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Non-classical phase transitions at a sonic point *

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Abstract

The relevant mathematical features of phase transition for a general hyperbolic nonlinear system near a sonic discontinuity are clarified. A well-posed Riemann's problem is obtained, including non-classical undercompressive shocks, defined by a geometrical kinetic relation. A counterpart is the geometrical rejection of some compressive shocks. The result is consistent with the structure profiles of the elasticity model of Shearer-Yang and the combustion model of Majda.

1 Introduction

Hyperbolic phase transitions occur in important physical contexts as elasticity and combustion. In the so-called Chapman-Jouguet regime, the waves are sonic on one side of their phase-transition discontinuity; here we call these waves semi-characteristic transition waves. Uniqueness fails in the Riemann problem near a sonic phase transition and the geometric-compressibility criterion of Lax [7], is not appropriate to select such large amplitude transition discontinuities.

Some admissibility criteria for particular physical systems have been suggested in many papers. Most of them require a simple phase transition to be asymptotic to travelling waves of appropriate augmented systems. The problem of existing travelling waves is that of heteroclinic connections of rest points for ordinary differential equations. At a sonic transition, one of the rest point is not hyperbolic. In addition to the non-planar general context obstruction, these are all reasons why there are so few responses to the wave structure problems.

Nevertheless, complete resolution is achieved near a sonic point in case of two significant models. In elasticity, for a cubic stress, Shearer and Yang [12], characterize the simple phase transition waves which satisfy the viscosity-capillarity criterion of Slemrod [10]. In combustion, for a qualitative model Majda [8], characterize the simple phase transition waves which satisfy a viscosity-reacting criterion with ignition temperature kinetics. These two models not only underline undercompressive admissible transition shocks but also compressive non-admissible ones. The undercompressive admissible shocks enter into the so-called kinetic relation of Abeyaratne and Knowles [1] and their set is called the

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kinetic set. The selection in each of the two models of non-compressive shocks as either kinetic or sonic appears as a mixed sonic-kinetic geometrical selection.

In the present paper, we carry out the local-mathematical analysis at a semi-characteristic hyperbolic nonlinear phase transition, up to the well-posedness of the mixed sonic-kinetic Riemann problem. Under a purely sonic-stability assumption, the local Riemann problem is well-posed and BV-stable. One could use it to get a global-existence result by a Glimm's type scheme, in the way of [4].

Section 2 provides the mathematical justification for the claims made in this paper. There, we identify the relevant phase boundary variables in regards to Hugoniot's large-jump equation, near a sonic phase transition, the speed of which is genuinely nonlinear. In this space, we underline the fundamental part of what we call a Hugoniot stationary vector field, as well as the Hugoniot nonlinear dipoles. The analysis leads to a geometrical admissibility criterion which explains in a natural way the rejection of compressible transition shocks as a counterpart of the admissibility of undercompressive ones.

In section 3 we show how the results of Shearer-Yang or Majda enter the sonic-kinetic general case.

Section 4 analyzes the entropy, so that as in Abeyaratne and Knowles [1], sonic-kinetic geometries can be defined when physical examples are not available.

Section 5 applies the analysis to the perturbations of Chapman-Jouguet detonations, for the complete compressible Euler system. At a Chapman-Jouguet detonation, the system is characteristic to the side of the burnt gas for a genuinely nonlinear mode but non-characteristic to the side of the unburnt gas. In a geometrical admissibility criterion, kinetic phase boundaries are weak detonations, and compressive non-admissible ones are strong detonations. The mixed sonic-kinetic Riemann problem is solved in the case of a particular geometry in order to show how the mathematical analysis applies in the absence of an explicit physical example.

2 Semi-characteristic hyperbolic transition waves

Let $\Omega^{\pm}$ be two disjoint open subsets of $\mathbb{R}^N$, $\underline{u}^-, \underline{u}^+$ two constant states in Ω^-, Ω^+, f a smooth function defined in $\Omega := \Omega^- \cup \Omega^+$ with values in $\mathbb{R}^N$; we are concerned with the system of conservation laws in one space dimension

$$\partial_t u + \partial_x f(u) = 0. \tag{2.1}$$

We assume that f satisfies the Lax conditions [7], in each open set $\Omega^{\pm}$:

- The eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$ of the differential $Df(u)$ are real and distinct (strict hyperbolicity),

- For $i = 1, \dots, N$, either $D\lambda_i(u) \cdot r_i(u) \neq 0$ for every u (genuine nonlinearity), or $D\lambda_i \cdot r_i = 0$ for every u (linear degeneracy), where $r_i(u)$ denotes a right eigenvector of $Df(u)$ associated to the eigenvalue $\lambda_i(u)$.

Let s be a fixed integer between 1 and N; we assume that

$$\lambda_s(\underline{u}^-) \neq \lambda_i(\underline{u}^+) \quad \forall i \tag{2.2}$$

and that the eigenvalue λ_s is genuinely nonlinear at $\underline{u}^-$

$$D\lambda_s(\underline{u}^-) \cdot r_s(\underline{u}^-) = 1. \tag{2.3}$$

We also assume that the open sets $\Omega^\pm$ are small enough to conserve these properties uniformly.

A *simple phase wave* with speed $\sigma \in \mathbb{R}$ is a solution u of the system (2.1) which is constant on each side of the linear curve $x = \sigma t$ and satisfies $u := u^\pm \in \Omega^\pm$ in $\pm(x - \sigma t) > 0$; we denote by (u^-, σ, u^+) such a phase wave. Let $(\underline{u}^-, \underline{\sigma}, \underline{u}^+)$ be an *s-sonic phase wave*, that is $\lambda_s(\underline{u}^-) = \underline{\sigma}$. Since by (2.2) $\underline{\sigma}$ is not an eigenvalue at $\underline{u}^+$, we also call this phase transition wave a *semi-characteristic*.

2.1 Hugoniot's stationary vector field

Let (u^-, σ, u^+) be a phase wave close to the s-sonic background phase wave $(\underline{u}^-, \underline{\sigma}, \underline{u}^+)$. Across the linear curve $x = \sigma t$ the Rankine Hugoniot equation is

$$J(u^-, \sigma, u^+) := -\sigma[u] + [f(u)] = 0 \tag{2.4}$$

where $[u] = u^+ - u^-$. Note that the function J satisfies

$$D_{u^+} J(u^-, \sigma, u^+) = Df(u^+) - \sigma I\,, \quad D_\sigma J(u^-, \sigma, u^+) = -[u]\,,$$
$$D_{u^-} J(u^-, \sigma, u^+) = -Df(u^-) + \sigma I\,.$$

From (2.2), $D_{u^+} J(\underline{u}^-, \underline{\sigma}, \underline{u}^+)$ is not singular. By the implicit function theorem, the jump condition (2.4) is locally equivalent to $u^+ = H(\sigma, u^-)$ for a smooth *Hugoniot function* H. A simple phase wave is thus defined by the phase-boundary variables (σ, u^-). In that phase-boundary space, we call *sonic manifold* the hypersurface

$$\Sigma_s := \{(\sigma, u^-) : \sigma = \lambda_s(u^-)\}\,.$$

The sonic hypersurface keeps apart the subsonic phase boundaries, which are *determinate* (as waves in the (t, x) independent variables):

$$\mathcal{PB}_{\text{det}} := \{(\sigma, u^-) : \sigma \leq \lambda_s(u^-)\}\,,$$

and the supersonic phase boundaries, which are *indeterminate* :

$$\mathcal{PB}_{\text{ind}} := \{(\sigma, u^-) : \sigma > \lambda_s(u^-)\}\,,$$

(because some s-shock or s-rarefaction small waves may appear at the left side of the phase discontinuity).

At the sonic manifold Σ_s, the eigenvalue $(\lambda_s(u^-) - \sigma)$ of the partial differential $D_{u^-}J(u^-, \sigma, u^+)$, (which does not depend on u^+), vanishes and the associated eigenspace is $\mathbb{R}.r_s(u^-)$; we denote by

$$\partial_{r_s} := r_{s,1}(u^-)\partial_{u_1^-} + \cdots + r_{s,N}(u^-)\partial_{u_N^-}\,,$$

the vector field defined by the eigenvector $r_s(u^-)$, as a vector field in the u^--space as well as in the (σ, u^-)-space. By (2.3), ∂_{r_s} is transversal to Σ_s. For $y \in \{\sigma, u_1^-, \dots, u_N^-\}$, the solution $u^+ = H(\sigma, u^-)$ of (2.4), satisfies

$$\begin{aligned} &D_{u^+}J(u^-, \sigma, H(\sigma, u^-))\partial_y H(\sigma, u^-) \\ &\quad = -(D_{u^-}J)(u^-, \sigma, H(\sigma, u^-))\partial_y u^- - (D_\sigma J)(u^-, \sigma, H(\sigma, u^-))\partial_y \sigma\,. \end{aligned}$$

In particular, we have

$$\begin{gathered} (\partial_{r_s} H)(\lambda_s(u^-), u^-) = 0\,, \\ \partial_\sigma H(\sigma, u^-) = (D_{u^+}J(u^-, \sigma, H(\sigma, u^-)))^{-1}(H(\sigma, u^-) - u^-) \neq 0. \end{gathered}$$

For applications, a *general Hugoniot function* is a smooth family of smooth functions $H(\alpha, \cdot) : \Omega^- \to \Omega^+$, with parameter $\alpha \in \omega$, such as:
0 is a simple eigenvalue of $D_{u^-}H(\underline{\alpha}, \underline{u}^-)$,
the set $\{det(D_{u^-}H(\alpha, u^-)) = 0\}$ is a graph $\Sigma_s := \{(\alpha, u^-) \in \omega \times \Omega^- : \alpha = \alpha_s(u^-)\}$.

A *Hugoniot stationary vector field* is a smooth family of smooth fields $X = X(\alpha, u^-, \partial_{u^-})$ on Ω^-, with parameter α, which satisfies $XH = 0$ on Σ_s.

A *Hugoniot stationary manifold* is a smooth hypersurface $\Sigma := \{(\alpha, u^-) : \Lambda(\alpha, u^-) = 0\}$ which is transversal to Σ_s and satisfies $X\Lambda(\alpha, u^-) = 0$ along $\Sigma \cap \Sigma_s$.

2.2 Hugoniot's nonlinear dipoles

In the phase boundary space, near $(\underline{\sigma}, \underline{u}^-)$, let us consider the equation

$$H(\sigma, v^-) - H(\sigma, u^-) = 0\,, \tag{2.5}$$

where v^- is the unknown, and $v^- = u^-$ is an obvious solution. This equation can be written as

$$f(v^-) - f(u^-) - \sigma(v^- - u^-) = 0\,, \tag{2.6}$$

which means that v^- is connected to u^- by a small amplitude s-shock with speed σ (because σ is close to $\lambda_s(\underline{u}^-)$). To define v^- as a function of the phase boundary variable (σ, u^-), we review Lax's analysis. The equation (2.6) is equivalent to,

$$A(\sigma, u^-, v^-)(v^- - u^-) := \int_0^1 (Df(u^- + t(v^- - u^-)) - \sigma I)dt\ (v^- - u^-) = 0\,,$$

which means that $(v^- - u^-)$ belongs to the kernel of $A(\sigma, u^-, v^-)$. The eigenvalues of $A(\underline{\sigma}, \underline{u}^-, \underline{u}^-)$ are $(\lambda_i(\underline{u}^-) - \lambda_s(\underline{u}^-))$. Near $(\underline{\sigma}, \underline{u}^-, \underline{u}^-)$, let $\mu_s(\sigma, u^-, v^-)$ denote the continuous eigenvalue of $A(\sigma, u^-, v^-)$ which satisfies $\mu_s(\sigma, u^-, u^-) = \lambda_s(u^-) - \sigma$, $\{r_i(\sigma, u^-, v^-) : 1 \le i \le N\}$ a basis of right eigenvectors of $A(\sigma, u^-, v^-)$, $\{\ell_i(\sigma, u^-, v^-) : 1 \le i \le N\}$ the dual basis, and $M_s(\sigma, u^-, v^-)$ the matrix composed of rows, with the left eigenvectors $\ell_i(\sigma, u^-, v^-)$, $i \neq s$. From the symmetry of $A(\sigma, u^-, v^-)$ in (u^-, v^-) we get

$$D_{v^-}\mu_s(\sigma, u^-, u^-) = \frac{1}{2} D_{u^-}(\mu_s(\sigma, u^-, u^-)) = \frac{1}{2} D\lambda_s(u^-)\,.$$

