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ON THE BEHAVIOR OF THE INTERFACE SEPARATING FRESH AND SALT GROUNDWATER IN A HETEROGENEOUS COASTAL AQUIFER

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ABSTRACT. We consider a flow of fresh and salt groundwater in a two-dimensional heterogeneous horizontal aquifer. Assuming the flow governed by a nonlinear Darcy law and the permeability depending only on the vertical coordinate, we show the existence of a unique monotone solution that increases (resp. decreases) with respect to the salt (resp. fresh) water discharge. For this solution we prove that the free boundary is represented by the graph $x = g(z)$ of a continuous function. Finally we prove a limit behavior at the end points of the interval of definition of g.

INTRODUCTION

Fresh water and sea water are actually miscible fluids and therefore the zone of contact between them takes the form of a transition zone caused by hydrodynamics dispersion. Across this zone the density of the mixed water varies from that of fresh water to that of sea water.

Under certain conditions the width of this zone is relatively small (when compared with the thickness of the aquifer) so that we assume that each liquid is confined to a well defined portion of the flow domain with an abrupt interface separating the two domains.

We consider here a two-dimensional model for fresh-salt water in a horizontally heterogeneous extended aquifer in the xz-plane. We suppose that the scale of the problem is sufficiently large so that the abrupt interface approximation is applicable. Moreover we consider the model of a flow obeying to a nonlinear Darcy law.

In section 1, we indicate briefly how to obtain the weak formulation and the existence and some properties of the solutions, the definition of the free boundary $\Gamma = [x = g(z)]$ and the continuity of g on its interval of definition $(-h^*, 0)$, $h^* > 0$. All these results generalize previous works in [5] and [4] (resp. [6]) where the aquifer was supposed homogeneous (resp. heterogeneous) and the flow governed by a nonlinear (resp. linear) Darcy law.

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The aim of this paper is to study the behavior of the free boundary when $z \to -h^*$ and $z \to 0$ under the assumption that the permeability of the porous medium depends only on z including the case of horizontal layers. Actually we establish in section 3 that $\lim_{z\to -h^*} g(z) = +\infty$ by generalizing the proof given in [6]. We also give a second simple proof which works only for a constant permeability. Then we prove that $\lim_{z\to 0} g(z) = g(0-)$ exists. We recall that in [1] the authors first proved (in the linear and homogeneous case) that $\liminf_{z\to 0} g(z) > -\infty$ by using blow up arguments. They also proved that $\limsup_{z\to 0} g(z) < 0$ and used this result to prove the existence of the limit $g(0-)$. Our proof does not assume $\limsup_{z\to 0} g(z) < 0$ and is valid in the general case. Moreover we prove that $g(0-)$ is finite in more general cases. Our proofs are systematically based on comparing the solution locally or globally with explicit functions satisfying similar equations. This method of comparison is developed in section 2 to show that the solution increases with respect to the salt water discharge Q_s and decreases with respect to the fresh water discharge Q_f. The uniqueness of the solution is obtained as a corollary of this monotonicity result. We also deduce a limit behavior of the solution when Q_f or Q_s goes to zero. Also by a comparison argument, when the permeability is constant, we give a simple proof of the fact that the set filled by fresh water is star shaped with the origin and the free boundary is non increasing in the region $[x > 0]$. These two last results were proved in the linear case in [1] by using blow up arguments. Our proofs based on monotonicity arguments are much simpler.

1. Description of the model

In this paper we are interested with the study of a stationary flow of fresh and salt water in a heterogeneous coastal aquifer $\Omega = \mathbb{R} \times (-h, 0)$, $h > 0$, with permeability $A(X)$, $X = (x, z) \in \mathbb{R}^2$. The velocity and the pressure of the fluid are related by the following nonlinear Darcy law

$$\mathbf{v} = -\left(\langle A(\nabla p + \gamma e_z), \nabla p + \gamma e_z\rangle\right)^{\frac{r-2}{2}} A(\nabla p + \gamma e_z) \tag{1.1}$$

with $r > 1$, $e_z = (0, 1)$ and γ is given by

$$\gamma = \gamma_f \chi(\Omega_f) + \gamma_s \chi(\Omega_s) \quad \text{with} \quad 0 < \gamma_f < \gamma_s \tag{1.2}$$

where γ_f (resp. γ_s) represents the specific weight of the fresh (resp. salt) water occupying the region Ω_f (resp. Ω_s) of Ω, $\chi(E)$ denotes the characteristic function of the set E.

Fresh water is injected over the segment $[OA]$ ($A = (0, a)$ with $a > 0$) uniformly (see Figure 1) with a total amount of Q_f. From infinity at the left of the aquifer, salt water arrives with a total discharge of Q_s. We assume that the two fluids are unmixed and separated by an interface Γ. The part of the boundary $\partial\Omega \setminus [OA]$ is assumed to be impervious and the flow incompressible. So the velocity satisfies

$$\begin{gathered} \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v} = -\frac{Q_f}{a} e_z \quad \text{on } [OA], \\ \mathbf{v} \cdot \nu = 0 \quad \text{on } \partial\Omega \setminus [OA], \quad \mathbf{v_i} \cdot \nu = 0 \quad \text{on } \Gamma \quad (i = s, f), \end{gathered} \tag{1.3}$$

where $\mathbf{v_i}$ is the restriction of $\mathbf{v}$ to Ω_i and ν is the outward unit normal to $\partial\Omega$ or Γ. We deduce from (1.3) that there exists a stream function ψ satisfying

$$\begin{gathered} \mathbf{v} = \operatorname{Rot} \psi = \Big(-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x}\Big) \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on} \quad \Gamma, \\ \psi(x, -h) = Q_s \quad \text{and} \quad \psi(x, 0) = \phi_0(x) \quad \text{for } x \in \mathbb{R} \end{gathered} \tag{1.4}$$

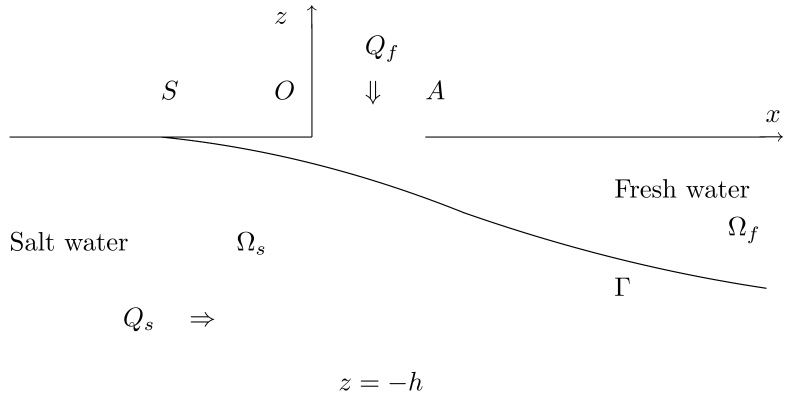

FIGURE 1

with

$$\phi_0(x) = -Q_f \min\big(\frac{x^+}{a}, 1\big). \tag{1.5}$$

We suppose the permeability matrix $A(X)$ such that

$$\begin{gathered} A(X) \in \big[L^\infty(\Omega)\big]^4, \\ \exists m_0 > 0 : \ \langle A(X)\xi, \xi\rangle \geq m_0|\xi|^2, \ \forall \xi \in \mathbb{R}^2, \quad \text{a.e. } X \in \Omega, \\ {}^tA = A. \end{gathered} \tag{1.6}$$

Then there exists a unique symmetric and strictly elliptic matrix $\mathcal{A}$ satisfying $A(X) = \mathcal{A}^t\mathcal{A} = \mathcal{A}^2$ (see [7]). Then (1.1) becomes

$$\begin{aligned} \mathbf{v} &= -\langle \mathcal{A}(X)(\nabla p + \gamma e_z), \mathcal{A}(X)(\nabla p + \gamma e_z)\rangle^{(r-2)/2}\mathcal{A}(X)\mathcal{A}(X)(\nabla p + \gamma e_z) \\ &= -|\mathcal{A}(X)(\nabla p + \gamma e_z)|^{r-2}\mathcal{A}(X)\mathcal{A}(X)(\nabla p + \gamma e_z). \end{aligned}$$

This leads to

$$|\mathcal{A}^{-1}(X)\mathbf{v}| = |\mathcal{A}(X)(\nabla p + \gamma e_z)|^{r-1}$$

and

$$\nabla p + \gamma e_z = -|\mathcal{A}^{-1}(X)\mathbf{v}|^{\frac{2-r}{r-1}}\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)\mathbf{v}). \tag{1.7}$$

Now for $\zeta \in \mathcal{D}(\Omega)$ we get by (1.4) and (1.7)

$$\int_\Omega (\nabla p + \gamma e_z)\operatorname{Rot}\zeta = -\int_\Omega |\mathcal{A}^{-1}(X)\operatorname{Rot}\psi|^{\frac{2-r}{r-1}}\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)\operatorname{Rot}\psi)\cdot\operatorname{Rot}\zeta.$$

If we set $\overline{A}(X) = \frac{1}{detA}A$, then there exists a unique symmetric and strictly elliptic matrix $B(X)$ such that $\overline{A}(X) = B^2 = B \cdot B$ and for which we have

$${}^t\mathcal{A}^{-1}(X)(\mathcal{A}^{-1}(X)\operatorname{Rot}\psi)\cdot\operatorname{Rot}\zeta = \langle\overline{A}(X)\nabla\psi, \nabla\zeta\rangle = B(X)\nabla\psi\cdot B(X)\nabla\zeta.$$

Therefore

$$\int_\Omega |B(X)\nabla\psi|^{\frac{2-r}{r-1}}B(X)\nabla\psi\cdot B(X)\nabla\zeta \ + \gamma e_x\nabla\zeta = 0 \quad \forall\zeta \in \mathcal{D}(\Omega). \tag{1.8}$$

Setting $q = r/(r-1)$, we deduce from (1.8) by taking $\zeta \in \mathcal{D}(\Omega_i)$ $(i = f, s)$

$$\operatorname{div}\big(\mathcal{B}(X, \psi)\big) = 0 \quad \text{in } \mathcal{D}'(\Omega_i) \quad i = f, s \tag{1.9}$$

with

$$\mathcal{B}(X,\xi) = \langle \overline{A}(X)\xi,\xi\rangle^{\frac{q-2}{2}}\overline{A}(X)\xi.$$

Moreover if we assume that

$$\lim_{x\to+\infty}\psi(x,z)\leq 0 \quad \text{for } (x,z)\in\Omega_f \quad \text{and} \quad \lim_{x\to\pm\infty}\psi(x,z)\geq 0 \quad \text{for } (x,z)\in\Omega_s \tag{1.10}$$

and since

$$\psi\leq 0 \quad \text{on } \partial\Omega_f \quad \text{and} \quad \psi\geq 0 \quad \text{on } \partial\Omega_s, \tag{1.11}$$

we deduce by (1.9)-(1.11) and the maximum principle for $\mathcal{B}$-harmonic functions in unbounded domains (see [9], [10]) that

$$\psi<0 \quad \text{in } \Omega_f \quad \text{and} \quad \psi>0 \quad \text{in } \Omega_s. \tag{1.12}$$

It follows from (1.2) and (1.12) that $\gamma\in H(\psi)$, where H is the maximal monotone graph defined by

$$H(t) = \gamma_f\chi([t<0]) + [\gamma_f,\gamma_s]\chi([t=0]) + \gamma_s\chi([t>0]),$$

$[\gamma_f,\gamma_s]$ being the closed interval of $\mathbb{R}$ with endpoints γ_f and γ_s.

