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EXISTENCE OF SOLUTIONS TO HIGHER-ORDER DISCRETE THREE-POINT PROBLEMS

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Abstract. We are concerned with the higher-order discrete three-point boundary-value problem

$$\begin{gathered}(\Delta^n x)(t) = f(t, x(t+\theta)), \quad t_1 \le t \le t_3 - 1, \quad -\tau \le \theta \le 1 \\ (\Delta^i x)(t_1) = 0, \quad 0 \le i \le n-4, \quad n \ge 4 \\ \alpha(\Delta^{n-3}x)(t) - \beta(\Delta^{n-2}x)(t) = \eta(t), \quad t_1 - \tau - 1 \le t \le t_1 \\ (\Delta^{n-2}x)(t_2) = (\Delta^{n-1}x)(t_3) = 0.\end{gathered}$$

By placing certain restrictions on the nonlinearity and the distance between boundary points, we prove the existence of at least one solution of the boundary value problem by applying the Krasnoselskii fixed point theorem.

1. Introduction

We are concerned with the existence of solutions to the higher-order discrete three-point problem

$$(\Delta^n x)(t) = f(t, x(t+\theta)), \quad t_1 \le t \le t_3 - 1, \quad -\tau \le \theta \le 1 \tag{1.1}$$

$$\begin{gathered}(\Delta^i x)(t_1) = 0, \quad 0 \le i \le n-4, \quad n \ge 4 \\ \alpha(\Delta^{n-3}x)(t) - \beta(\Delta^{n-2}x)(t) = \eta(t), \quad t_1 - \tau - 1 \le t \le t_1 \\ (\Delta^{n-2}x)(t_2) = (\Delta^{n-1}x)(t_3) = 0.\end{gathered} \tag{1.2}$$

Here we assume

(*i*) any interval $[a, b]$ is the set of integers $\{a, a+1, \cdots, b-1, b\}$;
(*ii*) $t_{i+1} > t_i + n - 1$ to avoid overlap in boundary conditions, $i \in \{1, 2\}$;
(*iii*) $f : [t_1, t_3 - 1] \times [0, \infty) \to [0, \infty)$;
(*iv*) $\alpha, \beta > 0, \quad t_3 - t_1 \ge \tau \ge -1$, and $\theta \in [-\tau, 1]$ is constant;
(*v*) $\eta : [t_1 - \tau - 1, t_1] \to \mathbb{R}$ with $\eta(t_1) = 0$;
(*vi*) x is defined on $[t_1 - \tau - 1, t_3 + n - 1]$.

For the rest of this paper we also have the hypotheses

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(H1) $G(t,s)$ on $[t_1,t_3+n-1]\times[t_1,t_3-1]$ is the Green's function for the difference equation

$$(\Delta^n u)(t)=0,\quad t\in[t_1,t_3-1]$$

subject to the boundary conditions (1.2) with $\tau=-1$.

(H2) $g(t,s)$ on $[t_1,t_3+2]\times[t_1,t_3-1]$ is the Green's function for the difference equation

$$(\Delta^3 u)(t)=0,\quad t\in[t_1,t_3-1]$$

subject to the boundary conditions

$$\begin{gathered}\alpha u(t_1)-\beta(\Delta u)(t_1)=0\\ (\Delta u)(t_2)=(\Delta^2 u)(t_3)=0\end{gathered}\tag{1.3}$$

for α,β as in (iv).

(H3) $\|x\|_{[t_1-\tau-1,t_3+2]}:=\sup_{t_1-\tau-1\le t\le t_3+2}|(\Delta^{n-3}x)(t)|$.

(H4) For $\Xi:=\{t\in[t_1,t_3+n-1]:t_1\le t+\theta\le t_3-1\}$,

$$\Xi_h:=\{t\in\Xi:t_2-h\le t+\theta\le t_2+h\}$$

is nonempty for some $h\in(0,t_3-t_2-2)$, which is nonempty by (ii).

The corresponding Green's function for the discrete homogeneous problem $(\Delta^3 u)(t)=0$ satisfying the boundary conditions (1.3), a slight generalization of that in [1, 2, 3, 4], is given via

$$g(t,s)=\begin{cases} s\in[t_1,t_2-1] & :\begin{cases}u_1(t,s) & :t\le s+1\\ v_1(t,s) & :t\ge s+1\end{cases}\\[2em] s\in[t_2-1,t_3-1] & :\begin{cases}u_2(t,s) & :t\le s+1\\ v_2(t,s) & :t\ge s+1\end{cases}\end{cases}\tag{1.4}$$

for $t\in[t_1,t_3+2]$ and $s\in[t_1,t_3-1]$, where

$$\begin{aligned}u_1(t,s)&:=\frac12(t-t_1)(2s-t-t_1+3)+\frac{\beta}{\alpha}(s-t_1+1),\\ v_1(t,s)&:=\frac12(s-t_1+2)(s-t_1+1)+\frac{\beta}{\alpha}(s-t_1+1),\\ u_2(t,s)&:=\frac12(t-t_1)(2t_2-t-t_1+1)+\frac{\beta}{\alpha}(t_2-t_1),\\ v_2(t,s)&:=\frac12(t-t_1)(2t_2-t-t_1+1)+\frac{\beta}{\alpha}(t_2-t_1)+\frac12(t-s-1)(t-s-2).\end{aligned}$$