As we have $D_{v^-}(M_s(\sigma, u^-, \cdot)(\cdot - u^-))(u^-) = M_s(\sigma, u^-, u^-)$, by the genuine non linearity property (2.3), the classical inverse function theorem applies, near $(\underline{\sigma}, \underline{u}^-, \underline{u}^-)$, to the equation

$$B(\sigma, u^-, v^-) := \begin{pmatrix} 2\mu_s(\sigma, u^-, v^-) \\ M_s(\sigma, u^-, v^-)(v^- - u^-) \end{pmatrix} = 0\,. \tag{2.7}$$

The smooth solution $v^- = v^-(\sigma, u^-)$ satisfies (2.5) and also $v^-(\lambda_s(u^-), u^-) = u^-$; if (σ, u^-) is sonic, that is, $\sigma = \lambda_s(u^-)$, then $(\sigma, v^-(\sigma, u^-))$ also is sonic, and coincides with $(\lambda_s(u^-), u^-)$. We may assume that $r_i(\sigma, u^-, u^-) = r_i(u^-)$. Then the first column of the matrix $(D_{v^-}B(\sigma, u^-, u^-))^{-1}$ is $r_s(u^-)$ and we have

$$D\lambda_s(u^-)\partial_\sigma v^-(\lambda_s(u^-), u^-) = 2\,, \tag{2.8}$$

$$\begin{aligned} D_{u^-}v^-(\lambda_s(u^-), u^-)r_s(u^-) = \\ & (D_{v^-}B(\lambda_s(u^-), u^-, u^-))^{-1} \begin{pmatrix} -D\lambda_s(u^-) \\ M_s(\lambda_s(u^-), u^-, u^-) \end{pmatrix} r_s(u^-) \\ & = (D_{v^-}B(\lambda_s(u^-), u^-, u^-))^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\, r_s(u^-), \end{aligned} \tag{2.9}$$

from which follows

$$\ell_s(u^-)(\partial_{r_s} v^-)(\lambda_s(u^-), u^-) \equiv \ell_s(u^-)(D_{u^-}v^-)(\lambda_s(u^-), u^-)r_s(u^-) = -1\,. \tag{2.10}$$

Now, differentiating the equation

$$-\sigma(v^-(\sigma, u^-) - u^-) + (f(v^-(\sigma, u^-) - f(u^-)) = 0\,, \tag{2.11}$$

we get

$$(\lambda_s(v^-) - \sigma)\ell_s(v^-)D_{u^-}v^-(\sigma, u^-)r_s(u^-) = (\lambda_s(u^-) - \sigma)\ell_s(v^-)r_s(u^-)$$

then, (2.10) implies that near $(\underline{\sigma}, \underline{u}^-)$, $(\lambda_s(v^-) - \sigma)$ and $(\lambda_s(u^-) - \sigma)$ have opposite signs (which is a classical result of Lax). As a consequence, if (σ, u^-) is subsonic, that is, $\sigma < \lambda_s(u^-)$, then $(\sigma, v^-(\sigma, u^-))$ is supersonic, that is $\sigma >$

$\lambda_s(v^-)$. The sonic twin solution of (2.5), $((\lambda_s(u^-), u^-), (\lambda_s(u^-), u^-))$, leaves the sonic manifold as a *subsonic-supersonic dipole* $((\sigma, u^-)_{sub}, (\sigma, v^-(\sigma, u^-))_{sup})$.

Denoting by $\epsilon_s \mapsto S_s(\epsilon_s, u^-)$ a smooth parametrization of the s-shock curve near $\underline{u}^-$, we can write

$$v^-(\sigma, u^-) = S_s(\epsilon_s^d(\sigma, u^-), u^-)\,, \tag{2.12}$$

for a smooth function $\epsilon_s^d(\sigma, u^-)$. For a classical parametrization [7, 6]

$$S_s(\epsilon_s, u^-) = u^- + \epsilon_s r_s(u^-) + \frac{\epsilon_s^2}{2} Dr_s(u^-).r_s(u^-) + O(\epsilon_s^3),$$

the speed $s_s(\epsilon_s, u^-)$ of the s-shock is

$$s_s(\epsilon_s, u^-) = \lambda_s(u^-) + \frac{\epsilon_s}{2} D\lambda_s(u^-).r_s(u^-) + O(\epsilon_s^2)\,.$$

When (σ, u^-) is subsonic, $\epsilon_s^d(\sigma, u^-) < 0$ is the strength of an admissible shock, called *(micro)-detonating strength*. Moreover we have, from (2.8) and (2.9),

$$\begin{gathered} \partial_\sigma \epsilon_s^d(\lambda_s(u^-), u^-) = 2\,, \\ (\partial_{r_s}\epsilon_s^d)(\lambda_s(u^-), u^-) \equiv (\partial_{u^-}\epsilon_s^d)(\lambda_s(u^-), u^-).r_s(u^-) = -2\,. \end{gathered} \tag{2.13}$$

2.3 The non-classical admissibility criterion

Let us consider a hypersurface Σ in the phase boundary space, near $(\underline{\sigma}, \underline{u}^-)$, parametrized by u^-,

$$\Sigma := \{(\sigma, u^-) : \sigma = \sigma_k(u^-)\}$$

where

$$\partial_{r_s}(\sigma_k - \lambda_s)(\underline{u}^-)) \neq 0\,. \tag{2.14}$$

Then Σ is transversal to Σ_s along the smooth manifold $\Sigma \cap \Sigma_s := \mathcal{C}_s$

$$\mathcal{C}_s = \{(\sigma, u^-) : \sigma_k(u^-) = \sigma = \lambda_s(u^-)\}\,.$$

Moreover, the u^--projection $\Pi\mathcal{C}_s$ of $\mathcal{C}_s$,

$$\Pi\mathcal{C}_s = \{u^- : \sigma_k(u^-) = \lambda_s(u^-)\}\,,$$

is a smooth hypersurface of $\mathbf{R}^N$ near $\underline{u}^-$. Let Σ_k be the indeterminate part (here supersonic) of Σ

$$\Sigma_k = \{(\sigma_k(u^-), u^-) : \sigma_k(u^-) > \lambda_s(u^-)\}.$$

Referring to [1], we call Σ_k the *kinetic hypersurface*. Let Σ_d be the (micro)-detonating set (determinate, here subsonic) associated to Σ_k

$$\Sigma_d := \{(\sigma, v^-(\sigma, u^-)) : (\sigma, u^-) \in \Sigma_k\}\,,$$

where $v^-(\sigma, u^-)$ is the solution of (2.7), or equivalently,

$$\begin{aligned}\Sigma_d &= \{(\sigma, u^-) : (\sigma, S_s(\epsilon_s^d(\sigma, u^-), u^-)) \in \Sigma_k\} \\ &= \{(\sigma, u^-) : \sigma_k(S_s(\epsilon_s^d(\sigma, u^-), u^-)) = \sigma\}\,.\end{aligned}$$

From (2.13), if

$$\partial_{r_s}\sigma_k(\underline{u}^-) \equiv D\sigma_k(\underline{u}^-)\,.\,r_s(\underline{u}^-) \neq \frac{1}{2}\,, \tag{2.15}$$

then Σ_d is an hypersurface with boundary $\mathcal{C}_s$, parametrized by u^-,

$$\Sigma_d := \{(\sigma, u^-) : \sigma = \sigma_d(u^-)\}\,.$$

In this case, we call Σ_d the *(micro)-detonating hypersurface.* From (2.13) again, the smooth function σ_d satisfies

$$\partial_{r_s}\sigma_d(u^-) \equiv D\sigma_d(u^-)\,.\,r_s(u^-) = \frac{\partial_{r_s}\sigma_k(u^-)}{2\partial_{r_s}\sigma_k(u^-) - 1}\,, \quad \text{along } \tilde{\mathcal{C}}_s\,. \tag{2.16}$$

In particular, Σ_d is transversal to Σ_s along $\mathcal{C}_s$.

In this geometry, we define the *admissible phase boundaries* by cutting up the supersonic phase boundaries but Σ_k, and deleting the subsonic ones belonging to

$$\mathcal{D} := \{(\sigma, S_s(\epsilon_s, u^-)) : (\sigma, u^-) \in \Sigma_d,\ \epsilon_s^d(\sigma, u^-) < \epsilon_s \leq 0\}\,.$$

Near the sonic phase boundary $(\underline{\sigma}, \underline{u}^-)$, the set $\mathcal{PB}_{ad}$ of the admissible phase boundaries writes

$$\mathcal{PB}_{ad} = \Sigma_k \cup (\mathcal{PB}_{\rm det} \setminus \mathcal{D})\,.$$

From (2.12) the S_s function pastes the hypersurfaces Σ_k and Σ_d as

$$\Sigma_d \ni (\sigma, u^-) \longmapsto (\sigma, S_s(\epsilon_s^d(\sigma, u^-), u^-)) \in \Sigma_k\,,$$

and the Hugoniot function H is defined on the pasted set,

$$H(\sigma, S_s(\epsilon_s^d(\sigma, u^-), u^-) = H(\sigma, u^-)\,. \tag{2.17}$$

2.4 A geometrical well-posedness criterion, with BV stability

Here we assume that Σ_k is not an Hugoniot stationary manifold :

$$(\partial_{r_s}\sigma_k)(u^-) \neq 0\,, \quad \forall u^- \in \Pi\mathcal{C}_s\,. \tag{2.18}$$

Σ_k is located above one side of $\Pi\mathcal{C}_s$; on that side we define $\tilde{\sigma}_k(u^-) = \sigma_k(u^-)$, on the other side we define $\tilde{\sigma}_k(u^-) = \lambda_s(u^-)$. The function $\tilde{\sigma}_k$ is continuous near $\underline{u}^-$ and piecewise smooth. Its graph $\tilde{\Sigma}_k$ is the union of Σ_k and of one part of Σ_s,

$$\tilde{\Sigma}_k = \Sigma_k \cup \{(\lambda_s(u^-), u^-) : \sigma_k(u^-) \leq \lambda_s(u^-)\}\,.$$

Denoting by $L_s(\epsilon_s, u^-)$ the s-Lax function which restriction to the shocks is $S_s(\epsilon_s, u^-)$, for a given (σ_0, u_0^-) close to $(\lambda_s(\underline{u}^-), \underline{u}^-)$, we obtain that the curve $\epsilon_s \mapsto (\sigma_0, L_s(\epsilon_s, u_0^-))$ is transversal to $\tilde{\Sigma}_k$ according to (2.18), and contained in the hyperplane $\{\sigma = \sigma_0\}$. It reaches the Lipschitz-continuous hypersurface $\tilde{\Sigma}_k$ for a defined $\epsilon_s = \tilde{\epsilon}_s(\sigma_0, u_0^-)$. The function $(\sigma, u^-) \mapsto \tilde{\epsilon}_s(\sigma, u^-)$ is Lipschitz-continuous and piecewise smooth; thus, the composition $H(\sigma, L_s(\tilde{\epsilon}_s(\sigma, u^-), u^-))$ has the same property