Then we are led to the following question:

problem (P) Find $(\psi,\gamma)\in W^{1,q}_{\rm loc}(\Omega)\times L^\infty(\Omega)$ such that

(i) $\int_\Omega |B(X)\nabla\psi|^{q-2}B(X)\nabla\psi\cdot B(X)\nabla\zeta+\gamma e_x\cdot\nabla\zeta = 0$ for all $\zeta\in W^{1,q}_0(\Omega)$ with compact support in $\bar\Omega$

(ii) $\gamma\in H(\psi)$ a.e. in Ω

(iii) $-Q_f\leq\psi\leq Q_s$ a.e. in Ω

(iv) $\psi(x,-h)=Q_s$, $\psi(x,0)=\phi_0(x)$ for all $x\in\mathbb{R}$.

Adapting technics in [4], [5] and [6] we prove the following theorems.

Theorem 1.1. (i) *There exists a solution (ψ,γ) of (P) that satisfies $\psi\in C^{0,\alpha}_{\rm loc}(\bar\Omega)$ for some $\alpha\in(0,1)$. Moreover if $A(X)\in C^{0,\sigma}_{\rm loc}(\Omega)$, then $\psi\in C^{1,\beta}_{\rm loc}(\Omega\setminus[\psi=0])$, $\sigma,\beta\in(0,1)$.*

(ii) *If $B(X)=B(z)$, then there exists a monotone solution (ψ,γ) of (P) in the following sense:*

$$\partial_x\psi\leq 0 \quad \text{and} \quad \partial_x\gamma\leq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.13}$$

For the rest of this article, we assume that

$$B(X)=B(z)=\big(B_{ij}(z)\big)_{1\leq i,j\leq 2} \quad \text{a.e. in } \Omega \tag{1.14}$$

and will consider only monotone solutions. Moreover we need to introduce the following two functions defined for $z\in[-h,0]$.

$$v_{+\infty}(z) = -Q_f+(Q_s+Q_f)\phi_1(z), \quad v_{-\infty}(z)=Q_s\phi_1(z) \tag{1.15}$$

where

$$\phi_1(z) = \frac{\int_z^0 \frac{ds}{(B_{12}^2+B_{22}^2)^{q'/2}(s)}}{\int_{-h}^0 \frac{ds}{(B_{12}^2+B_{22}^2)^{q'/2}(s)}}.$$

Then we have the following theorem.

Theorem 1.2. *Let (ψ,γ) be a solution of (P) and let $\Omega_{m,n}=(m,n)\times(-h,0)$ for $m,n\in\mathbb{R}$.*

(i) *For $r=\max(q,2)$, we have $\lim_{R\to\pm\infty}\int_{\Omega_{R,R+1}}|\nabla(\psi-v_{\pm\infty}|^r=0$.*

(ii) *For all* $z \in [-h, 0]$, $\psi(x, z) \to v_{\pm\infty}(z)$ *as* $x \to \pm\infty$.
(iii) $v_{+\infty} \le \psi \le v_{-\infty}$ *in* Ω.
(iv) $\gamma(x + R, z) \rightharpoonup \gamma_{\pm\infty}(z)$ *in* $L^{q'}(\Omega_{0,1})$ *as* $R \to \pm\infty$, *where* $\gamma_{\pm\infty} \in H(v_{\pm\infty})$.

Remark 1. From (iii), one can see that the strip $\mathbb{R} \times (-h, -h^*)$ is contained in $[\psi > 0]$, where $h^* \in (0, h)$ is defined by $\phi_1(-h^*) = \frac{Q_f}{Q_s+Q_f}$. Therefore, the free boundary $\Gamma = [\psi = 0]$ is contained in $\mathbb{R} \times [-h^*, 0)$.

Arguing as in [4] and [6] we can prove the continuity of the free boundary. The proof needs Lemma 15 (see Appendix) which requires the following regularity of B: If $q \neq 2$, then

$$B(z) \in C^{0,1}_{\text{loc}}(-h, 0). \tag{1.16}$$

Remark 2. Under assumption (1.16), the critical points of any $\mathcal{B}-$harmonic function in Ω are isolated (see [2]). For the rest of this article, we assume that (1.16) is satisfied.

Theorem 1.3. *There exists a continuous function* $g : (-h^*, 0) \to \mathbb{R}$ *such that*

$$[x = g(z)] \subset \Gamma \subset [x = g(z)] \cup [z = -h^*].$$

Corollary 1. (i) $\gamma = \gamma_s \chi([\psi > 0]) + \gamma_f \chi([\psi < 0])$ *a.e. in* Ω.
(ii) *The sets* $[\psi > 0]$ *and* $[\psi < 0]$ *are connected by arcs.*

2. Comparison and Uniqueness

In this section we prove that solutions of (P) increase with respect to Q_s and decrease with respect to Q_f. As a consequence we obtain the uniqueness of the solution of (P). Let us denote by $(P(Q_s, Q_f))$ the problem (P) corresponding to Q_s and Q_f. Then we have the following comparison result

Theorem 2.1. *Let* (ψ_1, γ_1) *and* (ψ_2, γ_2) *be solutions of Problems* $(P(Q_{s_1}, Q_{f_1}))$ *and* $(P(Q_{s_2}, Q_{f_2}))$ *respectively. If* $Q_{s_1} \le Q_{s_2}$ *and* $Q_{f_2} \le Q_{f_1}$, *then we have* $\psi_1 \le \psi_2$ *and* $\gamma_1 \le \gamma_2$ *a.e. in* Ω.

The proof of this theorem follows an idea in [4] and uses a recent result due to Alessandrini and Sigalotti [2] regarding isolation of critical points of $\mathcal{B}$-harmonic functions in planar domains. First we prove the following lemma

Lemma 1. *Under the assumptions of Theorem 2.1,*

$$T(\zeta) = \int_\Omega \Big(\mathcal{B}(z, \nabla\psi_1) - \mathcal{B}(z, \nabla\psi_0) + (\gamma_1 - \gamma_0)e_x\Big).\nabla\zeta = 0 \quad \forall \zeta \in \mathcal{D}(\mathbb{R}^2),$$

where $\psi_0 = \min(\psi_1, \psi_2)$ *and* $\gamma_0 = \min(\gamma_1, \gamma_2)$.

Proof. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$ and let $K = supp\zeta$, $M = \sup_K |\zeta|$. Then there exists $R_0 > a$ such that: $\forall R \ge R_0$, $K \subset (-R, R) \times \mathbb{R}$. Consider $\zeta_R = M. \min\left(1, (-|x| + R + 1)^+\right)$ and set $\zeta_1 = \zeta + \zeta_R$, $\zeta_2 = \zeta_R - \zeta$. Then for $\epsilon > 0$ and $i = 1, 2$, $\min(\zeta_i, \frac{\psi_1 - \psi_0}{\epsilon})$ is a test function and one has by the monotonicity of $\mathcal{B}(z, .)$

$$\begin{aligned} &\int_{[\psi_1 - \psi_0 \ge \epsilon\zeta_i]} \Big(\mathcal{B}(z, \nabla\psi_1) - \mathcal{B}(z, \nabla\psi_0)\Big)\nabla\zeta_i + \int_\Omega (\gamma_1 - \gamma_0).\zeta_{ix} \\ &\le \int_\Omega (\gamma_1 - \gamma_0)\Big(\zeta_i - \frac{\psi_1 - \psi_0}{\epsilon}\Big)^+_x. \end{aligned} \tag{2.1}$$

Since $Q_{s_1} \leq Q_{s_2}$ and $Q_{f_2} \leq Q_{f_1}$ one has

$$-h_1^* = \phi_1^{-1}\big(\frac{Q_{f_1}}{Q_{s_1}+Q_{f_1}}\big) \leq \phi_1^{-1}\big(\frac{Q_{f_2}}{Q_{s_2}+Q_{f_2}}\big) = -h_2^*$$

and then if we denote by $I = \{z \in (-h_2^*, 0) : g_2(z) < g_1(z)\}$

$$\begin{aligned}
&\int_\Omega (\gamma_1-\gamma_0)\big(\zeta_i - \frac{\psi_1-\psi_0}{\epsilon}\big)_x^+ \\
&= \int_{[\psi_0<0<\psi_1]} (\gamma_s-\gamma_f)\big(\zeta_i - \frac{\psi_1-\psi_0}{\epsilon}\big)_x^+ \\
&= (\gamma_s-\gamma_f)\int_I\int_{g_2(z)}^{g_1(z)} \big(\zeta_i - \frac{\psi_1-\psi_0}{\epsilon}\big)_x^+ \\
&= (\gamma_s-\gamma_f)\int_I \big(\zeta_i + \frac{\psi_2}{\epsilon}\big)^+(g_1(z),z) - \big(\zeta_i - \frac{\psi_1}{\epsilon}\big)^+(g_2(z),z)dz
\end{aligned}$$

which goes to zero when $\epsilon \to 0$. So from (2.1) we get $T(\zeta_i) \leq 0$ for $i = 1,2$. This leads to

$$T(\zeta_R) \leq T(\zeta) \leq -T(\zeta_R). \tag{2.2}$$

Moreover we have

$$\begin{aligned}
T(\zeta_R) = M\int_{\Omega_{-R-1,-R}} &(\mathcal{B}(z,\nabla\psi_1) - \mathcal{B}(z,\nabla\psi_0)).e_x + M\int_{\Omega_{-R-1,-R}}(\gamma_1-\gamma_0) \\
&- M\int_{\Omega_{R,R+1}} (\mathcal{B}(z,\nabla\psi_1) - \mathcal{B}(z,\nabla\psi_0)).e_x + M\int_{\Omega_{R,R+1}}(\gamma_1-\gamma_0).
\end{aligned}$$