Remark 1.1. As in [2], it can be shown that if

$$\frac{\beta}{\alpha}(t_2-t_1)+1>\frac12(t_3-t_1+2)(t_3+t_1-2t_2+1),$$

then

$$g(t,s)>0$$

for all $t\in[t_1,t_3+2]$, $s\in[t_1,t_3-1]$. Note that if the boundary points satisfy

$$t_3-t_2\le t_2-t_1-1,\tag{1.5}$$

then the above inequality holds for any choice of $\alpha,\beta>0$. Thus throughout this paper we assume that (1.5) holds. Moreover, as in [3], we have the following boundedness result.

Lemma 1.2. *For all* $t \in [t_1, t_3+2]$ *and* $s \in [t_1, t_3-1]$,

$$\ell(t) g(t_2, s) \le g(t,s) \le g(t_2,s) \tag{1.6}$$

where

$$\ell(t) := \min\left\{ \frac{t-t_1}{t_2-t_1}, \frac{t_3-t+2}{t_3-t_2+2} \right\}. \tag{1.7}$$

Remark 1.3. The following discussion is similar to that found in [6] for a continuous two-point problem on the unit interval. If x is a solution of (1.1), (1.2), it can be written as

$$x(t) = \begin{cases} x(-\tau; t) & : t_1 - \tau - 1 \le t \le t_1 \\ \sum_{s=t_1}^{t_3-1} G(t,s) f(s, x(s+\theta)) & : t_1 \le t \le t_3 + n - 1 \end{cases}$$

where, using standard first-order linear difference equation methods [7], $x(-\tau; t)$ satisfies

$$(\Delta^{n-3} x)(-\tau; t) = \left(1 + \frac{\alpha}{\beta}\right)^{t-t_1} (\Delta^{n-3}x)(t_1) + \frac{1}{\beta} \sum_{s=t}^{t_1-1} \left(1+\frac{\alpha}{\beta}\right)^{t-s-1} \eta(s)$$

for $t \in [t_1 - \tau - 1, t_1]$.

If u_0 is the solution of (1.1), (1.2) with $f \equiv 0$, then u_0 satisfies

$$(\Delta^{n-3} u_0)(t) = \begin{cases} \frac{1}{\beta} \sum_{s=t}^{t_1-1} \left(1+\frac{\alpha}{\beta}\right)^{t-s-1} \eta(s) & : t_1 - \tau - 1 \le t \le t_1 \\ 0 & : t_1 \le t \le t_3 + 2; \end{cases} \tag{1.8}$$

note that actually, using the Green's function, $u_0 \equiv 0$ on $[t_1, t_3+n-1]$. If x is any solution of (1.1), (1.2) set $u(t) := x(t) - u_0(t)$. Then $u(t) \equiv x(t)$ on $[t_1, t_3+n-1]$, and u satisfies

$$(\Delta^{n-3} u)(t) = \begin{cases} \left(1+\frac{\alpha}{\beta}\right)^{t-t_1} (\Delta^{n-3}u)(t_1) & : t_1 - \tau - 1 \le t \le t_1 \\ \sum_{s=t_1}^{t_3-1} g(t,s) f(s, u(s+\theta) + u_0(s+\theta)) & : t_1 \le t \le t_3 + 2. \end{cases}$$

But this implies

$$u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3} \left(1+\frac{\alpha}{\beta}\right)^{t-t_1} (\Delta^{n-3}u)(t_1) & : t_1 - \tau - 1 \le t \le t_1 \\ \sum_{s=t_1}^{t_3-1} G(t,s) f(s, u(s+\theta) + u_0(s+\theta)) & : t_1 \le t \le t_3 + n - 1. \end{cases}$$