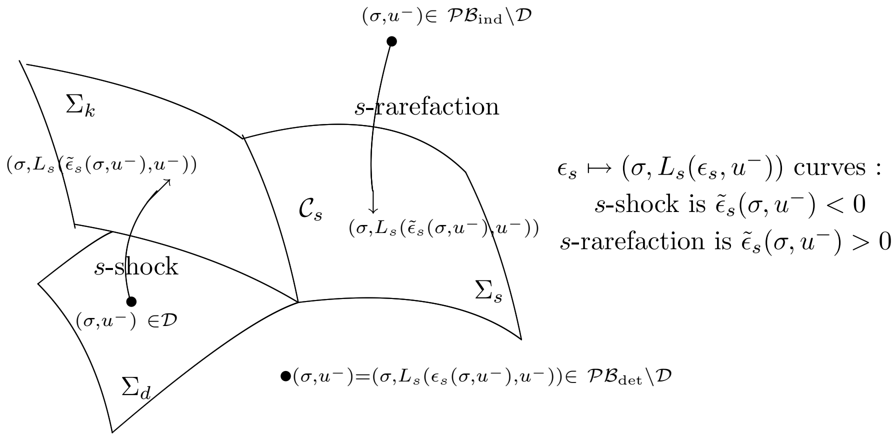


Figure 1: Mixed sonic-kinetic geometry: how to reach admissible phase boundaries

In regard to the pasting property (2.17), we already have a definition of Lipschitz-continuous for the phase boundary part in a mixed sonic-kinetic Riemann problem

$$u^+ = H(\sigma, L_s(\epsilon_s(\sigma, u^-), u^-))$$

where the discontinuous function

$$\epsilon_s(\sigma, u^-) := \begin{cases} \tilde{\epsilon}_s(\sigma, u^-), & \text{if } (\sigma, u^-) \in \mathcal{D} \text{ or } \sigma \geq \lambda_s(u^-) \\ 0, & \text{if } (\sigma, u^-) \notin \mathcal{D} \text{ and } \sigma \leq \lambda_s(u^-) \end{cases} \tag{2.19}$$

is defined in such a way that $(\sigma, L_s(\epsilon_s(\sigma, u^-), u^-))$ is an admissible phase boundary. To consider at most one indetermination in the Riemann problems, we assume that

$$\lambda_s(\underline{u}^+) < \lambda_s(\underline{u}^-) < \lambda_{s+1}(\underline{u}^+). \tag{2.20}$$

We use classical Lax functions $L_i^{\pm}(\epsilon_i, u^{\pm})$ to describe the small simple waves on the left $(-)$, or on the right $(+)$, of the phase transition curve $x = \sigma t$ in the space of the independent variables (t, x). We denote by $\epsilon_{nc}^- := (\epsilon_1, \dots, \epsilon_{s-1})$ the strength of the waves on the left, called *not causal*, that (surely) leave the discontinuity $x = \sigma t$; in the same way $\epsilon_{nc}^+ := (\epsilon_{s+1}, \dots, \epsilon_N)$. We write also

$$L_{nc}^-(\epsilon_{nc}^-, u^-) := L_{s-1}^-(\epsilon_{s-1}, \dots, L_1^-(\epsilon_1, u^-)\dots),$$
$$L_{nc}^+(\epsilon_{nc}^+, u^+) := L_N^+(\epsilon_N, \dots, L_{s+1}^+(\epsilon_{s+1}, u^+)\dots),$$

and $r_{nc}^\pm(u^\pm) := \partial_{\epsilon_{nc}^\pm} L_{nc}^\pm(0, u^\pm)$.
The mixed sonic-kinetic Riemann problem reads

$$\begin{gathered} u^+ = L_{nc}^+(\epsilon_{nc}^+, H(\sigma, L_s^-(\epsilon_s(\sigma, u^\#), L_{nc}^-(\epsilon_{nc}^-, u^-)))), \\ u^\# := L_{nc}^-(\epsilon_{nc}^-, u^-), \end{gathered} \tag{2.21}$$

where $\epsilon_s(\sigma, u^\#)$, defined by (2.19), forces the extra $s-$wave to reach transversally the mixed sonic-kinetic geometry when the phase boundary $(\sigma, u^\#)$ is not admissible. Its well-posedness may be obtained from the Lipschitz-continuous inverse function theorem of Clarke [2].

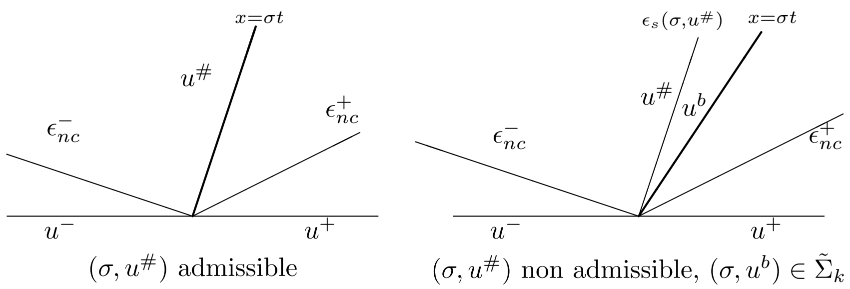


Figure 2:

A necessary condition for the well-posedness is the stability condition

$$\det\big(r_{nc}^-(\underline{u}^-)\,, \partial_\sigma H(\underline{\sigma}, \underline{u}^-)\,, r_{nc}^+(\underline{u}^+)\big) := \underline{D} \neq 0, \tag{2.22}$$

where

$$\partial_\sigma H(\underline{\sigma}, \underline{u}^-) = (Df(\underline{u}^+) - \lambda_s(\underline{u}^-)I)^{-1}[\,\underline{u}\,]\,.$$

To complete the proof of well-posedness we need to compute

$$D_{\epsilon_{nc}^-}(H(\underline{\sigma}, L_s^-(\tilde\epsilon_s(\underline{\sigma}, L_{nc}^-(\epsilon_{nc}^-, \underline{u}^-)), L_{nc}^-(\epsilon_{nc}^-, \underline{u}^-))))(0)$$

for the two smooth forms of $\tilde\epsilon_s(\sigma, u^-)$ associated to the impact of the $s-$Lax curves, starting from (σ, u^-), with Σ_s or Σ_k. This differential contains the term $\partial_{r_s} H(\underline{\sigma}, \underline{u}^-) = 0$ as a factor at each time where $\tilde\epsilon_s$ occurs genuinely. Thus, at point $(\epsilon_{nc}^-, \sigma, u^-) = (0, \underline{\sigma}, \underline{u}^-)$ the determinant of the differential of the composition associated to these two forms in (2.21) is still $\underline{D}$. Clarke's local inverse function theorem applies and the solution $(\epsilon_{nc}^-, \sigma, \epsilon_{nc}^+)(u^-, u^+)$ is Lipschitz-continuous.

To control the variation of the wave solution in the (t,x) independent variables, we need to estimate the strength

$$\tilde{\epsilon}_s(u^-,u^+) := \tilde{\epsilon}_s(\sigma(u^-,u^+), L^-_{nc}(\epsilon^-_{nc}(u^-,u^+),u^-))$$

of the possible s-wave at the left of the phase boundary $x=\sigma t$. Since, from (2.18), the function $\tilde{\epsilon}_s$ is Lipschitz-continuous we have

$$\begin{aligned}\tilde{\epsilon}_s(u^-,u^+) &= O(1)(|\sigma(u^-,u^+)-\underline{\sigma}| + |\epsilon^-_{nc}(u^-,u^+)| + |u^- - \underline{u}^-|)\\ &= O(1)(|u^- - \underline{u}^-| + |u^+ - \underline{u}^+|)\,.\end{aligned}$$

We have proved the following theorem.

Theorem 2.1 (mixed sonic-kinetic Riemann problem)
Let $(\underline{u}^-,\underline{\sigma},\underline{u}^+)$ be a sonic simple phase wave satisfying (2.2), (2.3), (2.20), (2.22). Let Σ a smooth hypersurface of $\mathbf{R}^{1+N}$ containing $(\underline{\sigma},\underline{u}^-)$ and satisfying (2.14), (2.15), (2.18). There exists a neighbourhood $\Omega^-\times\Omega^+$ of $(\underline{u}^-,\underline{u}^+)$, a neighbourhood $\omega^-_{nc}\times\omega\times\omega^+_{nc}$ of $(0,\underline{\sigma},0)$, such that for every $(u^-,u^+)\in\Omega^-\times\Omega^+$, there exists a unique $(\epsilon^-_{nc},\sigma,\epsilon^+_{nc})\in\omega^-_{nc}\times\omega\times\omega^+_{nc}$ satisfying

$$u^+ = L^+_{nc}(\epsilon^+_{nc}, H(\sigma, L^-_s(\epsilon_s(\sigma,u^\#), L^-_{nc}(\epsilon^-_{nc},u^-))))\,,\quad u^\# := L^-_{nc}(\epsilon^-_{nc},u^-)\,,$$

where $\epsilon_s(\sigma,u^-)$ is defined by (2.19), in such a way that $(\sigma, L^-_s(\epsilon_s(\sigma,u^\#))\in\mathcal{PB}_{ad}$.
The function $(u^-,u^+)\mapsto(\epsilon^-_{nc},\sigma,\epsilon^+_{nc})$ is Lipschitz-continuous and the variation of the phase wave solution is estimated by

$$|\epsilon^-_{nc}| + |\epsilon_s(\sigma,u^\#)| + |\sigma-\underline{\sigma}| + |\epsilon^+_{nc}| = O(1)(|u^- - \underline{u}^-| + |u^+ - \underline{u}^+|)\,.$$

Remark When Σ is an Hugoniot stationary manifold,

$$(\partial_{r_s}\sigma_k)(u^-) = 0\,,\quad \forall u^-\in\Pi\mathcal{C}_s\,,$$

which is not degenerate, $(\partial^2_{r_s}\sigma_k)(u^-)\neq 0$, for all $u^-\in\Pi\mathcal{C}_s$, Σ is located at one side of the hypersurface drawn from $\mathcal{C}_s$ by the integral curves of the Hugoniot stationary field ∂_{r_s}, with no intersection but $\mathcal{C}_s$, where they are tangent. Moreover, from (2.16), Σ_d also is an Hugoniot's stationary manifold. This geometry leads to an amplification of the variation of the solution of the mixed sonic-kinetic Riemann problem. The amplification concerns the polarization of u^- in the $r_s(u^-)$ direction. A simple example in the elasticity setting is studied in [11].

3 Two non-classical phase transition examples

3.1 Shearer-Yang model for dynamic elasticity

In [12], Shearer and Yang prove the existence of phase boundary profiles with finite viscosity and finite capillarity for a qualitative model in elasticity. The system

$$\begin{aligned} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x f(v) = \epsilon \partial_x^2 v - A\epsilon^2 \partial_x^3 u \end{aligned} \tag{3.1}$$

depends on a viscosity parameter $\epsilon > 0$ and a balance viscosity-capillarity fixed parameter $A > 0$. The stress f is cubic, $f(u) = u^3 - u$. Travelling waves solutions of (3.1), $(u, v)(t, x) = (U, V)(\frac{x-\sigma t}{\epsilon})$, with asymptotic conditions

$$(U, V)(-\infty) = (u_L, v_L)\,, \quad (U, V)(+\infty) = (u_R, v_R)\,, \quad (U', U'')(\pm\infty) = 0\,,$$

are defined by profiles $(U, V)(\xi)$, which satisfy $V(\xi) = v_L - \sigma(U(\xi) - U_L)$, and the ordinary differential equation

$$AU'' = -\sigma U' + f(U) - f(u_L) - \sigma^2(U - U_L)\,.$$

The state u_R is a rest point for this equation if (u_L, σ, u_R) satisfies the Rankine-Hugoniot condition

$$f(u_R) - f(u_L) - \sigma^2(u_R - u_L) = 0\,.$$

This is one of the jump equation for the discontinuous solution $(u, v) = (u_L, v_L)$ if $x < \sigma t$, $(u, v) = (u_R, v_R)$ if $x > \sigma t$, of the elasticity equation

$$\begin{aligned} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x f(v) = 0 \end{aligned} \tag{3.2}$$

The other jump condition, $(v_R - v_L) + \sigma(u_R - u_L) = 0$, is explicit in v_R; so, the relevant phase boundary variables are (σ, u_L). Phase transitions occur from $\{u_L < -1/\sqrt{3}\}$ to $\{u_R > 1/\sqrt{3}\}$ across a discontinuity with speed $\sigma < 0$. The boundary phase variables for the hyperbolic equation (3.2) are (σ, u_-), $\sigma < 0$, $u_- < -1/\sqrt{3}$. In this space the sonic manifold is the curve $\sigma = -\sqrt{3u_-^2 - 1}$.