Using Theorem 1.2 and the fact that we have either $v_{+\infty}^1 \equiv v_{+\infty}^2$ or $v_{+\infty}^1 < v_{+\infty}^2$ in $(-h,0)$, we deduce (see [6, 4]) that $\lim_{R\to+\infty} T(\zeta_R) = 0$ which leads by (2.2) to $T(\zeta) = 0$. □

Proof of Theorem 2.1. Let us denote by D the domain $[\psi_1 < 0]$ (see Corollary 1). First we remark from Lemma 1 that (ψ_0, γ_0) is also a solution of $(P(Q_{s_0}, Q_{f_0}))$, that ψ_0 and ψ_1 are $\mathcal{B}$-Harmonic in D and that $\mathcal{B}(z,\nabla\psi_0).\nu = \mathcal{B}(z,\nabla\psi_1).\nu$ on $(a,+\infty)\times\{0\}$.

Since we have $\psi_0 \leq \psi_1$ in D, $\psi_0 = \psi_1$ on $(a,+\infty)\times\{0\}$ and ψ_0, $\psi_1 \in C^1(D \cup (a,+\infty)\times\{0\})$, it is enough according to Lemma 14, to prove that $\nabla\psi_1$ does not vanish on some part Γ_0 of $(a,+\infty)\times\{0\}$. Assume that for some $x_1 > a$ and $0 < r < \frac{x_1-a}{2}$ we have $\nabla\psi_1(x,0) = 0 \quad \forall x \in (x_1-r, x_1+r)$. Let $B_r(x_1,0)$ be the ball of center $(x_1,0)$ and radius r. If we extend ψ_1 by $-Q_{f_1}$ and $\mathcal{B}(z,\xi)$ by $\mathcal{B}(0,\xi)$ to $B_r(x_1,0)\setminus\overline{D}$, it is clear that $\psi_1 \in W^{1,q}\big(B_r(x_1,0)\big)$ and is $\mathcal{B}$-Harmonic in $B_r(x_1,0)$. Since the zeros of the gradient of a nonconstant $\mathcal{B}$-Harmonic function are isolated (see [2]) and $\nabla\psi_1 = 0$ in $B_r(x_1,0)\cap[z>0]$, we deduce that $\psi_1 = -Q_{f_1}$ in $B_r(x_1,0)$. By the monotonicity of ψ_1 this leads to $\psi_1 = -Q_{f_1}$ in the strip $[x_1,+\infty)\times(-r,0)$ which contradicts the asymptotic behavior of ψ_1 at $+\infty$. So there exists $x_1' \in (x_1-r, x_1+r)$ such that $\nabla\psi_1(x_1',0) \neq 0$. Since $\psi_1 \in C^1(D\cup(a,+\infty)\times\{0\})$ there exists $r' > 0$ such that $\nabla\psi_1(x,0) \neq 0$ $\forall x \in (x_1'-r', x_1'+r')$. Therefore we get $\psi_0 = \psi_1$ in D. In particular $\psi_0 = \psi_1$ in $[\psi_0 < 0]$. Similarly one can prove that $\psi_0 = \psi_1$ in $[\psi_0 > 0]$ and then by continuity $\psi_0 = \psi_1$ in Ω. Using Corollary 1 we deduce that $\gamma_0 = \gamma_1$ in Ω. □

As a direct consequence of Theorem 2.1, we have

Corollary 2. *The solution of problem (P) is unique.*

According to Theorem 2.1 the solution of (P) decreases with respect to Q_f and increases with respect to Q_s. Intuitively one would expect that as $Q_f \to 0$ (resp. $Q_s \to 0$) the aquifer would be saturated by salt (resp. fresh) water only. More precisely we have

Theorem 2.2. *Let $(\psi_{Q_f}, \gamma_{Q_f})$ be the solution of $(P(Q_s, Q_f))$. We have*

$$(\psi_{Q_f}, \gamma_{Q_f}) \to (v_{-\infty}, \gamma_s) \quad \text{in } W^{1,q}_{\text{loc}}(\Omega) \times L^{q'}_{\text{loc}}(\Omega) \text{ as } Q_f \to 0,$$

where $v_{-\infty}$ is given by (1.15).

Proof. Using the monotonicity of ψ_{Q_f} and γ_{Q_f} with respect to Q_f and the fact that the two functions are uniformly bounded, we deduce by Beppo-Levi's theorem that there exists two functions ψ, γ such that

$$\begin{aligned} \psi_{Q_f} &\to \psi \quad \text{in } L^r_{\text{loc}}(\Omega) \text{ a.e. in } \Omega \\ \gamma_{Q_f} &\to \gamma \quad \text{in } L^r_{\text{loc}}(\Omega) \quad \forall r \geq 1. \end{aligned}$$

Now let $m > a$ and $\eta \in W^{1,\infty}(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $(-m, m)$, $\eta = 0$ for $|x| \geq m+1$ and $|\eta'| \leq 1$. Let $\Phi(x,z) = \phi_0(x) + \phi_1(z)(Q_s - \phi_0(x))$. Then $\eta^q(\psi_{Q_f} - \Phi)$ is a test function for $(P(Q_s, Q_f))$ and we have

$$\begin{aligned} &\int_{\Omega_{m+1}} \eta^q |B(z)\nabla\psi_{Q_f}|^q \\ &= \int_{\Omega_{m+1}} \eta^q |B(z)\nabla\psi_{Q_f}|^{q-2} B(z)\nabla\psi_{Q_f} \cdot B(z)\nabla\Phi \\ &\quad - \int_{\Omega_{m+1}} q\eta^{q-1}(\psi_{Q_f} - \Phi)|B(z)\nabla\psi_{Q_f}|^{q-2} B(z)\nabla\psi_{Q_f} \cdot B(z)\nabla\eta \\ &\quad - \int_{\Omega_{m+1}} \gamma_{Q_f}\eta^q \partial_x\psi_{Q_f} + \int_{\Omega_{m+1}} \gamma_{Q_f}\eta^q\partial_x\Phi - \int_{\Omega_{m+1}} \gamma_{Q_f}(\psi_{Q_f} - \Phi)q\eta^{q-1}\eta'. \end{aligned}$$

Since $\psi_{Q_f}, \Phi, \nabla\Phi, \eta, \eta', \gamma_{Q_f}$ are uniformly bounded, we deduce by using Hölder's inequality

$$|\psi_{Q_f}|_{1,q,\Omega_m} \leq c_m$$

where c_m is a constant depending on m. Then by using a diagonal process we get, up to a subsequence

$$\psi_{Q_f} \rightharpoonup \psi \quad \text{in } W^{1,q}_{\text{loc}}(\Omega).$$

Let us show that this convergence holds strongly. Let $m > a$ and let $\rho \in \mathcal{D}(\Omega_m)$, $\rho \geq 0$ in Ω_m. Then $\rho^q(\psi_{Q_f} - \psi)$ is a test function for $(P(Q_s, Q_f))$ and we have

$$\begin{aligned} &\int_{\Omega_m} \rho^q |B(z)\nabla\psi_{Q_f}|^q \\ &= \int_{\Omega_m} \rho^q |B(z)\nabla\psi_{Q_f}|^{q-2} B(z)\nabla\psi_{Q_f} \cdot B(z)\nabla\psi \\ &\quad - \int_{\Omega_m} q\rho^{q-1}(\psi_{Q_f} - \psi)|B(z)\nabla\psi_{Q_f}|^{q-2} B(z)\nabla\psi_{Q_f} \cdot B(z)\nabla\rho \\ &\quad - \int_{\Omega_m} \rho^q\gamma_{Q_f}\partial_x(\psi_{Q_f} - \psi) - \int_{\Omega_m} \gamma_{Q_f}(\psi_{Q_f} - \psi)q\rho^{q-1}\partial_x\rho. \end{aligned}$$

Applying Hölder's inequality and letting $Q_f \to 0$, we get

$$\limsup_{Q_f\to 0}\Big(\int_{\Omega_m}\rho^q|B(z)\nabla\psi_{Q_f}|^q\Big)^{1/q}\le\Big(\int_{\Omega_m}\rho^q|B(z)\nabla\psi|^q\Big)^{1/q}.$$

Hence $\rho^q B(z)\nabla\psi_{Q_f}\to\rho^q B(z)\nabla\psi$ in $L^q(\Omega_m)$ and in particular

$$\nabla\psi_{Q_f}\to\nabla\psi \quad \text{in } W^{1,q}_{\text{loc}}(\Omega).$$

Now using the monotonicity with respect to Q_f and the continuity of ψ_{Q_f}, it follows by Dini's theorem

$$\lim_{Q_f\to 0}\big(\lim_{R\to\pm\infty}\psi_{Q_f}(x+R,z)\big)=\lim_{R\to\pm\infty}\big(\lim_{Q_f\to 0}\psi_{Q_f}(x+R,z)\big)$$

and

$$\lim_{R\to\pm\infty}\psi(x+R,z)=\lim_{Q_f\to 0}v^{Q_f}_{\pm\infty}(z)=v_{-\infty}(z)\qquad\text{for } (x,z)\in\Omega_{0,1},$$

where $v^{Q_f}_{-\infty}$ and $v^{Q_f}_{+\infty}$ were defined by (1.15). Since $\partial_x\psi\le 0$, we deduce that $\psi(x,z)=v_{-\infty}(z)=Q_s\phi_1(z)$. Moreover $\gamma\in H(\psi)$ and $\psi>0$ in Ω, so $\gamma=\gamma_s$ a.e. in Ω. We conclude that when $Q_f\to 0$, the aquifer is completely saturated by salt water only. □

Theorem 2.3. *Let $(\psi_{Q_s},\gamma_{Q_s})$ be the solution of $(P(Q_s,Q_f))$. We have*

$$(\psi_{Q_s},\gamma_{Q_s})\to(\psi,\gamma_f)\quad \textit{in } W^{1,q}_{\text{loc}}(\Omega)\times L^{q'}_{\text{loc}}(\Omega)\ \textit{as } Q_s\to 0,$$

where ψ is the solution of the following problem:

Problem($P(Q_f)$):

(i) $\operatorname{div}\big(\mathcal{B}(z,\nabla\psi)\big)=0$ in $\mathcal{D}'(\Omega)$
(ii) $-Q_f<\psi<0$ in Ω
(iii) $\psi(x,-h)=0$, $\psi(x,0)=\phi_0(x)$ for all $x\in\mathbb{R}$
(iv) $\lim_{x\to-\infty}\psi(x,z)=0$, $\lim_{x\to+\infty}\psi(x,z)=Q_f(\phi_1(z)-1)$.