2. Existence of at Least One Solution

We are concerned with proving the existence of solutions of the higher-order discrete nonlinear boundary value problem (1.1), (1.2). In light of the above discussion in Remark 1.3, consider the fixed points of the operator $\mathcal{A}$ defined by

$$\mathcal{A}u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3} \left(1+\frac{\alpha}{\beta}\right)^{t-t_1} (\Delta^{n-3}u)(t_1) & : t_1 - \tau - 1 \le t \le t_1 \\ \sum_{s=t_1}^{t_3-1} G(t,s) f(s, u(s+\theta) + u_0(s+\theta)) & : t_1 \le t \le t_3 + n - 1, \end{cases}$$

with domain $\{u : [t_1-\tau-1, t_3+n-1] \to \mathbb{R}\}$. If $\mathcal{A}u = u$, then a solution x of (1.1), (1.2) would be given by

$$x(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3}\left(1+\frac{\alpha}{\beta}\right)^{t-t_1}(\Delta^{n-3}u)(t_1) + u_0(t) & : t_1-\tau-1 \le t \le t_1 \\ u(t) & : t_1 \le t \le t_3+n-1, \end{cases}$$

where u_0 satisfies (1.8).

Remark 2.1. In the following discussion we will need an $h \in (0, t_3-t_2-2)$; note that for all $t \in [t_2-h, t_2+h]$, we then have

$$\ell(t) \ge \ell(t_2+h+1) = 1 - \frac{h+1}{t_3-t_2+2} \tag{2.1}$$

for all $h \in (0, t_3-t_2-2)$, where ℓ is given in (1.7). Moreover, let $k, m > 0$ such that

$$\begin{aligned} k^{-1} := &\sum_{s=t_1}^{t_3-1} g(t_2,s) \\ = &\frac{1}{6}(t_2-t_1+1)(t_2-t_1)(3t_3-2t_2-t_1+2) \\ &+ \frac{\beta}{2\alpha}(t_2-t_1)(2t_3-t_2-t_1+1) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} m^{-1} := &\ \ell(t_2+h+1)\sum_{s=t_2-h}^{t_2+h} g(t_2,s) \\ = &\ \frac{1}{6}\left(1-\frac{h+1}{t_3-t_2+2}\right)\Big[(t_2-t_1+1)^{\underline{2}}(t_2-t_1+3h+5) \\ &-(t_2-t_1-h+2)^{\underline{3}} + \frac{3\beta}{\alpha}(4ht_2+2t_2-4ht_1-2t_1-h^2+h)\Big], \end{aligned} \tag{2.3}$$

where we have used the so-called falling factorial power [7]

$$b^{\underline{r}} := b(b-1)(b-2)\cdots(b-r+1).$$

Finally, set

$$M_0 := \|u_0\|_{[t_1-\tau-1, t_3+2]} \tag{2.4}$$

for u_0 as in (1.8).

We will employ the following fixed point theorem due to Krasnoselskii [8].

Theorem 2.2. *Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathcal{A} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|\mathcal{A}u\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, or*
(ii) *$\|\mathcal{A}u\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \le \|u\|$, $u \in P \cap \partial\Omega_2$*

holds. Then $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2.3. *Let k, m, M_0 be as in (2.2), (2.3), (2.4), respectively, and suppose the following conditions are satisfied.*

(C_1) *There exists $p > 0$ such that $f(t,w) \le kp$ for $t \in [t_1, t_3-1]$ and $0 \le \|w\| \le p + M_0$.*

(C_2) *There exists* $q > 0$ *such that* $f(t,w) \ge mq$ *for* $t \in \Xi_h$ *and* $q\ell(t_2+h+1) \le \|w\| \le q$, *for* $h \in (0, t_3 - t_2 - 2)$ *and* Ξ_h *as in* ($H4$).

Then (1.1), (1.2) *has a solution* $x = u + u_0$ *such that* $\|x\|_{[t_1-\tau-1,t_3+2]}$ *lies between* $\max\{0, p - M_0\}$ *and* $q + M_0$.

Proof. Many of the techniques employed here are as in [5, 6]. Let $\mathbb{B}$ denote the Banach space $\{u : [t_1 - \tau - 1, t_3 + n - 1] \to \mathbb{R}\}$ with the norm

$$\|u\|_{[t_1-\tau-1,t_3+2]} = \sup_{t\in[t_1-\tau-1,t_3+2]} |(\Delta^{n-3}u)(t)|.$$

Define the cone $\mathbb{P} \subset \mathbb{B}$ by

$$\mathbb{P} = \{u \in \mathbb{B} : \min_{t\in[t_2-h,t_2+h]} (\Delta^{n-3}u)(t) \ge \ell(t_2+h+1)\|u\|_{[t_1-\tau-1,t_3+2]}\}.$$

Consider the mapping $\mathcal{A} : \mathbb{P} \to \mathbb{B}$ via

$$\mathcal{A}u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3}\left(1+\frac{\alpha}{\beta}\right)^{t-t_1}(\Delta^{n-3}u)(t_1) & : t_1-\tau-1 \le t \le t_1 \\ \sum_{s=t_1}^{t_3-1} G(t,s)f(s, u(s+\theta)+u_0(s+\theta)) & : t_1 \le t \le t_3+n-1. \end{cases}$$