For every A in $]1/3, 4/9[$, near the sonic phase boundary, $(\underline{\sigma}, \underline{u}_-) = \big(-\sqrt{3\frac{2+A}{2-3A}}, -\frac{2}{3}\sqrt{\frac{6}{2-3A}}\big)$, Shearer and Yang obtain simple phase waves as limit $\epsilon \to 0$ of viscosity-capillarity profiles $(U, V)(\frac{x-\sigma t}{\epsilon})$ for the following cases:

Case 1: $\sigma = \sigma_k(u_-) := -3\sqrt{\frac{A}{2}}\frac{u_- - \sqrt{3(6A-1)u_-^2 - 2(9A-2)}}{2-9A}$ in the domain $\sigma > -\sqrt{3u_-^2 - 1}$, this is the curve Σ_k,

Case 2: $\sigma < \sigma_d(u_-)$ in the domain $\sigma \le -\sqrt{3u_-^2 - 1}$, $u_- \ge \underline{u}_-$,

Case 3: $\sigma \le -\sqrt{3u_-^2 - 1}$ in the domain $u_- \le \underline{u}_-$,

where σ_d is defined from the Hugoniot dipoles $((\sigma, \sigma_k(u_-)), (\sigma, \sigma_d(u_-)))$. The borderline of the second domain is the detonation curve Σ_d, which is not admissible, and the borderline of the third domain is the admissible part of the sonic curve Σ_s.

Remark The viscosity-capillarity admissibility criterion, in the elasticity setting, has been introduced by Slemrod [10].

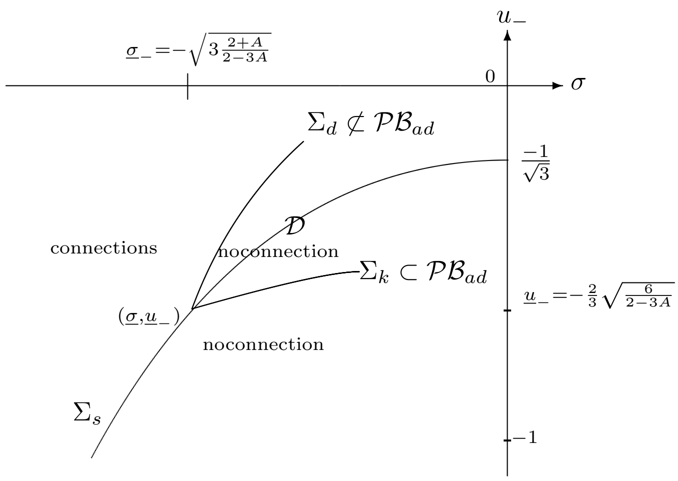


Figure 3: Viscosity-capillarity admissible phase boundaries of Shearer-Yang's model

3.2 Majda's model for dynamic combustion

In [8], Majda proves the existence of detonation profiles with finite reaction rate and finite diffusion for a qualitative model of combustion. The equation

$$\begin{gathered} \partial_t(u + q_0 z) + \partial_x f(u) = \beta \partial_x^2 u \\ \partial_t z = -K\varphi(u) z \end{gathered} \tag{3.3}$$

depends on a diffusion parameter $\beta > 0$ and a reaction rate parameter $K > 0$. The chemical reaction is exothermic, so that $q_0 > 0$. The condition satisfied by the smooth function φ,

$$\varphi(u) = \begin{cases} 0 & \text{for } u \leq 0 \\ \varphi(u) \in]0,1] & \text{for } u > 0 \\ 1 & \text{for } u \geq c_0 > 0 \end{cases}$$

is an ignition temperature kinetics condition. The flux f is smooth, strictly increasing and strictly convex. The travelling wave solutions $(u,z)(t,x) = (U,Z)(\frac{x-\sigma t}{\beta})$, with asymptotic conditions

$$(U,Z)(-\infty) = (u_L, 0), \quad (U,Z)(+\infty) = (u_R, 1), \quad (U',Z')(\pm\infty) = 0$$

satisfy the ordinary differential equation which depends on $K_0 := \beta K$,

$$\begin{gathered} U' = f(U) - \sigma(U + q_0 Z) + (\sigma u_L - f(u_L)) \\ Z' = \frac{1}{\sigma} K_0 \varphi(U) Z \end{gathered}$$

The state $(u_R, 1)$ is a rest point for this equation if $u_R \le 0$ and

$$f(u_R) - f(u_L) - \sigma(u_R + q_0 - u_L) = 0$$

which is the Rankine Hugoniot condition for the discontinuous solution $u = u_L$ if $x < \sigma t$, $u = u_R$ if $x > \sigma t$, of the combustion model

$$\begin{gathered} \partial_t(u + q_0 Y(u)) + \partial_x f(u) = 0, \\ Y(u) = 1 \text{ if } u < 0, \quad Y(u) = 0 \text{ if } u > 0\,. \end{gathered} \tag{3.4}$$

The boundary phase variables for the hyperbolic equation (3.4) are (σ, u_-), $\sigma > 0$, $u_- > c_0$. In this space the sonic manifold Σ_s is the curve $\sigma = f'(u_-)$.

In [8] we find the following result. First, a burnt state $(\underline{u}, \underline{z}) = (\underline{u}, 0)$, $\underline{u} > c_0$, being fixed, the Rankine-Hugoniot equation can be solved as $u_R = u_R(\sigma, u_L, q_0) \le 0$ for every (σ, u_L) close to the sonic phase boundary $(\underline{\sigma}, \underline{u}) = (f'(\underline{u}), \underline{u})$, if

$$q_0 > \hat{q}(\underline{\sigma}, \underline{u}) := \underline{u} - \frac{f(\underline{u}) - f(0)}{\underline{\sigma}}\,.$$

Let us remark that $\hat{q}$ is Hugoniot dipole invariant, that is $\hat{q}(\sigma, v(u, \sigma)) = \hat{q}(\sigma, u)$ for every Hugoniot dipole $((\sigma, v(\sigma, u)), (\sigma, u))$ near $(\underline{\sigma}, \underline{u})$. Moreover the level curves of $\hat{q}$, $\hat{q}(\sigma, u) = Cte$, are graphs of strictly convex smooth functions of u with minimum speed σ at the intersection with the sonic curve.

Second, K_0 being fixed, there exists a critical function $q_0^{cr}(\sigma, u_L, K_0)$ defined in the indeterminate (supersonic) domain $f'(u_L) \le \sigma$, which satisfies $q_0^{cr}(\sigma, u_L, K_0) \ge \hat{q}(\sigma, u)$, and such that:

- If $q_0 = q_0^{cr}(\sigma, u_L, K_0)$, it exists a unique connection $(U, Z)(\xi)$ between $(u_L, 0)$ and $(u_R(\sigma, u_L, q_0), 1)$,
- If $q_0 > q_0^{cr}(\sigma, u_L, K_0)$, it exists a unique connection between $(v(\sigma, u_L), 0)$ and $(u_R(\sigma, u_L, q_0), 1)$,
- If $q_0^{cr}(\sigma, u_L, K_0) > \hat{q}(\sigma, u)$ and $q_0^{cr}(\sigma, u_L, K_0) > q_0 > \hat{q}(\sigma, u)$, no connection between $(u_L, 0)$ or $(v(\sigma, u_L), 0)$ and $(u_R(\sigma, u_L, q_0), 1)$ exists.

For our purpose, we can read this result in the following way. An exothermic reaction parameter $q_0 > 0$ is fixed. For $K_0 > 0$, near a sonic phase boundary $(\underline{\sigma}, \underline{u})$ which satisfies

$$q_0^{cr}(f'(\underline{u}), \underline{u}, K_0) = q_0 > \hat{q}(f'(\underline{u}), \underline{u})$$

we define $\tilde{q}_0^{cr}$ as the extension by Hugoniot dipole invariance $\tilde{q}_0^{cr}(\sigma, v(\sigma, u)) = q_0^{cr}(\sigma, u)$. Simple phase waves are obtained as the limit, $\beta \to 0$, of viscosity-reacting travelling waves:

- For $q_0^{cr}(\sigma, u_L, K_0) = q_0$ in the domain $\sigma \ge f'(u_L)$,
- For $\tilde{q}_0^{cr}(\sigma, u_L, K_0) < q_0$ in the domain $\sigma \le f'(u_L)$.

We expect the function q_0^{cr} to be smooth enough, so the geometry of the phase boundaries space is again mixed sonic-kinetic.

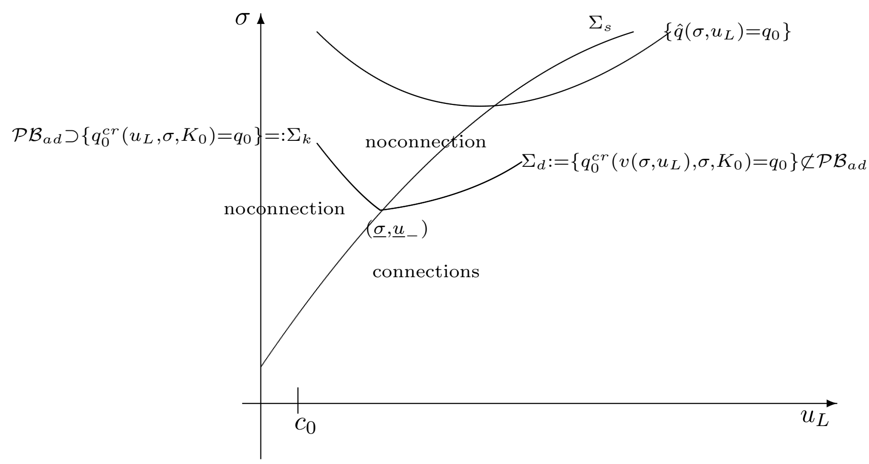


Figure 4: Viscosity-reaction admissible phase boundaries of Majda's model

4 The analytical entropy condition

We assume here that the system (2.1) admits an entropy η, convex in each domain $\Omega^{\pm}$. If q denotes the entropy-flux, $(Dq(u) = D\eta(u)Df(u))$, the admissible weak solutions of (2.1) with values in only one of the domains $\Omega^{\pm}$ satisfy the analytical entropy condition $\partial_t\eta(u) + \partial_x q(u) \le 0$, (which is equivalent to the geometrical compressive one [7]).

The only phase boundaries (σ, u^-) which satisfy the geometric entropy condition are the determinate (subsonic) ones. For $(u^-, \sigma, H(\sigma, u^-))$, a simple phase wave, the analytical entropy condition $\partial_t\eta(u) + \partial_x q(u) \le 0$ is equivalent to

$$\mathcal{E}(\sigma, u^-) := (\sigma\eta - q)(H(\sigma, u^-)) - (\sigma\eta - q)(u^-) \ge 0\,.$$

Since

$$\partial_{r_s}(\sigma\eta - q)(u^-) = (\sigma D\eta(u^-) - Dq(u^-))r_s(u^-) = D\eta(u^-)(\sigma - \lambda_s(u^-))r_s(u^-)$$

vanishes on Σ_s, as well as $\partial_{r_s}H$, we obtain $(\partial_{r_s}\mathcal{E})(\lambda_s(u^-), u^-) = 0$. Otherwise the derivation of the jump equation, we get that

$$(Df(H(\sigma, u^-)) - \sigma I)\partial_{r_s}H(\sigma, u^-) = (Df(u^-) - \sigma I)r_s(u^-) = (\lambda_s(u^-) - \sigma)r_s(u^-)$$

implies, using (2.3),

$$\begin{aligned}(Df(H(\lambda_s(u^-), u^-)) - \lambda_s(u^-)I)(\partial^2_{r_s}H)(\lambda_s(u^-), u^-) &= (\partial_{r_s}\lambda_s)(u^-)r_s(u^-)\\ &= r_s(u^-)\,.\end{aligned}$$

From what follows that for $u^+ = H(\lambda_s(u^-), u^-)$,

$$\begin{aligned}&\partial^2_{r_s}((\sigma\eta - q)\circ H)(\lambda_s(u^-), u^-)\\ &= D\eta(u^+)(\lambda_s(u^-)I - Df(u^+))(\partial^2_{r_s}H)(\lambda_s(u^-), u^-)\\ &= -D\eta(u^+)r_s(u^-)\,.\end{aligned}$$