Proof of Theorem 2.3. Taking into account the monotonicity of ψ_{Q_s} and γ_{Q_s} with respect to Q_s and arguing as in the proof of Theorem 2.2, we deduce the existence of two functions $\bar\psi$ and $\bar\gamma$ such that

$$\psi_{Q_s}\to\bar\psi\quad\text{in } W^{1,q}_{\text{loc}}(\Omega)\quad\text{and}\quad\gamma_{Q_s}\to\bar\gamma\quad\text{in } L^{q'}_{\text{loc}}(\Omega).$$

It follows that $(\bar\psi,\bar\gamma)$ satisfies:

(i) $\operatorname{div}\big(\mathcal{B}(z,\nabla\bar\psi)\big)=-\partial_x\bar\gamma$ in $\mathcal{D}'(\Omega)$
(ii) $\bar\gamma\in H(\bar\psi)$
(iii) $-Q_f\le\bar\psi\le 0$ in Ω
(iv) $\bar\psi(x,-h)=0$, $\bar\psi(x,0)=\phi_0(x)$ for all $x\in\mathbb{R}$
(v) $\lim_{x\to-\infty}\bar\psi(x,z)=0$, $\lim_{x\to+\infty}\bar\psi(x,z)=Q_f(\phi_1(z)-1)$
(vi) $\partial_x\bar\psi\le 0$ and $\partial_x\bar\gamma\le 0$ in $\mathcal{D}'(\Omega)$.

By the weak maximum principle, we can compare $\bar\psi$ with the solution ψ of $P(Q_f)$. This gives $\bar\psi\le\psi$ in Ω. Since ψ satisfies by the strong maximum principle $-Q_f<\psi<0$, we deduce that $\bar\psi<0$ in Ω and then $\bar\gamma=\gamma_f$ a.e in Ω. Consequently $\bar\psi=\psi$ in Ω. We conclude that when $Q_s\to 0$, the aquifer is completely saturated by fresh water only. □

The end of this section is devoted to study the set $\Omega_f = [\psi < 0]$. We would like to point out that it was proved in [1] in the linear case that this set is star shaped with the origin. The proof was based on blow-up arguments. Here we propose a different proof based on a comparison argument that works for the linear case as well as for the nonlinear one with constant permeability. So we assume that $\mathcal{B}$ does not depend on z. For any $r > 0$, we consider

$$\psi_r(x,z) = \frac{1}{r}\psi(rx,rz) \quad \text{and} \quad \gamma_r(x,z) = \frac{1}{r}\gamma(rx,rz)$$

defined on $\Omega_r = \mathbb{R} \times (\frac{-h}{r}, 0)$. It is easy to check that (ψ_r, γ_r) is the solution of the following problem:

Problem (P_r) Find $(\psi_r, \gamma_r) \in W^{1,q}_{\mathrm{loc}}(\Omega_r) \times L^\infty(\Omega_r)$ such that:

(i) $\int_{\Omega_r} |B\nabla\psi_r|^{q-2} B\nabla\psi_r \cdot B\nabla\zeta + \gamma_r e_x.\nabla\zeta = 0$ for all $\zeta \in W^{1,q}_0(\Omega_r)$ with compact support in $\overline{\Omega}_r$

(ii) $\gamma_r \in H(\psi_r)$ a.e. in Ω_r

(iii) $\psi_r(x, \frac{-h}{r}) = \frac{Q_s}{r}$, $\psi_r(x,0) = \frac{1}{r}\phi_0(rx)$ for all $x \in \mathbb{R}$

(iv) $\partial_x\psi_r \leq 0$ and $\partial_x\gamma_r \leq 0$ in $\mathcal{D}'(\Omega_r)$.

¿From the study of problem (P), we know that problem (P_r) has a unique solution with a continuous free boundary g_r and that $\lim_{x\to-\infty} \psi_r(x,z) = \frac{1}{r}v_{-\infty}(rz)$, $\lim_{x\to+\infty} \psi_r(x,z) = \frac{1}{r}v_{+\infty}(rz)$. Moreover since we assume that $\mathcal{B}$ does not depend on z, we have $v_{-\infty}(z) = -\frac{Q_s}{h}z$ and $v_{+\infty}(z) = -\frac{Q_f}{r} - \frac{Q_f+Q_s}{h}z$.

Theorem 2.4. *For $0 < r_1 < r_2$, we have $\psi_{r_1} \leq \psi_{r_2}$ in $\Omega_{r_2} \subset \Omega_{r_1}$.*

Proof. We remark that $(\psi_{r_1}, \gamma_{r_1})$ and $(\psi_{r_2}, \gamma_{r_2})$ satisfy the same equation on Ω_{r_2}. Moreover one can check that

$$\psi_{r_1} \leq \psi_{r_2} \quad \text{on } \partial\Omega_{r_2} \quad \text{and} \quad \lim_{x\to\pm\infty} \psi_{r_1}(x,z) \leq \lim_{x\to\pm\infty} \psi_{r_2}(x,z).$$

Then we can derive a similar result as in Lemma 1. Since $\psi_{r_1}(x,0) = \psi_0(x,0) = -Q_f/r_1$ where $\psi_0 = \min(\psi_{r_1}, \psi_{r_2})$, one can argue as in the proof of Theorem 2.1 to prove that $\psi_{r_1} = \psi_0$ in $[\psi_0 < 0]$.

To prove that $\psi_{r_1} = \psi_0$ in $[\psi_0 > 0]$ it is enough to verify that $\nabla\psi_{r_1}$ does not vanish on some part of the left hand side of the line $[z = -\frac{h}{r_2}]$. So assume that for some x_0, we have $\nabla\psi_{r_1}(x, -h/r_2) = 0$ for all $x \leq x_0$. Then $\psi_{r_1}(x_0, -h/r_2) = C \in \mathbb{R}$ for all $x \leq x_0$. By the asymptotic behavior $C = Q_s/r_2 > 0$ and by continuity and monotonicity, ψ_{r_1} is positive in a strip $D_\epsilon = (-\infty, x_0) \times (-\frac{h}{r_2} - \epsilon, -\frac{h}{r_2} + \epsilon)$ for some small $\epsilon > 0$. Therefore ψ_{r_1} is $\mathcal{B}$-Harmonic in D_ϵ and its gradient has non-isolated zeros. This means that $\psi_{r_1} = C$ in D_ϵ which contradicts the asymptotic behavior. □

Corollary 3. *Ω_f is star shaped with the origin.*

Proof. Let $X_0 \in \Omega_f$ and $t \in (0,1]$. We have by Theorem 2.4, $\psi_t(X_0) \leq \psi_1(X_0) = \psi(X_0) < 0$. So $\psi(tX_0) < 0$, which means that $tX_0 \in \Omega_f$. □

Corollary 4. *(i) There exists $z_0 \in (-h^*, 0)$ such that $g(z) \geq 0$ for all $z \in (-h^*, z_0)$.*
(ii) g is non-increasing where it is nonnegative.

Proof. (i) First note that the set $[g > 0]$ is nonempty. Indeed if $g(z) \leq 0$ $\forall z \in (-h^*, 0)$, then for all $(x,z) \in (0,+\infty) \times (-h^*, 0)$ we have $\psi(x,z) < 0$. Since we also have $\psi(x,z) > 0$ in $(0,+\infty) \times (-h, -h^*)$, we get a contradiction with Lemma 15.

Let $z_0 = \inf\{ z \in (-h^*, 0) \,/\, g(z) > 0\}$ and take $z \in (-h^*, z_0)$. Since $r = \frac{z_0}{z} < 1$, we have

$$\begin{aligned} 0 = \psi(g(z_0), z_0) &= \frac{1}{r}\psi\Big(r\Big(\frac{g(z_0)}{r}\Big), rz\Big) = \psi_r\Big(\frac{g(z_0)}{r}, z\Big) \\ &\le \psi_1\Big(\frac{g(z_0)}{r}, z\Big) = \psi\Big(\frac{g(z_0)}{r}, z\Big) \le \psi(g(z_0), z) \end{aligned}$$

since $g(z_0) \ge 0$ and $\frac{1}{r} > 1$. Thus $\psi(g(z_0), z) \ge 0$ and then $g(z) \ge g(z_0) \ge 0$.
(ii) Let $z_1, z_2 \in (-h^*, z_0)$ such that $z_1 < z_2$. If $g(z_2) \ge 0$, we can argue as in $i)$ and obtain $g(z_1) \ge g(z_2)$. □

3. Behavior of the free boundary near $z = -h^*$

In [6] we proved in the linear and heterogeneous case ($q = 2$) that the free boundary has the line $[z = -h^*]$ as an asymptote. Here we generalize the proof to the nonlinear case. Before this, we give a second proof which is much simpler but works only when the permeability is constant.

Theorem 3.1. *(i) The set $S = \{x \in \mathbb{R} : \psi(x, -h^*) = 0\}$ is empty and $\Gamma = [x = g(z)]$.*
(ii) $\lim_{z\to -h^*} g(z) = +\infty$.

