}[a,b]$ *such that* $\lim_{n\to\infty}\|\widetilde{f}_n\| = \infty$ *and* $\lim_{n\to\infty} s_{\mathcal{N}}(\widetilde{f}_n) = g_0$.

Proof In [6, Corollary 2] it is shown the existence of solutions for $n = 1$ whenever $g = g(v)$ is continuous. Hence, it is enough to observe that if g does not depend on the variable t then $\Theta = g(\mathbb{R})$. ♢

Theorem 3.3 *Let us assume that* g *is bounded and set* $\Theta = \{\frac{1}{b-a}\int_a^b g(t, v_0)dt : v_0 \in \mathbb{R}\}$. *Then for each* $g_0 \in \overline{\Theta}$, *there exists a sequence* $\{\widetilde{f}_n\}_{n=1}^{\infty} \subset \widetilde{C}[a,b]$ *such that* $\lim_{n\to\infty}\|\widetilde{f}_n\| = \infty$ *and* $\lim_{n\to\infty} s_{\mathcal{P}}(\widetilde{f}_n) = g_0$.

Proof We define for each $n > 2(b-a)^{-1}$ a function $\varphi_n : [a,b] \to \mathbb{R}$ which satisfies the following conditions:

a) $\varphi_n \in C^1[a,b]$

b) $\varphi_n(a) = \varphi_n(b)$

c) $\varphi_n(a + \frac{1}{2n}) = \varphi_n(b - \frac{1}{2n}) = 1$

d) $\varphi_n(t) = v_0$ for all $t \in [a + \frac{1}{n}, b - \frac{1}{n}]$

e) $\int_a^b \varphi_n(t)dt = 0$

Clearly, these functions exist. The rest of the proof is analogous to that of Theorem 3.1. We just change w_n by φ_n and $s_{\mathcal{N}}(\widetilde{f})$ by $s_{\mathcal{P}}(\widetilde{f})$. The only difference with the other proof is that now the graphs of the functions φ_n are not uniformly bounded, and this is the reason because we need now to assume that g is bounded. ♢

Corollary 3.4 *Assume that $g = g(v) \in C(\mathbb{R})$ is bounded and $g_0 \in \overline{g(\mathbb{R})}$. Then there exists a sequence $\{\widetilde{f}_n\}_{n=1}^{\infty} \subset \widetilde{C}[a,b]$ such that $\lim_{n\to\infty} \|\widetilde{f}_n\| = \infty$ and $\lim_{n\to\infty} s_{\mathcal{P}}(\widetilde{f}_n) = g_0$.*

Proof In [6, Corollary 4] it is shown the existence of solutions for $n = 1$ whenever $g = g(v)$ is continuous. Hence, it is enough to observe that if g does not depend on the variable t then $\Theta = g(\mathbb{R})$. ♢

We have proved that the limits $\lim_{\|\widetilde{f}\|\to\infty} s_{\mathcal{N}}(\widetilde{f})$ and $\lim_{\|\widetilde{f}\|\to\infty} s_{\mathcal{P}}(\widetilde{f})$ never exist if $\overline{\Theta}$ is not a single point. This makes natural to ask if some weaker asymptotic results are possible. For example, for which functions $\widetilde{f} \in \widetilde{C}[a,b]$ do the radial limits $\lim_{k\to\infty} s_{\mathcal{N}}(k\widetilde{f})$ or $\lim_{k\to\infty} s_{\mathcal{P}}(k\widetilde{f})$ exist? Now we prove a comparison result which will be helpful for the computation of these limits.

Lemma 3.5 (Comparison Principle) *Let $k > 0$ and $\widetilde{f} \in \widetilde{C}[a,b]$. If $w_{\mathcal{N}}$ is a solution of the problem*

$$\begin{gathered} w'(t) + g(t, w(t)) = s_{\mathcal{N}}(k\widetilde{f}) + k\widetilde{f}(t), \quad t \in (a,b) \\ w(a) = w(b) = 0, \end{gathered} \tag{3.3}$$

where $w_{\mathcal{P}}$ is a solution of the problem

$$\begin{gathered} w'(t) + g(t, w(t)) = s_{\mathcal{P}}(k\widetilde{f}) + k\widetilde{f}(t), \quad t \in (a,b) \\ w(a) = w(b); \quad \int_a^b w(t)dt = 0, \end{gathered} \tag{3.4}$$