Since $(\partial^2_{r_s}(\sigma\eta - q))(\lambda_s(u^-), u^-) = -D\eta(u^-)r_s(u^-)$, we obtain

$$(\partial^2_{r_s}\mathcal{E})(\lambda_s(u^-), u^-) = -(D\eta(u^+)) - D\eta(u^-))r_s(u^-)\,.$$

Therefore, the analytical entropy condition

$$\mathcal{E}(\underline{\sigma}, \underline{u}^-) > 0\,, \quad (D\eta(\underline{u}^+) - D\eta(\underline{u}^-))r_s(\underline{u}^-) < 0 \tag{4.1}$$

implies that near $(\underline{\sigma}, \underline{u}^-)$, along the integral curves of ∂_{r_s}, the entropy dissipation rate $\mathcal{E}(\sigma, u^-)$ is minimum at the sonic point, where it is positive. As a consequence, every phase boundary (σ, u^-) close to $(\underline{\sigma}, \underline{u}^-)$ satisfies the analytical entropy condition. Moreover, for an Hugoniot dipole $((\sigma, u^-), (\sigma, v(\sigma, u^-)))$ where (σ, u^-) is determinate (subsonic), we have

$$\mathcal{E}(\sigma, u^-) - \mathcal{E}(\sigma, v(\sigma, u^-)) = (\sigma\eta - q)(u^-) - (\sigma\eta - q)(v(\sigma, u^-)) \geq 0\,,$$

because σ is the speed of the admissible small amplitude s-shock from u^- to $v(\sigma, u^-)$. So, for an Hugoniot dipole, the entropy dissipation rate $\mathcal{E}$ is larger at the determinate phase boundary than at the indeterminate one; the excess is that of the small amplitude shock wave with the same speed, called above micro-detonation.

For lack of analytical entropy selection, and also lack of structure profiles for complex physical systems, but indications on some relevant models, we can use the properties of the entropy dissipation function $\mathcal{E}$ to define an admissible set of indeterminate (supersonic) phase boundaries near a sonic one $(\underline{\sigma}, \underline{u}^-)$. This is an idea of Abeyaratne and Knowles [1], in the elasticity setting. We consider an extra function ϕ which is constant along the integral curves of ∂_{r_s}, negative on Σ_s, and satisfies

$$\phi(\underline{\sigma}, \underline{u}^-) = -\mathcal{E}(\underline{\sigma}, \underline{u}^-)\,.$$

As $\partial_{r_s}(\mathcal{E} + \phi)$ vanishes on Σ_s, if $(\underline{\sigma}, \underline{u}^-)$ is not a critical point of $\mathcal{E} + \phi$, the set

$$K_\phi := \{(\sigma, u^-) : \mathcal{E}(\sigma, u^-) = -\phi(\sigma, u^-)\}$$

is an Hugoniot-stationary manifold, which has to be rejected, (see the remark in the previous section). So we assume that $(\underline{\sigma}, \underline{u}^-)$ is a critical point of $\mathcal{E} + \phi$, that is $D(\mathcal{E} + \phi)(\underline{\sigma}, \underline{u}^-) = 0$, and we use the Morse theory [9], in the only physically reasonable following case. The differential of $\mathcal{E} + \phi$ vanishes not only at $(\underline{\sigma}, \underline{u}^-)$ but all along an hypersurface $\mathcal{C}_s$ of Σ_s which contains $(\underline{\sigma}, \underline{u}^-)$,

$$D(\mathcal{E} + \phi)(\sigma, u^-) = 0, \quad \forall (\sigma, u^-) \in \mathcal{C}_s\,.$$

Also the second differential at point $(\underline{\sigma}, \underline{u}^-)$ of the restriction of $\mathcal{E} + \phi$ to a plane transversal to $\mathcal{C}_s$ has signature $(1, 1)$. Since every vector field which is tangent to Σ_s, is orthogonal to r_s at every point of $\mathcal{C}_s$, relatively to the second differential of $\mathcal{E} + \phi$, the set K_ϕ then is the union of two smooth hypersurfaces transversal along $\mathcal{C}_s$, not Hugoniot-stationary, every one being transversal to Σ_s.

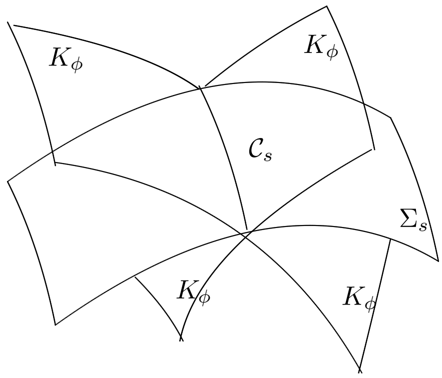


Figure 5:

The indeterminate (supersonic) part of one of these hypersurfaces can be selected as admissible phase boundaries subset Σ_k. This set defines a (micro)-detonating hypersurface Σ_d to be rejected, and a open domain $\mathcal{D}$ (with borderline $\Sigma_k \cup \Sigma_d$) also to be rejected. Then only the sonic phase boundaries which do not belong to $\mathcal{D}$ may be admissible, in a mixed sonic-kinetic setting.

5 A non-classical Riemann problem near a sonic detonation wave.

5.1 Chapman-Jouguet detonations

We consider the one-dimensional combustion model, involving an infinite reaction rate between ideal polytropic gases with the same γ-law, $\gamma > 1$, and constant heat of complete exothermic reaction $Q > 0$. This model is governed by the Euler equations of gas dynamics. In Eulerian coordinates $V = (\rho, \rho u, \rho e)$, the conservative form is

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0 \\ \partial_t(\rho e) + \partial_x((\rho e + p)u) = 0 \end{aligned}$$

A discontinuity curve $x = \chi(t)$ moving to the right, separates the burnt gas, which is on its left, from the unburnt gas which is on its right. We denote with the index $-$ the functions and variables in the burnt gas, by the index $+$ those in the unburnt gas.

In the following, notation is taken from the book by [6]. The specific internal energy is $\epsilon = e - \frac{u^2}{2}$, the specific volume is $\tau = 1/\rho$; the equations of state, with constant energies of formation $Q_\pm$, are

$$\epsilon_-(\tau_-, p_-) = \frac{\tau_- p_-}{\gamma - 1} + Q_- \,, \quad \epsilon_+(\tau_+, p_+) = \frac{\tau_+ p_+}{\gamma - 1} + Q_+ \,, \quad Q := Q_+ - Q_- > 0 \,.$$

The eigenvalue which occurs for detonations is

$$\lambda_3 = u_- + \sqrt{\gamma p_- \tau_-}\,.$$

We select a Chapman-Jouguet background detonation wave $(\underline{V}^-, \underline{\sigma}, \underline{V}^+)$. The states $\underline{V}^\pm$ are constant and they are related on each side of the discontinuity $x = \underline{\sigma} t$ by the jump conditions, where $[x] := x^+ - x^-$,

$$\begin{aligned} \sigma[\rho] &= [\rho u] \\ \sigma[\rho u] &= [\rho u^2 + p] \\ \sigma[\rho e] &= [(\rho e + p)u] \end{aligned} \tag{5.1}$$

Moreover, the wave is 3-sonic on the left hand side; it satisfies the Chapman-Jouguet condition

$$\sigma - u_- = \sqrt{\gamma p_- \tau_-}$$

with the detonation property

$$\sigma - u_+ > \sqrt{\gamma p_- \tau_-}\,,$$

from which follows $[p] < 0 < [\tau]$. Using the dependent variables

$$M := \frac{u_- - \sigma}{\tau_-}\,, \quad U^\pm = (u_\pm, \tau_\pm, p_\pm)\,,$$

where M is negative, the equation (5.1) for the simple phase waves is equivalent to

$$\tilde{J}(U^-, M, U^+) := \begin{pmatrix} [u] - M[\tau] \\ [p] + M^2[\tau] \\ [\epsilon] + \frac{1}{2}[\tau](p_- + p_+) \end{pmatrix} = 0\,, \tag{5.2}$$

from which $M = \frac{u_+ - \sigma}{\tau_+}$ follows. The third row also reads

$$(\frac{\gamma+1}{\gamma-1}\tau_- - \tau_+)p_- - (\frac{\gamma+1}{\gamma-1}\tau_+ - \tau_-)p_+ - 2Q = 0\,.$$

We have

$$D_{U^+}\tilde{J}(U^-, M, U^+) = \begin{pmatrix} 1 & -M & 0 \\ 0 & M^2 & 1 \\ 0 & (-p_- - \frac{\gamma+1}{\gamma-1}p_+) & -(\frac{\gamma+1}{\gamma-1}\tau_+ - \tau_-) \end{pmatrix},$$

$$\det D_{U^+}\tilde{J}(U^-, M, U^+) = \frac{2\tau_+}{\gamma-1}(\gamma\frac{p_+}{\tau_+} - M^2) \neq 0\,,$$

near the Chapman-Jouguet background detonation $(\underline{U}^-, \underline{M}, \underline{U}^+)$. So, detonations are semi-characteristic phase transition waves and the jump equation (5.2) is locally equivalent to $U^+ = \tilde{H}(M, U^-)$ for a smooth Hugoniot function $\tilde{H}$. One eigenvalue of

$$D_{U^-}\tilde{J}(U^-, M, U^+) = \begin{pmatrix} -1 & M & 0 \\ 0 & -M^2 & -1 \\ 0 & (\frac{\gamma+1}{\gamma-1}p_- + p_+) & (\frac{\gamma+1}{\gamma-1}\tau_- - \tau_+) \end{pmatrix},$$

is -1, the others are the solutions λ of the equation

$$\lambda^2 + (M^2 + [\tau] - \frac{2\tau_-}{\gamma - 1})\lambda + \frac{2\tau_-}{\gamma - 1}(\gamma\frac{p_-}{\tau_-} - M^2) = 0\,.$$

So, at a Chapman-Jouguet detonation, 0 is a simple eigenvalue of $D_{U^-}\tilde{H}$ and the (sonic) set $\{det(D_{U^-}\tilde{H}(M,U^-)) = 0\}$ is the graph $\tilde{\Sigma}_s = \{(M,U^-) : M = -\sqrt{\gamma\frac{p_-}{\tau_-}}\}$. For $(M,U^-) \in \tilde{\Sigma}_s$, the kernel of $D_{U^-}\tilde{H}(M,U^-)$ is $\mathbb{R}^t(M,1,-M^2)$. Then in the (M,U^-) variables, a canonical Hugoniot's stationary vector field is

$$\tilde{X} := M\partial_{u_-} + \partial_{\tau_-} - M^2\partial_{p_-}\,.$$

5.2 Hugoniot calculus, reduced phase boundary variables

The solution $U^+ = \tilde{H}(M,U^-)$ reads

$$(u_+,\tau_+,p_+) = (u_- + M(\tau_+ - \tau_-), H(M,\tau_-,p_-))$$

and the set $\tilde{\Sigma}_s$ is defined by a function independent of u_-. Then a phase transition may be identified by the *reduced variables* $R := (M,\tau_-,p_-)$. In this space, the *Chapman-Jouguet or sonic set* is the surface

$$\Sigma_s := \big\{(M,\tau_-,p_-) : -M = \sqrt{\gamma\frac{p_-}{\tau_-}}\big\}$$

the *indeterminate* (supersonic) combustion boundaries $R := (M,\tau_-,p_-)$ are defined by $-M > \sqrt{\gamma\frac{p_-}{\tau_-}}$ and the *determinate* (subsonic) ones by $-M < \sqrt{\gamma\frac{p_-}{\tau_-}}$. Let

$$\beta_\pm := \sqrt{\gamma\frac{p_\pm}{\tau_\pm}}\,, \quad q := \sqrt{2Q\frac{\gamma-1}{\gamma+1}}\,.$$