Case of a constant permeability: (ii) Since g is non-increasing in $(-h^*, z_0)$ (see Corollary 4), there exists a limit L for g as $z \to -h^*$. Assume that L is finite. By the monotonicity of g we get $g(z) \le L$ $\forall z \in (-h^*, z_0)$ and then

$$\forall x > L, \quad \forall z \in (-h^*, z_0), \quad \psi(x, z) < 0.$$

Since $\psi > 0$ for $z < -h^*$ we deduce by continuity that we have necessarily $\psi(x, -h^*) = 0$ for all $x > L$. This leads to a contradiction with Lemma 15. Thus $L = +\infty$.
(i) Assume that $S \ne \emptyset$. Then there exists $x_0 \in \mathbb{R}$ such that $\psi(x_0, -h^*) = 0$. For $A_0 > x_0$ there exists $\delta > 0$ by (ii) such that: for all $z \in (-h^*, -h^* + \delta)$, $g(z) > A_0$. So for $(x, z) \in (-\infty, A_0)\times(-h^*, -h^*+\delta)$ we have $\psi(x, z) > 0$. By monotonicity of ψ, we deduce that $\psi(x, -h^*) = 0$ $\forall x \ge x_0$ since one has $v_{+\infty}(-h^*) = 0 \le \psi(x, -h^*) \le \psi(x_0, -h^*) = 0$. Hence we have $\psi > 0$ in $(x_0, A_0) \times \Big((-h^* - \delta, -h^* + \delta) \setminus \{-h^*\}\Big)$ and $\psi(x, -h^*) = 0$ $\forall x \in (x_0, A_0)$ which contradicts Lemma 15.
The general case: Since we follow the proof given in [6] for the linear case, we will only give an outline of it.
(i) Assume that $S \ne \emptyset$. Then $S = [\alpha, +\infty)$ with $\alpha = \inf S > -\infty$. We need some lemmas.

Lemma 2. *There exists $u \in L^\infty(-h, 0)$ such that $u > 0$ for a.e $z \in (-h, 0)$ and*

$$E\big(u(z)\big) = \big(\overline{A}_{22}u^2(z) - 2\overline{A}_{12}u(z) + \overline{A}_{11}\big)^{\frac{q-2}{2}} \big(\overline{A}_{22}u(z) - \overline{A}_{12}\big) - C_0 = 0$$

for some constant $C_0 > -\big(\overline{A}_{11}^{\frac{q-2}{2}}\overline{A}_{12}\big)(z)$ for a.e $z \in (-h, 0)$. $\overline{A}_{ij}$ being the entries of the matrix $\overline{A}$ introduced in Section 1.

Proof. The function $E : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{t\to+\infty} E(t) = +\infty$, $E(0) = -\overline{A}_{11}^{\frac{q-2}{2}}\overline{A}_{12} - C_0 < 0$. So we deduce that for a.e $z \in (-h, 0)$, there exists $u(z) > 0$ such that $E\big(u(z)\big) = 0$. This clearly defines a positive and uniformly bounded function u on $(-h, 0)$. □

Set $\alpha' = \alpha + \frac{1}{2}$ and define $f(z) = \int_{-h}^{z} u(s)ds + \alpha'$. For $k > 0$ define v and θ by

$$v(x,z) = \big(k(\gamma_s - \gamma_f)\big)^{1/(q-1)}(f(z) - x)^+$$
$$\theta(x,z) = \gamma_s\chi\big([x < f(z)]\big) + \gamma_f\chi\big([x > f(z)]\big)$$

for $(x,z) \in D(z_1) = (\alpha', +\infty) \times (-h^*, z_1)$ with $z_1 \in (-h^*, 0)$. Then we have the following lemma.

Lemma 3. *There exists $k > 0$ such that*

$$\int_{D(z_1)} \Big(\mathcal{B}(z, \nabla v) + \theta e_x\Big)\nabla\xi \geq 0 \quad \forall \xi \in \mathcal{D}\big(D(z_1)\big).$$

To prove this lemma, one can adapt the proof of [6, Lemma 6.3].

Lemma 4. *Let (ψ, γ) be the solution of (P). Then there exists $z_0 \in (-h^*, 0)$ such that*

$$\int_{D(z_0)} \big(\mathcal{B}(z, \nabla\psi^+) - \mathcal{B}(z, \nabla v_0) + (\gamma - \theta_0)e_x\big)\nabla\zeta = 0 \quad \forall \zeta \in \mathcal{D}\big(\mathbb{R}^2\big)$$

where $v_0 = \min(\psi^+, v)$ and $\theta_0 = \min(\gamma, \theta)$.

To prove this lemma, one can adapt the proof in [6, Lemma 6.4].

completion of the proof for Theorem 3.1. Let $z_* \in (-h^*, z_0)$.
• If $\psi^+(\alpha', z_*) = 0$, then by monotonicity we have $\psi^+(x, z_*) = 0\ \forall x \geq \alpha'$.
• If $\psi^+(\alpha', z_*) > 0$, then since we can choose k such that $\psi^+(\alpha', z_*) < v(\alpha', z_*)$, we deduce by continuity that there exists a small ball $B_r(\alpha', z_*)$ in which one has $\psi^+ > 0$ and $\psi^+ < v$. Then $\psi^+ = v_0$ in $B_r(\alpha', z_*)$. Let us denote by C_* the connected component of $D(z_0) \cap [\psi^+ > 0]$ which contains $B_r(\alpha', z_*) \cap D(z_0)$.

Let $D^+(z_0) = D(z_0) \cap [x < f(z)]$. By Lemma 4 we have

$$\operatorname{div}\Big(\mathcal{B}(z, \nabla\psi^+) - \mathcal{B}(z, \nabla v_0)\Big) + (\gamma - \theta_0)_x = 0 \quad \text{in } \mathcal{D}'\big(C_* \cap D^+(z_0)\big).$$

But since in $C_* \cap D^+(z_0)$ we have $\psi^+ > 0$, $v > 0$, it follows that $\gamma = \theta = \theta_0 = \gamma_s$ and then

$$\operatorname{div}\big(\mathcal{B}(z, \nabla\psi^+)\big) = \operatorname{div}\big(\mathcal{B}(z, \nabla v_0)\big) = 0 \quad \text{in } \mathcal{D}'\big(C_* \cap D^+(z_0)\big).$$

So we have

$$\operatorname{div}\big(\mathcal{B}(z, \nabla\psi^+)\big) = 0, \quad \operatorname{div}\big(\mathcal{B}(z, \nabla v_0) = 0 \quad \text{in } \mathcal{D}'\big(C_* \cap D^+(z_0)\big)$$
$$\psi^+ \geq v_0 \quad \text{in } C_* \cap D^+(z_0)$$
$$\psi^+ = v_0 \quad \text{in } B_r(\alpha', z_*) \cap D^+(z_0)$$

which leads by Lemma 13 (see Appendix) to $\psi^+ = v_0$ in $C_* \cap D^+(z_0)$. We then conclude as in the end of the proof of Theorem 6.1 in [6].
(ii) See Corollary 6.6 of [6]. □

4. Behavior of the free boundary near $z = 0$

Now we study the free boundary near $z = 0$. We first prove the existence of a limit $g(0-) \leq 0$ as z approaches zero. Then we prove that this limit is finite in two cases.

4.1. **Existence of the limit of g at $z = 0$.**

Theorem 4.1. *The function g has a limit when z approaches zero.*

Proof. It suffices to show that $\liminf_{z\to 0} g(z) = \limsup_{z\to 0} g(z)$.
First, if $\limsup_{z\to 0} g(z) = -\infty$, then

$$\liminf_{z\to 0} g(z) = \limsup_{z\to 0} g(z) = \lim_{z\to 0} g(z) = -\infty.$$

Now assume that $\limsup_{z\to 0} g(z) = a^+ > -\infty$. Note that $a^+ \le 0$. Indeed if $a^+ > 0$, then $\psi(a^+, 0) < 0$ and by continuity of ψ there exists $\epsilon > 0$ such that $\psi < 0$ in $(a^+ - \epsilon, a^+ + \epsilon) \times (-\epsilon, 0)$. So $g(z) \le a^+ - \epsilon \quad \forall z \in (-\epsilon, 0)$ and then $a^+ \le a^+ - \epsilon$ which is impossible.

Set $a^- = \liminf_{z\to 0} g(z)$ and assume that $a^- < a^+$. Let $x_1 \in (a^-, a^+)$, $x_2 \in (x_1, a^+)$ and let $(z_n)_n$ be a sequence that satisfies

$$\lim_{n\to +\infty} z_n = 0 \quad \text{and} \quad \lim_{n\to +\infty} g(z_n) = a^-.$$

So there exists $n_1 \in \mathbb{N}$ such that $g(z_n) \le x_1$ for all $n \ge n_1$ and then

$$\psi^+(x_1, z_n) = 0 \quad \forall n \ge n_1. \tag{4.1}$$

Arguing as in the proof of Lemma 2, we prove the existence of a negative function $u \in L^\infty(-h, 0)$ such that for a.e $z \in (-h, 0)$

$$E\big(u(z)\big) = \big(\overline{A}_{22}u^2(z) - 2\overline{A}_{12}u(z) + \overline{A}_{11}\big)^{\frac{q-2}{2}}\big(\overline{A}_{22}u(z) - \overline{A}_{12}\big) - C_1 = 0 \tag{4.2}$$

for some constant $C_1 < -\big(\overline{A}_{11}^{\frac{q-2}{2}}\overline{A}_{12}\big)(z)$ for a.e $z \in (-h, 0)$. We then set

$$f(z) = \int_0^z u(s)ds + x_1' \quad \text{for } x_1' \in (x_1, x_2).$$

Since f is continuous and non-increasing there exists $n_2 \in \mathbb{N}$ such that

$$f(0) = x_1' < f(z) < x_2 \quad \forall z \in (z_n, 0)\ \forall n \ge n_2. \tag{4.3}$$