$v_{\mathcal{N}}$ is the unique solution of

$$\begin{gathered} v'(t) = \widetilde{f}(t), \quad t \in (a,b) \\ v(a) = v(b) = 0 \end{gathered}, \tag{3.5}$$

and $v_{\mathcal{P}}$ is the unique solution of

$$\begin{aligned} v'(t) &= \widetilde{f}(t), \quad t \in (a,b) \\ v(a) = v(b); &\quad \int_a^b v(s)ds = 0, \end{aligned} \tag{3.6}$$

then $\|w_{\mathcal{N}} - kv_{\mathcal{N}}\| \leq (b-a)(M-m)$ *and* $\|w_{\mathcal{P}} - kv_{\mathcal{P}}\| \leq \frac{1}{2}(b-a)(M-m)$*, where* $m := \inf_{(t,s)\in[a,b]\times\mathbb{R}} g(t,s)$ *and* $M := \sup_{(t,s)\in[a,b]\times\mathbb{R}} g(t,s)$*.*

Proof: Let $w_{\mathcal{N}}$ be a solution of (3.3) and let $v_{\mathcal{N}}(t) = \int_a^t \widetilde{f}(s)ds$ be the solution of (3.5). Then

$$w_{\mathcal{N}}(t) = k\int_a^t \widetilde{f}(s)ds + s_{\mathcal{N}}(k\widetilde{f})(t-a) - \int_a^t g(s, w_{\mathcal{N}}(s))ds$$

and

$$\begin{aligned} w_{\mathcal{N}}(t) - kv_{\mathcal{N}}(t) &= s_{\mathcal{N}}(k\widetilde{f})(t-a) - \int_a^t g(s, w_{\mathcal{N}}(s))ds \\ &= \frac{t-a}{b-a}\int_a^b g(s, w_{\mathcal{N}}(s))ds - \int_a^t g(s, w_{\mathcal{N}}(s))ds. \end{aligned}$$

Hence

$$|w_{\mathcal{N}}(t) - kv_{\mathcal{N}}(t)| \leq (b-a)(M-m), \quad \text{for all } t \in [a,b]$$

since

$$(t-a)m \leq \frac{t-a}{b-a}\int_a^b g(s, w_{\mathcal{N}}(s))ds \leq (t-a)M$$

and

$$(t-a)m \leq \int_a^t g(s, w_{\mathcal{N}}(s))ds \leq (t-a)M.$$

This completes the proof for the Neumann problem. For the periodic case we must take into account that if $w_{\mathcal{P}}$ is a solution of (3.4) and

$$v_{\mathcal{P}}(t) = \int_a^t \widetilde{f}(s)ds - \frac{1}{b-a}\int_a^b \int_a^t \widetilde{f}(s)ds\,dt$$

is the solution of (3.6) then

$$\begin{aligned} w_{\mathcal{P}}(t) = kv_{\mathcal{P}}(t) + s_{\mathcal{P}}(k\widetilde{f})(t - \frac{a+b}{2}) + \frac{1}{b-a}\int_a^b \int_a^t g(s, w_{\mathcal{P}}(s))ds\,dt \\ - \int_a^t g(s, w_{\mathcal{P}}(s)ds. \end{aligned}$$

After this, the proof is quite similar to that of the Neumann problem. ♢

In what follows we denote by $|A|$ the Lebesgue measure of the set A.

Theorem 3.6 *Assume that the limits $g(t,\pm\infty) := \lim_{s\to\pm\infty} g(t,s)$ exist uniformly in $t \in [a,b]$. Given $\widetilde{f} \in \widetilde{C}[a,b]$ and $F(t) = \int_a^t \widetilde{f}(s)ds$, we have that*
(i) If $|\{t \in [a,b] : F(t) = 0\}| = 0$ then

$$\lim_{k\to\infty} s_{\mathcal{N}}(k\widetilde{f}) = \frac{\int_{F^{-1}(0,+\infty)} g(t,+\infty)dt + \int_{F^{-1}(-\infty,0)} g(t,-\infty)dt}{b-a}$$

(ii) If $|\{t \in [a,b] : F(t) = 0\}| > 0$ and $g(t,s) = g(t,0)$ for all (t,s) in $[a,b] \times [-(b-a)(M-m), (b-a)(M-m)]$, then

$$\begin{aligned}&\lim_{k\to\infty} s_{\mathcal{N}}(k\widetilde{f})\\ &= \frac{1}{b-a}\Big(\int_{F^{-1}(0,+\infty)} g(t,+\infty)dt + \int_{F^{-1}(-\infty,0)} g(t,-\infty)dt + \int_{F^{-1}(0)} g(t,0)dt\Big).\end{aligned}$$