At a sonic point $R_s \in \Sigma_s$, the thermodynamical part

$$X := \partial_{\tau_-} - M^2\partial_{p_-}$$

of the Hugoniot's stationary vector field writes $X_s := \partial_{\tau_-} - \beta_-^2\partial_{p_-}$ and is transversal to Σ_s. We also denote by $J(R,\tau_+,p_+)$ the thermodynamic part of $\tilde{J}$, $(\tau_+(R),p_+(R)) = H(R)$ the local solution of $J(R,\tau_+,p_+) = 0$. Using differential calculus, we obtain

$$\begin{gathered}
\partial_M\tau_+(R) = \frac{M[\tau]((\gamma+1)\tau_+ - (\gamma-1)\tau_-)}{\tau_+(\beta_+^2 - M^2)}\,,\\
\partial_M p_+(R) = \frac{-M[\tau]((\gamma+1)p_+ + (\gamma-1)p_-)}{\tau_+(\beta_+^2 - M^2)}\,,\\
\partial_{\tau_-}\tau_+(R) = \frac{\tau_+\beta_+^2 - \tau_-M^2}{\tau_+(\beta_+^2 - M^2)}\,, \quad \partial_{\tau_-}p_+(R) = \frac{M^2[p]}{\tau_+(\beta_+^2 - M^2)}\,,\\
\partial_{p_-}\tau_+(R) = \frac{-\gamma[\tau]}{\tau_+(\beta_+^2 - M^2)}\,, \quad \partial_{p_-}p_+(R) = \frac{\tau_-\beta_-^2 - \tau_+M^2}{\tau_+(\beta_+^2 - M^2)}\,,\\
X\tau_+(R) = \frac{\tau_-(\beta_-^2 - M^2)}{\tau_+(\beta_+^2 - M^2)}\,, \quad Xp_+(R) = \frac{\tau_-M^2(M^2 - \beta_-^2)}{\tau_+(\beta_+^2 - M^2)}\,.
\end{gathered} \tag{5.3}$$

5.3 Entropy analysis

The entropy inequality required for physical solutions of (5.1) reads $\partial_t(\rho s) + \partial_x(\rho s u) \geq 0$, where the entropy s is

$$s_\pm = s_{0,\pm} + \frac{r_0}{\gamma - 1} \log(\frac{\tau_\pm^\gamma p_\pm}{\gamma - 1}) .$$

For the simple detonation waves, this inequality is equivalent to

$$[\rho s(u - \sigma)] = M[s] \geq 0 , \quad \text{or} \quad [s] \leq 0 ,$$

that is

$$\mathcal{E}(R) := s_+(\tau_+(R), p_+(R)) - s_-(\tau_-, p_-) \leq 0 .$$

Everywhere we have

$$Xs_-(R) = \frac{r}{\gamma - 1} \frac{1}{p_-}(\beta_-^2 - M^2) ,$$

and at a sonic point R_s,

$$X\tau_+(R_s) = 0 , \quad Xp_+(R_s) = 0 , \quad Xs_-(R_s) = 0 . \tag{5.4}$$

Then $X\mathcal{E}(R_s) = 0$. Moreover we have

$$\begin{gathered} \partial_M \mathcal{E}(R) = r_0 \frac{M[\tau]^2}{\tau_+ p_+} , \quad \partial_{\tau_-} \mathcal{E}(R) = \frac{r_0}{\gamma - 1} \frac{[\tau](M^2\tau_- - \beta_+^2 \tau_+)}{\tau_- \tau_+ p_+} , \\ X\mathcal{E}(R) = \frac{r_0}{\gamma - 1} \frac{[\tau]}{\tau_+ p_- p_+}(\beta_-^2 - M^2)(\tau_+ M^2 - p_-) , \end{gathered} \tag{5.5}$$

and on Σ_s,

$$M^2[\tau]^2 = q^2 , \quad 2\tau_- - \gamma[\tau] = \gamma \frac{p_- + p_+}{M^2} > 0 , \quad X\beta_-^2 = -\frac{\gamma + 1}{\tau_-} \beta_-^2 . \tag{5.6}$$

As a consequence, at a sonic point $R_s \in \Sigma_s$,

$$X^2\mathcal{E}(R_s) = -r_0 \frac{\gamma + 1}{\gamma - 1} \frac{[\tau]}{\tau_- \tau_+ p_- p_+} ([\tau] + \frac{\gamma - 1}{\gamma} \tau_-)\beta_-^4 < 0 .$$

Thus, along the integral curves of X, the entropy dissipation rate $\mathcal{E}$ is locally maximum at the Chapman-Jouguet detonation. From (5.6) we also get at a sonic point

$$\frac{\tau_+^\gamma p_+}{\tau_-^\gamma p_-} = (1 + \frac{q}{\sqrt{\gamma p_- \tau_-}})^\gamma (1 - \frac{\gamma q}{\sqrt{\gamma p_- \tau_-}}) .$$

Then the entropy inequality is satisfied near the background detonation if we assume that its sonic speed $\underline{\sigma}$ satisfies

$$(1 + \frac{q}{\underline{\sigma} - \underline{u}_-})^\gamma \, (1 - \frac{\gamma q}{\underline{\sigma} - \underline{u}_-}) < e^{-[s_0]\frac{\gamma - 1}{r_0}} . \tag{5.7}$$

5.4 Entropic selection of admissible weak detonations

Only partial results are known about the existence of structure profiles for reacting compressible Navier-Stokes equations. In our knowledge, the most advanced analysis is that of Gardner [5], in Lagrangian coordinates, with an ignition temperature assumption, (see also Wagner [13]). A rewriting of his existence theorem [13, Theorem 2.1, p. 438], where assumptions concern the burnt states (in place of the unburnt ones, which is not correct), leads to the existence of a critical liberated energy q^{cr} in the way of [8]. The function q^{cr} defines a borderline of an existence domain for detonation structure profiles. Then, in the reduced phase boundary variables $(\sigma, (\tau, T))$, admissible indeterminate phase transitions have to belong to a "kinetic" surface. The geometric properties of this surface at the intersection with the sonic surface should require some estimates for the asymptotic unburnt critical point $C(z_*)$ of the weak connection of Gardner. In lack of detailed structure profiles, but with the indication of Majda's model in section 3.2 and Gardner's result, we present here an example using the entropy to select the indeterminate (supersonic) combustion boundaries near a sonic one $\underline{R} = (\underline{M}, \underline{\tau_-}, \underline{p_-})$, following section 4.

With (M, τ_-) as parameters of the sonic surface Σ_s near $\underline{R} = (\underline{M}, \underline{\tau_-}, \underline{p_-})$, we select the simplest curve $\mathcal{C}_s \subset \Sigma_s$ in regards to calculus and geometry to illustrate our previous theory.

$$\Sigma_s = \{R_- : p_- = p_s(M, \tau_-) := \frac{1}{\gamma} M^2 \tau_-\},$$
$$\mathcal{C}_s = \{R_- : R_- = R_s(\tau_-) := (\underline{M}, \tau_-, p_s(\underline{M}, \tau_-))\}.$$

We consider an extra constitutive function $\phi(R)$, of thermodynamical nature, which is constant along the integral curves of the field X and defined by a smooth positive function ϕ_s on the sonic surface

$$X\phi = 0, \quad \phi(M, \tau_-, \frac{\tau_- M^2}{\gamma}) = \phi_s(M, \tau_-), \quad \phi(\underline{R}) = \phi_s(\underline{M}, \underline{\tau_-}) = -\mathcal{E}(\underline{R}).$$

We assume that the function $\mathcal{E} + \phi$ is critical all along the curve $\mathcal{C}_s$ of the sonic surface. Using (5.5), and $\partial_{\tau_-}\mathcal{E}(R_s) = \frac{\gamma r_0}{\gamma - 1} \frac{[\tau]^2 M^2}{\tau_- \tau_+ p_+}$, we get the zero and first order conditions to be satisfied by ϕ_s along $\mathcal{C}_s$:

$$\phi_s(\underline{M}, \tau_-) = -\mathcal{E}(R_s(\tau_-)).$$
$$0 = Z_s(\mathcal{E} + \phi)(R_s(\tau_-)) \equiv r_0 \gamma \frac{\gamma + 1}{\gamma - 1} \frac{q^2}{\underline{M}(\underline{M}\tau_- + q)(\underline{M}\tau_- - \gamma q)} + \partial_M \phi_s(\underline{M}, \tau_-)),$$

where

$$Z_s := \partial_M + \frac{2\tau_- M}{\gamma} \partial_{p_-}.$$

(The vector field Z_s is tangent to Σ_s, transversal to $\mathcal{C}_s$ and $(Z_s\phi)(M, \tau_-, \frac{\tau_- M^2}{\gamma}) = \partial_M \phi_s(M, \tau_-))$.

Also we assume that, at point $\underline{R}$, the second differential of the restriction of $\mathcal{E}+\phi$ to the plane generated by the components $x := (0,1,-M^2)$ and $z := (1,0,\frac{2\tau_- M}{\gamma})$ of X and Z_s has signature $(1,1)$. As along the curve $\mathcal{C}_s$, we have

$$\begin{aligned}
D^2(\mathcal{E}+\phi)(R_s(\tau_-))(x,x) &= X^2(\mathcal{E}+\phi)(R_s(\tau_-)) = X^2\mathcal{E}(R_s(\tau_-)) < 0\,,\\
D^2(\mathcal{E}+\phi)(R_s(\tau_-))(x,z) &= Z_sX(\mathcal{E}+\phi)(R_s(\tau_-)) = 0\,,\\
D^2(\mathcal{E}+\phi)(R_s(\tau_-))(z,z) &= Z_s^2(\mathcal{E}+\phi)(R_s(\tau_-))\\
&= Z_s^2\mathcal{E}(R_s(\tau_-)) + \partial_M^2\phi_s(\underline{M},\tau_-)\,,\\
Z_s^2\mathcal{E}(R_s(\tau_-)) &= -r_0\gamma\frac{\gamma+1}{\gamma-1}\frac{q^2}{\underline{M}^2(\underline{M}\tau_-+q)(\underline{M}\tau_--\gamma q)}\\
&\quad\times\Big(1+\frac{M\tau_-}{(\underline{M}\tau_-+q)}+\frac{M\tau_-}{(\underline{M}\tau_--\gamma q)}\Big)\,,
\end{aligned}$$

the second order condition $X^2\mathcal{E}(\underline{R})(Z_s^2\mathcal{E}(\underline{R})+\partial_M^2\phi_s(\underline{M},\underline{\tau}_-))<0$, needs to be satisfied by ϕ_s; that is,

$$\begin{aligned}
\partial_M^2\phi_s(\underline{M},\underline{\tau}_-)) > &r_0\gamma\frac{\gamma+1}{\gamma-1}\frac{q^2}{\underline{M}^2(\underline{M\tau}_-+q)(\underline{M\tau}_--\gamma q)}\\
&\times\Big(1+\frac{\underline{M\tau}_-}{(\underline{M\tau}_-+q)}+\frac{\underline{M\tau}_-}{(\underline{M\tau}_--\gamma q)}\Big)\,.
\end{aligned}\tag{5.8}$$

Under conditions (5.4), (5.4), (5.8) the set $\{R : (\mathcal{E}+\phi)(R)=0\}$ is the union of two smooth surfaces $\Sigma^\pm$, transversal along $\mathcal{C}_s$, which are not Hugoniot stationary (relatively to the field X).