Now for $k > 0$ we define v and θ by

$$\begin{aligned} v(x,z) &= \big(k(\gamma_s - \gamma_f)\big)^{\frac{1}{q-1}}(f(z) - x)^+ \\ \theta(x,z) &= \gamma_s\chi\big([x < f(z)]\big) + \gamma_f\chi\big([x > f(z)]\big) \end{aligned} \tag{4.4}$$

for $(x,z) \in D(z_n) = (x_1, +\infty) \times (z_n, 0)$ with $n \ge n_2$. Then as in Lemma 3, one can deduce from (4.2) and(4.4) the existence of $k > 0$ such that

$$\int_{D(z_n)} \Big(\mathcal{B}(z, \nabla v) + \theta e_x\Big)\nabla\xi \ge 0 \quad \forall \xi \in \mathcal{D}\big(D(z_n)\big).$$

Using the fact that $\psi^+(x_1, 0) = 0$, the continuity of ψ, the monotonicity of f, (4.1) and (4.3), there exists $n \ge \sup(n_1, n_2)$ such that $\psi^+(x_1, z) \le v(x_1, z) \quad \forall z \in (z_n, 0)$. Moreover one can check that

$$\psi^+ \le v \quad \text{on } \partial D(z_n) \quad \text{and} \quad \lim_{x\to +\infty} \psi^+(x,z) = \lim_{x\to +\infty} v(x,z) = 0.$$

Arguing as in the proof of Theorem 3.1, we obtain $\psi^+ \le v$ in $D(z_n)$ from which we deduce that $\psi^+(x,z) = 0 \quad \forall (x,z) \in (x_2, a^+) \times (z_n, 0)$ and then $g(z) \le x_2 \quad \forall z \in (z_n, 0)$. So $a^+ = \limsup_{z\to 0} g(z) \le x_2 < a^+$ and we get a contradiction. Thus we have $a^+ = a^-$ and $\lim_{z\to 0} g(z)$ exists. □

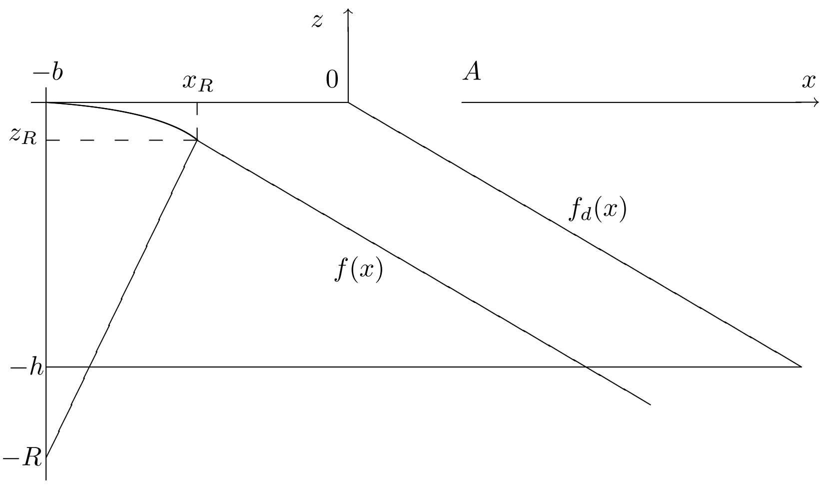

FIGURE 2

The remainder of this section is devoted to show that $g(0-)$ is finite in two general cases.

4.2. **The Linear Nonhomogeneous Case.** We assume that $q = 2$ and $B'_{21}(z) \leq 0$ in $\mathcal{D}'(-h, 0)$, then we have the following theorem.

Theorem 4.2.
$$\lim_{z \to 0} g(z) > -\infty.$$

The idea of the proof is to compare ψ with a suitable function. First let $R, b > 0$ and consider

$$f(x) = \begin{cases} -R + \sqrt{R^2 - (x+b)^{+2}} & \text{if } x \leq x_R = \frac{R(a+b)}{\sqrt{R^2+(a+b)^2}} - b, \\ z_R - \frac{a+b}{R}(x - x_R) & \text{if } x \geq x_R \end{cases}$$

with $z_R = \frac{R(a+b)}{\sqrt{R^2+(a+b)^2}} - R$ (see Figure 2). Set

$$\omega(z) = h \frac{\int_0^z \frac{ds}{B_{22}(s)}}{\int_{-h}^0 \frac{ds}{B_{22}(s)}} = \kappa \int_0^z \frac{ds}{B_{22}(s)} \quad \text{for } z \in (-h, 0).$$

For $t > 0$, consider

$$G(t) = \lambda[(t+1)^2 - 1] \quad \text{with} \quad \lambda = \frac{Q_s}{h^2 + 2h}$$

and define v_1 by

$$v_1(x, z) = G(f(x) - \omega(z)) \quad \text{for } (x, z) \in D_1 = \{(x, z) \in \Omega / -h < z < \omega^{-1}(x)\}.$$

For $d > 0$, set $f_d(x) = f(x) + d$ and for $t > 0$ consider

$$K(t) = \mu \log(1 + t) \quad \text{with} \quad \mu = \frac{Q_f}{\log(1 + d)}.$$

Then we define v_2 by

$$v_2(x, z) = -K(\omega(z) - f(x)) \quad \text{for } (x, z) \in D_2$$

with $D_2 = \{(x,z) \in \Omega : \omega^{-1}(x) < z < \omega^{-1} o f_d(x)\}$. Now set

$$v = \chi(D_1)v_1 + \chi(D_2)v_2 \quad \text{and} \quad \theta = \chi(D_1)\gamma_s + \chi(D_1)\gamma_f.$$

Remark that $v \in H^1_{\rm loc}(D)$ since $v_1(x,\omega^{-1}(x)) = G(0) = 0$ and $v_2(x,\omega^{-1}(x)) = -K(0) = 0$. We will compare (ψ,γ) with (v,θ) on the domain $D = (\overline{D_1 \cup D_2}) \cap \Omega$. The proof needs some preliminary Lemmas.

Lemma 5. *There exists* $\alpha_1 > 0$ *such that*

$$\forall \alpha \in (0,\alpha_1),\ \forall R \geq \max\Big(\frac{2M^2}{\kappa^2}, \frac{a}{\alpha}\Big), \quad \operatorname{div}\big(B(z)\nabla v_1\big) \geq 0 \quad \text{in } \mathcal{D}'(D_1). \tag{4.5}$$

Proof. Indeed let C_1 be a constant such that $B_{21}(z) + C_1 > 0$ for a.e $z \in (-h,0)$. Then we have

$$\begin{aligned}
&\operatorname{div}\big(B(z)\nabla v_1\big)\\
&= \frac{\partial}{\partial x}\big(B_{11}\frac{\partial v_1}{\partial x} + B_{12}\frac{\partial v_1}{\partial z}\big) + \frac{\partial}{\partial z}\big(-C_1\frac{\partial v_1}{\partial x} + B_{22}\frac{\partial v_1}{\partial z}\big) + \frac{\partial}{\partial z}\big((B_{21}+C_1)\frac{\partial v_1}{\partial x}\big)\\
&= B_{11}\big(G''(f(x)-\omega(z))(f')^2 + G'(f(x)-\omega(z))f''\big)\\
&\quad + (B_{12}-C_1)\big(\frac{-\kappa}{B_{22}} f' G''(f(x)-\omega(z))\big) + \frac{\kappa^2}{B_{22}} G''(f(x)-\omega(z))\\
&\quad - \frac{\kappa}{B_{22}}(B_{21}+C_1)G''(f(x)-\omega(z))f' + B'_{21}G'(f(x)-\omega(z))f'\\
&\geq 2\lambda B_{11}\big((f')^2 + \frac{\kappa^2}{B_{22}B_{11}} - \frac{\kappa}{B_{22}B_{11}}(B_{12}-C_1)f' + (f(x)-\omega(z)+1)f''\big)\\
&= 2\lambda B_{11} I_1
\end{aligned} \tag{4.6}$$

since $B_{21}+C_1 > 0$, $G'' = 2\lambda$, $f'(x) \leq 0$, $B'_{21} \leq 0$ in $\mathcal{D}'(-h,0)$ and $G'(t) = 2\lambda(t+1) > 0$ for $t > 0$.

We shall distinguish three cases:

- For $x \leq -b$, we have $f(x) = 0$ and then $I_1 = \frac{\kappa^2}{B_{22}B_{11}} > 0$.
- For $x \in (x_R, f^{-1}(-h))$, we have $f''(x) = 0$, $f'(x) = -\frac{a+b}{R} = -\alpha$, $m \leq B_{11}, B_{22} \leq M$, $|B_{12}| \leq M$, where m and M are two positive constants such that

$$m|\xi|^2 \leq \langle B(z)\xi,\xi\rangle \leq M|\xi|^2 \quad \forall \xi \in \mathbb{R}^2,\ \text{a.e. } z \in (-h,0). \tag{4.7}$$

  So

$$\begin{aligned}
I_1 &= \alpha^2 + \frac{\kappa^2}{B_{22}B_{11}} + \frac{\alpha\kappa}{B_{22}B_{11}}(B_{12}-C_1)\\
&\geq \alpha^2 + \frac{\kappa^2}{M^2} - \frac{\alpha\kappa}{m^2}(M+C_1) \to \frac{\kappa^2}{M^2} > 0 \quad \text{as } \alpha \to 0.
\end{aligned}$$

  Therefore, there exists $\alpha_0 > 0$ such that $\forall \alpha \in (0,\alpha_0)$ $\quad I_1 \geq 0$.
- For $x \in (-b, x_R)$, we have

$$f'(x) = \frac{-(x+b)}{\sqrt{R^2-(x+b)^2}}, \quad f''(x) = -\frac{R^2}{(R^2-(x+b)^2)^{3/2}},$$
$$1 + {f'}^2(x) = \frac{R^2}{R^2-(x+b)^2}.$$

Then

$$\begin{aligned} I_1 = \frac{R^2}{(R^2-(x+b)^2)^{3/2}}\Big(&\sqrt{R^2-(x+b)^2} \\ &- f(x) + \frac{(R^2-(x+b)^2)^{3/2}}{R^2}\big(\frac{\kappa^2}{B_{22}B_{11}}-1\big) \\ &+ \frac{\kappa(B_{12}-C_1)}{B_{22}B_{11}}\frac{(R^2-(x+b)^2)}{R^2}(x+b)+\omega(z)-1\Big). \end{aligned}$$

Note that

- $\sqrt{R^2-(x+b)^2}-f(x)=R$.
- $\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\Big(\frac{\kappa^2}{B_{22}B_{11}}-1\Big) \geq R\Big(\frac{\kappa^2}{M^2}-1\Big)$ provided that $M>\kappa$.
- $\frac{\kappa(B_{12}-C_1)}{B_{22}B_{11}}\frac{(R^2-(x+b)^2)}{R^2}(x+b) \geq -C_2\frac{\alpha R}{\sqrt{1+\alpha^2}} \geq -C_2(x_R+b)$ since $\frac{\kappa(B_{12}-C_1)}{B_{22}B_{11}} > -C_2$ for some positive constant C_2.
- $\omega(z)\geq -h$.