Proof It follows from Lemma 3.5 that

$$kF(t)-(b-a)(M-m) \le w_{\mathcal{N}}(t) \le kF(t)+(b-a)(M-m), \text{ for all } t \in [a,b]; \quad (3.7)$$

where $F(t) = \int_a^t \widetilde{f}(s)ds$. We define the sets:

$$\begin{aligned}A^+ &= \{t \in [a,b] : F(t) > 0\} = F^{-1}(0,+\infty)\\ A^0 &= \{t \in [a,b] : F(t) = 0\} = F^{-1}(0)\\ A^- &= \{t \in [a,b] : F(t) < 0\} = F^{-1}(-\infty,0)\end{aligned}$$

Then

$$\begin{aligned}s_{\mathcal{N}}(k\widetilde{f}) &= \frac{1}{b-a}\int_a^b g(t,w_{\mathcal{N}}(t))dt\\ &= \frac{1}{b-a}\int_{A^0} g(t,w_{\mathcal{N}}(t))dt + \frac{1}{b-a}\int_{A^+} g(t,w_{\mathcal{N}}(t))dt\\ &\quad + \frac{1}{b-a}\int_{A^-} g(t,w_{\mathcal{N}}(t))dt\end{aligned}$$

Now we will estimate each one of the integrals which appear in the equality above. First, using (3.7) and the Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{b-a}\int_{A^+} g(t,w_{\mathcal{N}}(t))dt &= \frac{1}{b-a}\int_{A^+} g(t,+\infty)dt\\ \lim_{k\to\infty}\frac{1}{b-a}\int_{A^-} g(t,w_{\mathcal{N}}(t))dt &= \frac{1}{b-a}\int_{A^-} g(t,-\infty)dt.\end{aligned}$$

Second, under the assumption (i) (i.e. $|A^0| = 0$) we have

$$\frac{1}{b-a}\int_{A^0} g(t,w_{\mathcal{N}}(t))dt = 0.$$

On the other hand, under the hypotheses of (ii) (i.e. $g(t,s)=g(t,0)$ for all $(t,s)\in[a,b]\times[-(b-a)(M-m),(b-a)(M-m)]$), we obtain from (3.7) that

$$-(b-a)(M-m)\le w_{\mathcal{N}}(t)\le(b-a)(M-m)$$

for all $t\in A^0$. Hence,

$$\frac{1}{b-a}\int_{A^0}g(t,w_{\mathcal{N}}(t))dt=\frac{1}{b-a}\int_{A^0}g(t,0)dt.$$

Taking into account the two items above we complete the proof. ♢

Theorem 3.7 *Assume that $g(t,s)$ is bounded and that the limits $g(t,\pm\infty):=\lim_{s\to\pm\infty}g(t,s)$ exist uniformly in $t\in[a,b]$. Given $\widetilde{f}\in\widetilde{C}[a,b]$ and*

$$H(t)=\int_a^t\widetilde{f}(s)ds-\frac{1}{b-a}\int_a^b\Big(\int_a^t\widetilde{f}(s)ds\Big)dt,$$

we have that:
(i) If $|\{t\in[a,b]:H(t)=0\}|=0$ then

$$\lim_{k\to\infty}s_{\mathcal{P}}(k\widetilde{f})=\frac{1}{b-a}\Big(\int_{H^{-1}(0,+\infty)}g(t,+\infty)dt+\int_{H^{-1}(-\infty,0)}g(t,-\infty)dt\Big)$$

(ii) If $|\{t\in[a,b]:H(t)=0\}|>0$ and $g(t,s)=g(t,0)$ for all (t,s) in $[a,b]\times[-\frac{b-a}{2}(M-m),\frac{b-a}{2}(M-m)]$ then

$$\begin{aligned}&\lim_{k\to\infty}s_{\mathcal{P}}(k\widetilde{f})\\&=\frac{1}{b-a}\Big(\int_{H^{-1}(0,+\infty)}g(t,+\infty)dt+\int_{H^{-1}(-\infty,0)}g(t,-\infty)dt+\int_{H^{-1}(0)}g(t,0)dt.\Big)\end{aligned}$$

The proof of this theorem is analogous to that of Theorem 3.6, using the periodic case of the comparison principle. The following result is a direct consequence of the theorems above:

Corollary 3.8 *With the notation of Theorems 3.6 and 3.7, if $g=g(s)$ does not depend on the variable t and there exists the limits $g(\pm\infty):=\lim_{s\to\pm\infty}g(s)$ then*

$$\lim_{k\to\infty}s_{\mathcal{N}}(k\widetilde{f})=\frac{g(+\infty)\left|F^{-1}(0,+\infty)\right|+g(-\infty)\left|F^{-1}(-\infty,0)\right|}{b-a}$$

whenever $\left|F^{-1}(0)\right|=0$ and

$$\lim_{k\to\infty}s_{\mathcal{P}}(k\widetilde{f})=\frac{g(+\infty)\left|H^{-1}(0,+\infty)\right|+g(-\infty)\left|H^{-1}(-\infty,0)\right|}{b-a}$$

whenever $\left|H^{-1}(0)\right|=0$.