By $Z_s^2(\mathcal{E}+\phi)>0$ on $\mathcal{C}_s$, the zeroes of $(\mathcal{E}+\phi)$ leave the sonic surface in the Z_s direction, and they leave by pair, subsonic and supersonic, along the integral curves of X, because $\mathcal{E}+\phi$ is maximum at the sonic points. Therefore the $\Sigma^\pm$ may be parametrized by (M,τ_-) near $\underline{R}$ as

$$\Sigma^\pm = \{(M,\tau_-,p_-) : p_- = p^\pm(M,\tau_-)\}$$

for smooth functions $p^\pm$ which satisfy $p^\pm(\underline{M},\tau_-)=p_s(\underline{M},\tau_-)$ for every τ_- and also

$$\begin{aligned}
\partial_M p^\pm(\underline{M},\tau_-) &= \partial_M p_s(\underline{M},\tau_-)\\
&\pm\frac{\gamma+1}{\gamma}\frac{\underline{M}^2}{(X^2\mathcal{E})(R_s(\tau_-))}\sqrt{-(X^2\mathcal{E}\,.\,Z_s^2(\mathcal{E}+\phi))(R_s(\tau_-))}\,.
\end{aligned}\tag{5.9}$$

Hence, the $\Sigma^\pm$ intersect the sonic surface Σ_s transversally, and they are located on either side of the plane $\{M=\underline{M}\}$. We choose as kinetic one indeterminate (supersonic) part Σ_k of $\Sigma^-\cup\Sigma^+$, for example $\Sigma^+\cap\{M\le\underline{M}\}$, and we denote by p_k the restriction of p^+ to the set $M\le\underline{M}$. Then the chosen *kinetic manifold* is

$$\Sigma_k = \{R : p_- = p_k(M,\tau_-)\}\,,$$

where the smooth function p_k is defined for $M\le\underline{M}$ satisfying: $p_k(\underline{M},\tau_-) = p_s(\underline{M},\tau_-)$, $p_k(M,\tau_-)\le p_s(M,\tau_-)$, and the (+) part of (5.9).

5.5 Entropic excision of non-admissible strong detonations

The counterpart of the admissibility of the weak detonation defined by Σ_k is to reject some strong detonations as non-admissible. This set is defined from the (micro)-detonating set Σ_d associated to Σ_k.

To describe this (micro)-detonating set, which is subsonic, we use the (micro)-detonating strength defined in section 2.3. In the combustion setting, the speed of the 3-shocks in the burnt gas is

$$s_3(\epsilon_3, U^-) = u_- + \tau_- \sqrt{\beta_-^2 + \frac{(\gamma+1)\epsilon_3}{2\tau_-}}\,, \quad \text{where} \quad \epsilon_3 = p_+ - p_- \le 0\,.$$

If we denote by $\eta := (\tau, p)$ the thermodynamic variable, and

$$m(\epsilon_3, \eta^-) := \frac{u_- - s_3(\epsilon_3; U^-)}{\tau_-} = -\sqrt{\beta_-^2 + \frac{(\gamma+1)\epsilon_3}{2\tau_-}}\,,$$

recalling that $M = \frac{u_- - \sigma}{\tau_-}$, we see that $s_3(\epsilon_3; U^-) = \sigma$ is equivalent to $m(\epsilon_3, \eta^-) = M$. Let

$$\epsilon_3^d(M, \eta^-) := \frac{2\tau_-}{\gamma+1}(M^2 - \beta_-^2)$$

be the solution ϵ_3 of the equation $m(\epsilon_3, \eta^-) = M$. The (micro)-detonating set Σ_d associated to Σ_k is $\Sigma_d := \{R : (M, \ell_3(\epsilon_3^d(M, \eta^-), \eta^-)) \in \Sigma_k\}$, $\ell_3(\epsilon_3, \eta^-)$ being the thermodynamic part of the Lax function for the 3-shocks,

$$\ell_3(\epsilon_3, \eta^-) := \begin{pmatrix} \frac{\tau_-(2\gamma p_- + (\gamma-1)\epsilon_3)}{2\gamma p_- + (\gamma+1)\epsilon_3} \\ p_- + \epsilon_3 \end{pmatrix}.$$

Thus,

$$\Sigma_d = \big\{R : p_- + \epsilon_3^d(M, \tau_-, p_-) - p_k(M, \frac{1}{\gamma+1}((\gamma-1)\tau_- + \frac{2\gamma p_-}{M^2}) = 0\big\}$$

since at point $(\underline{M}, \underline{\eta}^-)$ the derivative ∂_{p_-} of the vanishing function which defines Σ_d is -1, Σ_d is an hypersurface parametrized by (M, τ_-). The *micro-detonating manifold* reads

$$\Sigma_d = \big\{(M, \tau_-, p_-) : p_- = p_d(M, \tau_-)\big\}\,,$$

where the smooth function p_d is defined for $M \le \underline{M}$ and satisfies

$$\begin{gathered} p_d(\underline{M}, \tau_-) = p_s(\underline{M}, \tau_-), \quad p_d(M, \tau_-) \ge p_s(M, \tau_-)\,, \\ \partial_M p_d(\underline{M}, \tau_-) = 2\partial_M p_s(\underline{M}, \tau_-) - \partial_M p_k(\underline{M}, \tau_-)\,. \end{gathered}$$

The detonations which belong to the set

$$\mathcal{D} := \big\{(M, \ell_3(\epsilon_3, \eta^-)) \ : \ \epsilon_3^d(M, \eta^-) < \epsilon_3 \le 0\,, (M, \eta) \in \Sigma_d\big\}$$

have to be rejected. The sonic and subsonic parts of $\mathcal{D}$ are the sets of the non-admissible Chapman-Jouguet or strong detonations induced by the entropic selection Σ_k for the weak detonations.

5.6 The phase boundary part in a non-classical combustion Riemann problem

Since $p_k(M,\tau_-) \le p_s(M,\tau_-) \le p_d(M,\tau_-)$, a simple combustion wave (U^b,σ) with speed σ and state $U^b = (u_b,\tau_b,p_b)$ on the left, close to a Chapman-Jouguet detonation $((\underline{u}_-,\underline{\tau}_-,\underline{p}_-),\underline{u}_- + \sqrt{\gamma\underline{\tau}_-\underline{p}_-})$, satisfies the properties as shown on Table 1.

Sonic	$p_b = p_s(M,\tau_b) = \frac{\tau_b M^2}{\gamma}$	
(strictly)supersonic	$p_b < p_s(M,\tau_b)$	
(strictly)subsonic	$p_b > p_s(M,\tau_b)$	
Kinetic	$M \le \underline{M}$ and $p_b = p_k(M,\tau_-)$	then it is indeterminate, admissible, and supersonic
(micro)-detonating	$M \le \underline{M}$ and $p_b = p_d(M,\tau_-)$	then it is determinate, non admissible, and subsonic
Determinate and admissible	$M \le \underline{M}$ and $p_b > p_d(M,\tau_-)$ or $M \ge \underline{M}$ and $p_b \ge p_s(M,\tau_-)$	then it is strictly subsonic, or subsonic

Table 1: Properties of a simple combustion wave

We denote by $\Sigma_s^a := \{R \in \Sigma_s : M \ge \underline{M}\}$ the set of admissible sonic phase boundaries. By $\Sigma_k \cup \Sigma_s^a$ we denote the graph of the function

$$\tilde{p}_k(M,\tau_-) = \begin{cases} p_k(M,\tau_-) & \text{if } M \le \underline{M} \\ p_s(M,\tau_-) & \text{if } M \ge \underline{M}' \,. \end{cases}$$

By $\Sigma_d \cup \Sigma_s^a$ we denote the graph of the function

$$\tilde{p}_d(M,\tau_-) = \begin{cases} p_d(M,\tau_-) & \text{if } M \le \underline{M}\,, \\ p_s(M,\tau_-) & \text{if } M \ge \underline{M}\,. \end{cases}$$

Coming back to the U-variable, $U = (u,\tau,p) = (u,\eta)$, we denote by

$$\tilde{\Sigma}_s^a = \{(M,U^-) : (M,\eta^-) \in \Sigma_s^a\}\,, \quad \tilde{\Sigma}_d = \{(M,U^-) : (M,\eta^-) \in \Sigma_d\}\,,$$
$$\tilde{\Sigma}_k = \{(M,U^-) : (M,\eta^-) \in \Sigma_k\}$$

the unfolded admissible sonic, (micro)-detonating, and kinetic manifolds. We paste $\tilde{\Sigma}_d$ and $\tilde{\Sigma}_k$ with the map

$$\tilde{\Sigma}_d \ni (M,U^-) \longmapsto (M, S_3(\epsilon_3^d(M,\eta^-),U^-)) \in \tilde{\Sigma}_k\,,$$

where $S_3(\epsilon_3,U^-)$ is, for $\epsilon_3 \le 0$, the admissible shock half-curve of the Lax curve

$$\epsilon_3 \mapsto L_3(\epsilon_3,U^-) := (\nu_3(\epsilon_3,U^-),\ell_3(\epsilon_3,\eta^-)) := (\nu_3(\epsilon_3,U^-),\mu_3(\epsilon_3,\eta^-),p_- + \epsilon_3)\,,$$

$$\nu_3(\epsilon_3, U^-) := \begin{cases} u_- + \epsilon_3\sqrt{\frac{2\tau_-}{2\gamma p_- + (\gamma+1)\epsilon_3}} & \text{if } \epsilon_3 \le 0, \text{ (shock)} \\ u_- + \frac{2\sqrt{\gamma}}{\gamma-1}(p_-\tau_-^\gamma)^{\frac{1}{2\gamma}}((p_- + \epsilon_3)^{\frac{\gamma-1}{2\gamma}} - p_-^{\frac{\gamma-1}{2\gamma}}) & \text{if } \epsilon_3 \ge 0, \text{ (rarefaction)}, \end{cases}$$

$$\mu_3(\epsilon_3, \eta^-) := \begin{cases} \tau_- \frac{2\gamma p_- + (\gamma-1)\epsilon_3}{2\gamma p_- + (\gamma+1)\epsilon_3} & \text{if } \epsilon_3 \le 0, \text{(shock)} \\ \tau_-(\frac{p_-}{p_- + \epsilon_3})^{\frac{1}{\gamma}} & \text{if } \epsilon_3 \ge 0, \text{ (rarefaction).} \end{cases}$$

Using the arguments in section 2.3, the Hugoniot function $\tilde{H}$ satisfies the pasting property

$$\tilde{H}(M, U^-) = \tilde{H}(M, S_3(\epsilon_3^d(M, \eta^-), U^-)) \quad \forall (M, U^-) \in \tilde{\Sigma}_d .$$

(M, η^-) being fixed in the phase boundary space, the transversal curve $\epsilon_3 \mapsto (M, \ell_3(\epsilon_3, \eta^-))$ reaches $\Sigma_k \cup \Sigma_s^a$ at a defined point $\epsilon_3 = \tilde{\epsilon}_3(M, \eta^-)$ which is the solution of

$$p_- + \epsilon_3 - \tilde{p}_k(M, \mu_3(\epsilon_3, \eta^-)) = 0 , \tag{5.10}$$

and the function $\tilde{\epsilon}_3$ is Lipschitz-continuous. Defining the *discontinuous* function

$$\epsilon_3^{sk}(M, \eta^-) = \begin{cases} \tilde{\epsilon}_3(M, \eta^-) & \text{if} \quad p_- \le \tilde{p}_d(M, \tau_-) \\ 0 & \text{if} \quad p_- > \tilde{p}_d(M, \tau_-) , \end{cases} \tag{5.11}$$

by the pasting property, we get the result that the composition

$$(M, U^-) \longmapsto \tilde{H}(M, L_3(\epsilon_3^{sk}(M, \eta^-), U^-))$$

is *Lipschitz continuous* near $(\underline{M}, \underline{U}^-)$. The succession $\tilde{H}(M, L_3(\epsilon_3^{sk}(M, \eta^-), U^-))$ of a small amplitude 3-wave, and of a large amplitude phase transition which is kinetic or 3-sonic on the left when $\epsilon_3^{sk}(M, \eta^-) \ne 0$, is the *phase boundary part* in our mixed sonic-kinetic Riemann problem. In this fan, the admissible large amplitude detonations belong to the set

$$\mathcal{PB}_{ad}^{sk} := \tilde{\Sigma}_k \cup \tilde{\Sigma}_s^a \cup \{(M, U^b) : p_b > \tilde{p}_d(M, \tau_b)\} .$$

Remark The availability of a small 3-shock or non attached 3-rarefaction wave in a detonation process is pointed out at the end of section 90 of [3].

5.7 Two well-posed Riemann problem near a sonic detonation

We focus here on the properties required for the well-posedness of the Riemann problem near a sonic detonation, using or not supersonic phase boundaries described in the previous section.