Therefore

$$\begin{aligned} I_1 &\geq \frac{R^2}{(R^2-(x+b)^2)^{3/2}}\big(R\frac{\kappa^2}{M^2}-\frac{C_2\alpha}{\sqrt{1+\alpha^2}}R-h-1\big) \\ &\geq \frac{R^3}{(R^2-(x+b)^2)^{3/2}}\Big(\big(\frac{\kappa^2}{2M^2}-C_2\alpha\big)+\big(\frac{\kappa^2}{2M^2}-\frac{h+1}{R}\big)\Big) \qquad \geq 0 \end{aligned}$$

provided that $\alpha \leq \frac{\kappa^2}{2M^2C_2}$ and $R > \frac{2M^2}{\kappa^2}(h+1)$.

Finally for $\alpha \in \big(0, \alpha_1 = \min\big(\alpha_0, \frac{\kappa^2}{2M^2C_2}\big)\big)$ and $R > \frac{2M^2}{\kappa^2}(h+1)$, one has $I_1 \geq 0$ and then by (4.6) we obtain (4.5). □

Lemma 6. *There exists $\alpha_2 > 0$ such that*

$$\forall \alpha \in (0,\alpha_2),\ \forall R \geq \max\big(\frac{2M^2}{\kappa^2}, \frac{a}{\alpha}\big), \quad \operatorname{div}(B(z)\nabla v_2) \geq 0 \quad \text{in } \mathcal{D}'(D_2). \tag{4.8}$$

Proof. Indeed we have

$$\begin{aligned} &\operatorname{div}(B(z)\nabla v_2) \\ &= \frac{\partial}{\partial x}\big(B_{11}\frac{\partial v_2}{\partial x}+B_{12}\frac{\partial v_2}{\partial z}\big)+\frac{\partial}{\partial z}\big(-C_1\frac{\partial v_2}{\partial x}+B_{22}\frac{\partial v_2}{\partial z}\big)+\frac{\partial}{\partial z}\big((B_{21}+C_1)\frac{\partial v_2}{\partial x}\big) \\ &= B_{11}\big(-K''(\omega(z)-f(x))(f')^2+K'(\omega(z)-f(x))f''\big) \\ &\quad + (B_{12}-C_1)\frac{\kappa}{B_{22}}f'K''(\omega(z)-f(x)) - \frac{\kappa^2}{B_{22}}K''(\omega(z)-f(x)) \\ &\quad + \frac{\kappa}{B_{22}}(B_{21}+C_1)K''(\omega(z)-f(x))f' + B_{21}'K'(\omega(z)-f(x))f' \\ &\geq \frac{\mu B_{11}}{(1+\omega(z)-f(x))^2}\Big((f')^2+\frac{\kappa^2}{B_{22}B_{11}}-\frac{\kappa(B_{12}-C_1)}{B_{22}B_{11}}f'+(1+\omega(z)-f(x))f''\Big) \\ &= \frac{\mu B_{11}}{(1+\omega(z)-f(x))^2}I_2, \end{aligned} \tag{4.9}$$

since $B_{21}+C_1>0$, $K''(t)=-\frac{\mu}{(1+t)^2}<0$, $K'(t)=\frac{\mu}{1+t}>0$, $f'(x)\leq 0$, $B_{21}'\leq 0$ in $\mathcal{D}'(-h,0)$.

We distinguish two cases:

• For $x \in (x_R, f_d^{-1}(-h))$, we have $f''(x) = 0$, $f'(x) = -\frac{a+b}{R} = -\alpha$ and then

$$\begin{aligned} I_2 &= \alpha^2 + \frac{\kappa^2}{B_{22}B_{11}} + \frac{\alpha\kappa}{B_{22}B_{11}}(B_{12} - C_1) \\ &\geq \alpha^2 + \frac{\kappa^2}{M^2} - \frac{\alpha\kappa}{m^2}(M + C_1) \to \frac{\kappa^2}{M^2} > 0 \quad \text{as } \alpha \to 0. \end{aligned}$$

So there exists $\alpha_0 > 0$ such that $I_2 \geq 0 \quad \forall \alpha \in (0, \alpha_0)$.

• For $x \in (-b, x_R)$, we have

$$\begin{aligned} I_2 &= \frac{R^2}{(R^2 - (x+b)^2)} + \frac{\kappa^2}{B_{22}B_{11}} - 1 + \frac{\kappa(B_{12} - C_1)}{B_{22}B_{11}} \frac{(x+b)}{\sqrt{R^2 - (x+b)^2}} \\ &\quad - \frac{R^2}{(R^2 - (x+b)^2)^{3/2}}(1 + \omega(z) - f(x)) \\ &= \frac{R^2}{(R^2 - (x+b)^2)^{3/2}} \Big(\sqrt{R^2 - (x+b)^2} + \frac{(R^2 - (x+b)^2)^{3/2}}{R^2} \big(\frac{\kappa^2}{B_{22}B_{11}} - 1 \big) \\ &\quad + \frac{\kappa(B_{12} - C_1)}{B_{22}B_{11}}(x+b) \frac{(R^2 - (x+b)^2)}{R^2} - (1 + \omega(z) - f(x)) \Big). \end{aligned}$$

Taking into account the fact that $x \in (-b, x_R]$ and (4.7), it follows that

$$\begin{aligned} &I_2 \\ &\geq \frac{R^2}{(R^2 - (x+b)^2)^{3/2}} \Big(-R + 2\sqrt{R^2 - (x_R+b)^2} + R\big(\frac{\kappa^2}{M^2} - 1\big) - C_2(x_R + b) - 1 \Big) \\ &= \frac{R^2}{(R^2 - (x+b)^2)^{3/2}} \Big(-R + \frac{2R}{\sqrt{1+\alpha^2}} + R\big(\frac{\kappa^2}{M^2} - 1\big) - C_2 \frac{\alpha}{\sqrt{1+\alpha^2}} R - 1 \Big) \\ &= \frac{R^3}{(R^2 - (x+b)^2)^{3/2}} \Big(\frac{2}{\sqrt{1+\alpha^2}} - 2 + \big(\frac{\kappa^2}{2M^2} - \frac{C_2\alpha}{\sqrt{1+\alpha^2}} \big) + \frac{\kappa^2}{2M^2} - \frac{1}{R} \Big) \geq 0 \end{aligned}$$

provided that $R > \frac{2M^2}{\kappa^2}$ and $\alpha \in (0, \alpha_0')$ for a small $\alpha_0' > 0$. Finally for $\alpha \in (0, \alpha_2 = \min(\alpha_0, {\alpha_0}'))$, we have $I_2 \geq 0$ and from (4.9) we obtain (4.8). □

Lemma 7. *There exists $\alpha_* > 0$ and $R_* > 0$ such that*

$$\forall \alpha \in (0, \alpha_*),\ \forall R \geq R_*, \quad \operatorname{div}\big(B(z)\nabla v + \theta e_x\big) \geq 0 \quad \text{in } \mathcal{D}'(D). \tag{4.10}$$

Proof. Let $\xi \in \mathcal{D}(D)$, $\xi \geq 0$. Then by (4.5) and (4.8),

$$\begin{aligned} I(\xi) &= \int_D \big(B(z)\nabla v + \theta e_x\big)\nabla\xi \\ &= \int_{D_1} \big(B(z)\nabla v_1 + \gamma_s e_x\big)\nabla\xi + \int_{D_2} \big(B(z)\nabla v_2 + \gamma_f e_x\big)\nabla\xi \\ &= \langle -\operatorname{div}\big(B(z)\nabla v_1\big), \xi\rangle + \langle -\operatorname{div}\big(B(z)\nabla v_2\big), \xi\rangle \\ &\quad + \int_{[z=\omega^{-1}(x)]\cap\Omega} \big(B(z)(\nabla v_1 - \nabla v_2).\nu + (\gamma_s - \gamma_f)\nu_x\big)\xi \\ &\leq \int_{[z=\omega^{-1}(x)]\cap\Omega} \big(B(z)(\nabla v_1 - \nabla v_2).\nu + (\gamma_s - \gamma_f)\nu_x\big)\xi \end{aligned}$$

provided that $\alpha \in (0, \min(\alpha_1, \alpha_2))$ and $R > \frac{2M^2}{\kappa^2}(h+1)$. Moreover one has

$$\nabla v_1(x, \omega^{-1}(x)) = G'(0)^t \big(f'(x), -\frac{\kappa}{B_{22}(\omega^{-1}(x))}\big)$$

$$= \frac{2Q_s}{h^2+2h}{}^t\big(f'(x), -\frac{\kappa}{B_{22}(\omega^{-1}(x))}\big)$$

$$\begin{aligned}\nabla v_2(x,\omega^{-1}(x)) &= K'(0)^t\big(f'(x), -\frac{\kappa}{B_{22}(\omega^{-1}(x))}\big)\\ &= \frac{Q_f}{\log(1+d)}{}^t\big(f'(x), -\frac{\kappa}{B_{22}(\omega^{-1}(x))}\big)\end{aligned}$$

$$\begin{aligned}\nu(x,\omega^{-1}(x)) &= \frac{1}{\sqrt{1+(\omega^{-1}of)'^2(x)}}{}^t\big(-(\omega^{-1}of)'(x), 1\big)\\ &= \frac{1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}{}^t\big(-f'(x), \frac{\kappa}{B_{22}(\omega^{-1}(x))}\big).\end{aligned}$$