The following proposition gives an estimation of the size of the sets of functions with the property that the radial limits exists.

Proposition 3.9 *The sets*

$$\mathcal{F} = \{\widetilde{f} \in \widetilde{C}[a,b] : F(t) = \int_a^t \widetilde{f}(s)ds \text{ satisfies } |F^{-1}(0)| = 0\}$$

and

$$\mathcal{H} = \Big\{\widetilde{f} \in \widetilde{C}[a,b] : H(t) = \int_a^t \widetilde{f}(s)ds - \frac{1}{b-a}\int_a^b \Big(\int_a^t \widetilde{f}(s)ds\Big)dt \text{ satisfies } |H^{-1}(0)| = 0\Big\}$$

are dense non-meager subsets of the Banach space $\widetilde{C}[a,b]$.

Proof Clearly, $\mathcal{F}$ is a dense subset of $\widetilde{C}[a,b]$, since $\widetilde{\Pi} = \Pi \cap \widetilde{C}[a,b]$ is dense in $\widetilde{C}[a,b]$, where Π denotes the set of algebraic polynomials, and $\widetilde{\Pi} \setminus \{0\} \subset \mathcal{F}$. Now, we are going to prove that $\mathcal{F}$ has nonempty interior, which implies that $\mathcal{F}$ is non-meager. Of course, there is no loss of generality if we assume that $[a,b] = [-1,1]$. Then $\widetilde{f}(t) = t$ belongs to $\mathcal{F}$. Let $\widetilde{g} \in \widetilde{C}[-1,1]$ be such that $\|\widetilde{f} - \widetilde{g}\| < \frac{1}{4}$ and let $G(t) = \int_{-1}^t \widetilde{g}(s)ds$. Then

$$\frac{t^2}{2} - \frac{t}{4} - \frac{3}{4} \le G(t) \le \frac{t^2}{2} + \frac{t}{4} - \frac{1}{4} \quad \text{for all } t \in [-1,1].$$

Thus, $G^{-1}(0) \subset \{-1\} \cup [1/2,1]$. If $\#G^{-1}(0) \ge 3$ then there are two points $x, y \in [1/2,1]$ such that $G(x) = G(y) = 0$ and Rolle's theorem implies that $G'(t) = \widetilde{g}(t)$ vanishes at some point $\xi \in [1/2,1]$, which is impossible since $\|\widetilde{f} - \widetilde{g}\| < \frac{1}{4}$. This implies that $\#G^{-1}(0) \le 2$, so that $\widetilde{g} \in \mathcal{F}$ and $\mathcal{F}$ has nonempty interior and proves the claim for the set $\mathcal{F}$. Finally, the proof of the claim for the set $\mathcal{H}$ follows from similar arguments. ♢

Remark Note that when $g \in C^1(\mathbb{R})$, it follows from [1, Theorems 3.3 and 3.4] that $s_{\mathcal{N}}(\cdot)$ and $s_{\mathcal{P}}(\cdot)$ are continuous functionals so that $\lim_{\|\widetilde{f}\| \to 0} s_{\mathcal{N}}(\widetilde{f}) = s_{\mathcal{N}}(0)$ and $\lim_{\|\widetilde{f}\| \to 0} s_{\mathcal{P}}(\widetilde{f}) = s_{\mathcal{P}}(0)$. Now, $s_{\mathcal{N}}(0) = s_{\mathcal{P}}(0) = g(0)$ since [6, Theorem 3] guarantees that $w(t) = 0$ is the unique solution of the systems

$$w'(t) + g(w(t)) = g(0), \quad t \in (a,b)$$
$$w(a) = w(b) = 0 \quad \text{or} \quad w(a) = w(b); \quad \int_a^b w(t)dt = 0$$

Acknowledgements The authors are very grateful to Prof. A. Cañada both for proposing the problems considered in this note and for several useful suggestions.

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