Then

$$\Delta^{n-3}(\mathcal{A}u)(t) = \begin{cases} \left(1+\frac{\alpha}{\beta}\right)^{t-t_1} \sum_{s=t_1}^{t_3-1} g(t_1,s)f(s, u(s+\theta)+u_0(s+\theta)) \\ \sum_{s=t_1}^{t_3-1} g(t,s)f(s, u(s+\theta)+u_0(s+\theta)) \end{cases}$$

so that $\Delta^{n-3}(\mathcal{A}u)(t) \le \Delta^{n-3}(\mathcal{A}u)(t_1)$ for $t_1-\tau-1 \le t \le t_1$. In other words, $\|\mathcal{A}u\|_{[t_1-\tau-1,t_3+2]} = \|\mathcal{A}u\|_{[t_1,t_3+2]}$. It follows for $h \in (0, t_3-t_2-2)$ and $t \in [t_2 - h, t_2+h]$ that

$$\begin{aligned} \Delta^{n-3}(\mathcal{A}u)(t) &= \sum_{s=t_1}^{t_3-1} g(t,s)f(s, u(s+\theta)+u_0(s+\theta)) \\ &\ge \ell(t) \sum_{s=t_1}^{t_3-1} g(t_2,s)f(s, u(s+\theta)+u_0(s+\theta)) \\ &\ge \ell(t_2+h+1)\|\mathcal{A}u\|_{[t_1-\tau-1,t_3+2]} \end{aligned}$$

by properties of the Green's function (1.6), so that $\mathcal{A} : \mathbb{P} \to \mathbb{P}$.

Without loss of generality, we may assume $0 < p < q$. Define the bounded open balls

$$\Omega_p = \{u \in \mathbb{B} : \|u\|_{[t_1-\tau-1,t_3+2]} < p\},$$

and

$$\Omega_q = \{u \in \mathbb{B} : \|u\|_{[t_1-\tau-1,t_3+2]} < q\};$$

then $0 \in \Omega_p \subset \Omega_q$. If $u \in \mathbb{P} \cap \partial\Omega_p$, then $\|u\| = p$ and

$$|(\Delta^{n-3}u)(t) + (\Delta^{n-3}u_0)(t)| \le p + M_0$$

for all $t \in [t_1, t_3+2]$. As a result,

$$\begin{aligned} \|\mathcal{A}u\| &= \sum_{s=t_1}^{t_3-1} g(t_2,s) f(s, u(s+\theta)+u_0(s+\theta)) \\ &\le kp \sum_{s=t_1}^{t_3-1} g(t_2,s) \\ &= p \\ &= \|u\| \end{aligned}$$

using (C_1) and (2.2). Thus, $\|\mathcal{A}u\| \le \|u\|$ for $u \in \mathbb{P} \cap \partial\Omega_p$.

Similarly, let $u \in \mathbb{P} \cap \partial\Omega_q$, so that $\|u\| = q$. Then for $s \in \Xi_h$,

$$(\Delta^{n-3}u)(s+\theta) \ge \min_{t\in[t_2-h,t_2+h]} (\Delta^{n-3}u)(t) \ge \|u\| \ell(t_2+h+1)$$

for all $h \in (0, t_3-t_2-2)$ and $\ell(\cdot)$ as in (2.1). As a result,

$$q\ell(t_2+h+1) \le (\Delta^{n-3}u)(s+\theta) + (\Delta^{n-3}u_0)(s+\theta) \le q$$

for $s \in \Xi_h$, since $\Delta^{n-3}u_0 \equiv 0$ on $[t_1, t_3+2]$ by Remark 1.3. It follows that

$$\begin{aligned} \|\mathcal{A}u\| &= \sum_{s=t_1}^{t_3-1} g(t_2,s) f(s, u(s+\theta)+u_0(s+\theta)) \\ &\ge \sum_{s\in\Xi_h} g(t_2,s) f(s, u(s+\theta)+u_0(s+\theta)) \\ &\ge mq\ell(t_2+h+1) \sum_{s=t_2-h}^{t_2+h} g(t_2,s) \\ &= q \\ &= \|u\| \end{aligned}$$

by (C_2) and (2.3). Consequently, $\|\mathcal{A}u\| \ge \|u\|$ for $u \in \mathbb{P} \cap \partial\Omega_q$. By Theorem 2.2, $\mathcal{A}$ has a fixed point $u \in \mathbb{P} \cap (\overline{\Omega}_q \setminus \Omega_p)$; i.e., $p \le \|u\| \le q$. Therefore the discrete problem (1.1), (1.2) has a solution $x = u + u_0$ such that $p - M_0 \le \|x\| \le q + M_0$, if $M_0 < p$. □

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