In the unburnt gas, there is no wave in a Riemann problem near a Chapman-Jouguet detonation. In the burnt gas, we use the jump of p from left to right as a parameter for the C^2-Lax's curves $\epsilon_i \mapsto L_i(\epsilon_i, U)$ for genuinely nonlinear waves. L_3 is defined above, L_1 writes

$$L_1(\epsilon_1, U) := (\nu_1(\epsilon_1, U), \ell_1(\epsilon_1, \eta)) := (\nu_1(\epsilon_1, U), \mu_1(\epsilon_1, \eta), p + \epsilon_1)\,,$$

$$\nu_1(\epsilon_1, U) := \begin{cases} u - \epsilon_1 \sqrt{\frac{2\tau}{2\gamma p + (\gamma+1)\epsilon_1}} & \text{if } \epsilon_1 \geq 0, \text{ (shock}\,, \\ u - \frac{2\sqrt{\gamma}}{\gamma-1}(p\tau^\gamma)^{\frac{1}{2\gamma}}((p+\epsilon_1)^{\frac{\gamma-1}{2\gamma}} - p^{\frac{\gamma-1}{2\gamma}}) & \text{if } \epsilon_1 \leq 0, \text{ (rarefaction)} \end{cases}$$

$$\mu_1(\epsilon_1, \eta) := \begin{cases} \tau \frac{2\gamma p + (\gamma-1)\epsilon_1}{2\gamma p + (\gamma+1)\epsilon_1} & \text{if } \epsilon_1 \geq 0, \text{ (shock)}\,, \\ \tau(\frac{p}{p+\epsilon_1})^{\frac{1}{\gamma}} & \text{if } \epsilon_1 \leq 0, \text{ (rarefaction).} \end{cases}$$

The $2-$contact discontinuities are parametrized by the jump ϵ_2 of τ as

$$L_2(\epsilon_2, U) := (u, \tau + \epsilon_2, p)\,.$$

For a given burnt datum U^- close to $\underline{U}^-$, an unburnt datum U^+ close to $\underline{U}^+$, the general form of the Riemann problem is

$$U^+ = \tilde{H}(M, L_3(\epsilon_3^\#(M, \eta^\#), U^\#))\,, \quad U^\# := (u_\#, \eta^\#) = L_2(\epsilon_2, L_1(\epsilon_1, U^-))\,. \tag{5.12}$$

Choosing the function $\epsilon_3^\#(M, \eta)$, we describe two well posed Riemann problems, under a unique stability condition.

Mixed sonic-kinetic Riemann problem, non-classical theory

The mixed sonic-kinetic Riemann problem selects the supersonic detonations of $\Sigma_s^a \cup \Sigma_k$. The function $\epsilon_3^\#(M, \eta)$ is here $\epsilon_3^{sk}(M, \eta)$ function defined in (5.11). The composition

$$(\epsilon_1, \epsilon_2, M, U^-) \mapsto \tilde{H}(M, L_3(\epsilon_3^{sk}(M, \eta^\#), U^\#))\,,$$

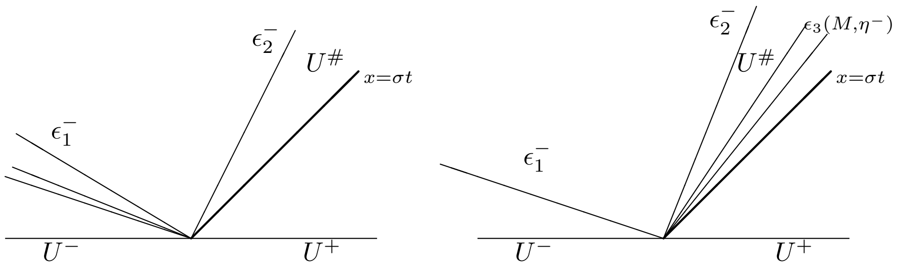
where $U^\# = (u_\#, \eta^\#) = L_2(\epsilon_2, L_1(\epsilon_1, U^-))$ is Lipschitz-continuous, and as for theorem 2.1, well posedness is a consequence of the invertibility of the differential

$$\begin{aligned} D_s &:= D_{(\epsilon_1, \epsilon_2, M)}(\tilde{H}(M, L_2(\epsilon_2, L_1(\epsilon_1, U^-)))(0, 0, \underline{M}, \underline{U}^-) \\ &= \begin{pmatrix} 2/M & * & * \\ 0 & \partial_{\tau_-}\tau_+ & \partial_M \tau_+ \\ 0 & \partial_{\tau_-} p_+ & \partial_M p_+ \end{pmatrix} (\underline{M}, \underline{U}^-)\,, \end{aligned}$$

where $\partial\tau_+$, ∂p_+ are the functions in (5.3). At the Chapman-Jouguet detonation point $(\underline{U}^-, \underline{M}, \underline{U}^+)$, we have

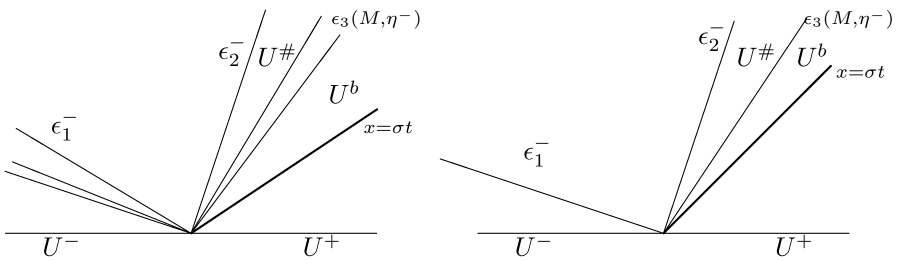
$$\det D_s = -2(\gamma+1)[\tau][p]\frac{\gamma p_+ + M^2 \tau_+}{\beta_+^2 - M^2} \neq 0\,.$$

Clarke's inverse function theorem can be applied near $(\underline{U}^-, \underline{M}, \underline{U}^+)$ and the solution $(\epsilon_1, \epsilon_2, M)(U^-, U^+)$ is Lipschitz-continuous. Moreover, the mapping $(U^-, U^+) \mapsto \epsilon_3^{sk}(M, \eta^\#)$ is Lipschitz-continuous function on the open set where $(M, U^\#)(U^-, U^+) \notin \tilde{\Sigma}_d$. Therefore, we have proved the following theorem.



subsonic case sonic case, attached rarefaction

Figure 6:



supersonic case, non attached rarefaction supersonic case, shock

Figure 7:

Theorem 5.1 (sonic-kinetic combustion Riemann problem)
Let $(\underline{U}^-, \underline{M}, \underline{U}^+)$ be a Chapman-Jouguet detonation satisfying the entropy condition (5.7). Let ϕ_s a smooth positive function satisfying (5.4), (5.4), (5.8), which defines by entropic selection the admissible detonations

$$\mathcal{PB}_{ad}^{sk} = \tilde{\Sigma}_k \cup \tilde{\Sigma}_s^a \cup \{(M, U^b) : p_b > \tilde{p}_d(M, \tau_b)\}\,.$$

There exists a neighbourhood $\Omega^- \times \Omega^+$ of $(\underline{U}^-, \underline{U}^+)$, a neighbourhood $\omega_1 \times \omega_2 \times \omega$ of $(0, 0, \underline{M})$, such that for every $(U^-, U^+) \in \Omega^- \times \Omega^+$, it exists a unique solution $(\epsilon_1, \epsilon_2, M) \in \omega_1 \times \omega_2 \times \omega$ of the Riemann problem

$$U^+ = \tilde{H}(M, L_3(\epsilon_3^{sk}(M, \eta^\#), U^\#))\,, \quad U^\# := (u_\#, \eta^\#) = L_2(\epsilon_2, L_1(\epsilon_1, U^-))\,.$$

where $\epsilon_3^{sk}(M, \eta^\#)$ is defined by (5.11), in such a way that $(M, L_3^-(\epsilon_3^{sk}(M, U^\#)) \in \mathcal{PB}_{ad}^{sk}$. The function $(U^-, U^+) \mapsto (\epsilon_1, \epsilon_2, M)$ is Lipschitz-continuous and the variation of the wave solution is estimated by

$$|\epsilon_1| + |\epsilon_2| + |\epsilon_3^{sk}(M, \eta^\#)| + |M - \underline{M}| = O(1)(|U^- - \underline{U}^-| + |U^+ - \underline{U}^+|)\,.$$

Remark We find again these pictures as the numerical diagrams of figures 4 or 5 in Wood [15].

Sonic Riemann problem, Chapman-Jouguet theory

The fully sonic Riemann problem selects the supersonic combustion boundaries of Σ_s. This is the usual configuration accepted by the physicists. Its definition uses as $\epsilon_3^{\#}$ the continuous function

$$\epsilon_3^s(M,\eta^-) := \begin{cases} (\frac{\tau_- M^2}{\gamma})^{\frac{\gamma}{\gamma+1}} p_-^{\frac{1}{\gamma+1}} - p_- & \text{if } p_- \le p_s(M,\tau_-) \equiv \frac{\tau_- M^2}{\gamma} \\ 0 & \text{if } p_- \ge p_s(M,\tau_-)\,, \end{cases}$$

where $(\frac{\tau_- M^2}{\gamma})^{\frac{\gamma}{\gamma+1}} p_-^{\frac{1}{\gamma+1}} - p_- := \hat{\epsilon}_3(M,\eta^-)$ is the solution of the equation

$$p_- + \epsilon_3 - p_s(M, \tau_-(\frac{p_-}{p_- + \epsilon_3})^{1/\gamma}) = 0\,.$$

The composition $(\epsilon_1,\epsilon_2,M,U^-) \mapsto \tilde{H}(M, L_3(\epsilon_3^s(M,\eta^{\#}),U^{\#}))$, with

$$U^{\#} = (u_{\#},\eta^{\#}) = L_2(\epsilon_2, L_1(\epsilon_1,U^-))$$

is C^1. Its well posedness comes from

$$\begin{aligned} &\det\{D_{(\epsilon_1,\epsilon_2,M)}(\tilde{H}(M, L_3(\hat{\epsilon}_3(M,\eta^{\#}), L_2(\epsilon_2, L_1(\epsilon_1,U^-)))(0,0,\underline{M},\underline{U}^-)\} \\ &= \det D_s \neq 0\,; \end{aligned}$$

so that the classical inverse function theorem applies near the sonic detonation $(\underline{U}^-,\underline{M},\underline{U}^+)$ and the solution $(\epsilon_1,\epsilon_2,M)(U^-,U^+)$ is C^1. Moreover $(U^-,U^+) \mapsto \epsilon_3^s(M,\eta^{\#})$ is Lipschitz-continuous. In a wave solution, the phase transition discontinuity is strictly subsonic (strong detonation), or sonic with or without a rarefaction attached on its left. Only the two first pictures above may appear.

Proposition 5.2 (sonic combustion Riemann problem)
Let $(\underline{U}^-,\underline{M},\underline{U}^+)$ be a Chapman-Jouguet detonation. There exists a neighbourhood $\Omega^-\times\Omega^+$ of $(\underline{U}^-,\underline{U}^+)$, a neighbourhood $\omega_1\times\omega_2\times\omega$ of $(0,0,\underline{M})$, such that for every $(U^-,U^+)\in\Omega^-\times\Omega^+$, it exists a unique solution $(\epsilon_1,\epsilon_2,M)\in\omega_1\times\omega_2\times\omega$ of the Riemann problem

$$U^+ = \tilde{H}(M, L_3(\epsilon_3^s(M,\eta^{\#}),U^{\#}))\,, \quad U^{\#} := (u_{\#},\eta^{\#}) = L_2(\epsilon_2, L_1(\epsilon_1,U^-))\,.$$

The detonation part of the solution is a sonic or a strong detonation. The function $(U^-,U^+) \mapsto (\epsilon_1,\epsilon_2,M)$ is C^1 and the variation of the wave solution is estimated by

$$|\epsilon_1| + |\epsilon_2| + |\epsilon_3^s(M,\eta^{\#})| + |M-\underline{M}| = O(1)(|U^- - \underline{U}^-| + |U^+ - \underline{U}^+|)\,.$$

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