Then

$$\begin{aligned}J_1 &= B(z)(\nabla v_1-\nabla v_2).\nu + (\gamma_s-\gamma_f)\nu_x\\ &= \frac{-1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}\Big(\frac{2Q_s}{h^2+2h}-\frac{Q_f}{\log(1+d)}\Big)\\ &\times B(z)^t\Big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}(x))}\Big)^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}(x))}\big)\\ &- \frac{f'(x)}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}(\gamma_s-\gamma_f)\\ &= \frac{-1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}\Big(\frac{Q_s}{h^2+2h}-\frac{Q_f}{\log(1+d)}\Big)\\ &\times B(z)^t\Big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}(x))}\Big)^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}(x))}\big)\\ &- \frac{1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}\Big(\frac{Q_s}{h^2+2h}B(z)^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}of(x))}\big)\\ &\times {}^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}of(x))}\big) + (\gamma_s-\gamma_f)f'(x)\Big).\end{aligned}$$

Using (4.7), we obtain

$$\begin{aligned}&J_2\\ &= -\frac{1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}\Big(\frac{Q_s}{h^2+2h}B(z)^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}of(x))}\big)\\ &\times {}^t\big(f'(x), \frac{-\kappa}{B_{22}(\omega^{-1}of(x))}\big) + (\gamma_s-\gamma_f)f'(x)\Big)\\ &\le \sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}\Big(\frac{-mQ_s}{h^2+2h}+(\gamma_s-\gamma_f)\frac{-f'(x)}{f'^2(x)+\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2}\Big).\end{aligned}$$

Note that

$$F(x)=\frac{-f'(x)}{f'^2(x)+\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2}\le -\frac{M^2}{\kappa^2}f'(x)$$

and then

- If $x\in(x_R,f_d^{-1}(-h))$ we have $f'(x)=-\alpha$ and $F(x)\le\frac{\alpha M^2}{\kappa^2}$.
- If $x\in(-b,x_R)$

$$\begin{aligned}F(x)&\le -\frac{M^2}{\kappa^2}f'(x)=\frac{M^2}{\kappa^2}\frac{x+b}{\sqrt{(R^2-(x+b)^2)}}\\ &\le\frac{M^2}{\kappa^2}\frac{x_R+b}{R}=\frac{M^2}{\kappa^2}\frac{\alpha}{\sqrt{1+\alpha^2}}\le\alpha\frac{M^2}{\kappa^2}.\end{aligned}$$

So if $\alpha<\frac{mQ_s}{h^2+2h}\frac{1}{\gamma_s-\gamma_f}\frac{\kappa^2}{M^2}=\alpha_3$, we get $J_2\le 0$ and then

$$\begin{aligned}J_1\le&\frac{1}{\sqrt{\big(\frac{\kappa}{B_{22}(\omega^{-1}of(x))}\big)^2+f'^2(x)}}\Big(\frac{Q_f}{\log(1+d)}-\frac{Q_s}{h^2+2h}\Big)\\ &\times B(z)^t\Big(f'(x),\frac{-\kappa}{B_{22}(\omega^{-1}(x))}\Big)^t\Big(f'(x),\frac{-\kappa}{B_{22}(\omega^{-1}(x))}\Big).\end{aligned}$$

Note that

$$\frac{Q_f}{\log(1+d)}<\frac{Q_s}{h^2+2h}\Leftrightarrow R>\frac{e^{\frac{Q_f}{Q_s}(h^2+2h)}-1+a\alpha}{\sqrt{1+\alpha^2}\big(\sqrt{1+\alpha^2}-1\big)}.$$

Thus if $\alpha\in(0,\alpha_*=\min(\alpha_1,\alpha_2,\alpha_3))$ and

$$R\ge R_*(\alpha)=\max\Big(\frac{a}{\alpha},\frac{2M^2}{\kappa^2}(h+1),\frac{e^{\frac{Q_f}{Q_s}(h^2+2h)}-1+a\alpha}{\sqrt{1+\alpha^2}\big(\sqrt{1+\alpha^2}-1\big)}\Big),$$

we get $I(\xi)\le 0$ for all $\xi\in\mathcal{D}(D)$, $\xi\ge 0$. Therefore (4.10) holds.

□

Lemma 8. *For* $R>\max\big(\frac{a}{\alpha},\frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\big)$,

$$v\le\psi\quad\text{on }\partial D\quad\text{and}\quad\lim_{x\to-\infty}v(x,z)\le\lim_{x\to-\infty}\psi(x,z).\tag{4.11}$$

Proof. Indeed we have

- $v(x,-h)=G(f(x)-\omega(-h))=G(f(x)+h)\le G(h)=Q_s=\psi(x,-h)$ for $x<f^{-1}(-h)$.
- $v(x,-h)=-K(\omega(-h)-f(x))=-K(-h-f(x))\le 0<Q_s=\psi(x,-h)$ for $x\in(f^{-1}(-h),f_d^{-1}(-h))$.
- $v(x,0)=G(f(x)-\omega(0))=G(0)=0=\psi(x,0)$ for $x\le -b$.
- $v(x,0)=-K(\omega(0)-f(x))=-K(-f(x))\le 0=\psi(x,0)$ for $x\in(-b,x_d)$, where x_d is defined by $f_d(x_d)=0$ and is to be chosen such that $x_d=0$. In fact $f_d(x_d)=0$ is equivalent to

$$\begin{aligned}x_d&=(z_R+d)\frac{R}{a+b}+x_R\\ &=\frac{R}{a+b}\Big(\frac{R}{a+b}\frac{R}{\sqrt{1+\big(\frac{R}{a+b}\big)^2}}-R+d+\frac{a+b}{\sqrt{1+\big(\frac{R}{a+b}\big)^2}}\end{aligned}$$

$$-\frac{(a+b)^2}{R}+\frac{a(a+b)}{R}\Big)=\frac{1}{\alpha}\big(R\sqrt{1+\alpha^2}-(1+\alpha^2)R+d+a\alpha\big).$$

Then we can choose x_d=0 if $d=-a\alpha+R\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)>0$ which holds if

$$R>\frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}.$$

- $v(x,\omega^{-1}(x))=-K(f_d(x)-f(x))=-K(d)=-Q_f\le\psi(x,\omega^{-1}(x))$
- $\lim_{x\to-\infty}v(x,z)=\lim_{x\to-\infty}v_1(x,z)=\lim_{x\to-\infty}G(f(x)-\omega(z))$ $=G(-\omega(z))\le\frac{Q_s}{h}\omega(z)=v_{-\infty}(z)=\lim_{x\to-\infty}\psi(x,z)$.

This (4.11) holds. □

Lemma 9. *For all* $\alpha\in(0,\alpha_*)$, $R>\max\big(R_*(\alpha),\frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\big)$, *one has*

$$\int_D(B(z)\nabla(v-\psi_0)+(\theta-\gamma_0)e_x)\nabla\zeta=0\quad\forall\zeta\in\mathcal{D}(\mathbb{R}^2)\tag{4.12}$$

where $\psi_0=\min(\psi,v)$ *and* $\gamma_0=\min(\gamma,\theta)$.

Proof. Let $\zeta\in\mathcal{D}(\mathbb{R}^2)$ and consider for $i=1,2$ the function ζ_i defined as in the proof of Lemma 1. Then for $\epsilon>0$, $\min\big(\zeta_i,\frac{v-\psi_0}{\epsilon}\big)\in H_0^1(D)$ by (4.11) and is nonnegative with compact support. By (4.10) and $(P)i$), we get

$$\begin{aligned}&\int_{D\cap[v-\psi_0\ge\epsilon\zeta_i]}B(z)\nabla(v-\psi_0)\nabla\zeta_i+\int_D(\theta-\gamma_0)\zeta_{ix}\\&\le\int_D(\theta-\gamma_0)\big(\zeta_i-\frac{v-\psi_0}{\epsilon}\big)_x^+\\&\le(\gamma_s-\gamma_f)\int_{I_0}\big(\zeta_i+\frac{\psi(f^{-1}o\omega(z),z)}{\epsilon}\big)^+-\big(\zeta_i-\frac{v(g(z),z)}{\epsilon}\big)^+\end{aligned}$$

where $I_0=\{z\in(-h^*,0)/\,g(z)<f^{-1}o\omega(z)\}$. Letting $\epsilon\to0$, we get

$$\int_D\big(B(z)\nabla(v-\psi_0)+(\theta-\gamma_0)e_x\big)\nabla\zeta_i\le0.$$

To obtain (4.12), we argue as in the proof of Lemma 1. □

Proof of Theorem 4.2. Let $u=(v-\psi_0)\chi(D)$ and $\zeta\in\mathcal{D}([z>\omega^{-1}(x)]\cap\Omega)$. Note that $\theta=\gamma_f=\gamma_0$ in D_2. We deduce from (4.12)

$$\int_{[z>\omega^{-1}(x)]\cap D}B(z)\nabla u.\nabla\zeta=0.$$

Since u vanishes on $[z=\omega^{-1}of_d(x)]$, then $u\in H^1_{\rm loc}(\Omega)$ and

$$\int_{[z>\omega^{-1}(x)]\cap\Omega}B(z)\nabla u.\nabla\zeta=0.$$

Moreover $u\ge0$ and $u=0$ in $\Omega\setminus D$. So by the strong maximum principle we get $u=0$ in $[z>\omega^{-1}(x)]\cap\Omega$ which leads to $v\le\psi$ in D_2. In particular we obtain $\psi(f^{-1}o\omega(z),z)\ge v(f^{-1}o\omega(z),z)=v(x,\omega^{-1}(x))=0$. So $g(z)\ge f^{-1}o\omega(z)\quad\forall z\in(-h^*,0)$ and then

$$g(0-)\ge f^{-1}o\omega(0)=f^{-1}(0)=-b>-\infty.$$

□

4.3. **The Nonlinear Homogeneous Case.** Assume that $\mathcal{B}(z,\xi)=|\xi|^{q-2}\xi$. Then we have the following theorem.

Theorem 4.3. *$g(0-)=\lim_{z\to-\infty} g(z)>-\infty$.*

We follow the proof of Theorem 4.2 and we use the same notation with $\omega(z)=z$. We will need the following three Lemmas.

Lemma 10. *We have*

$$\Delta_q v_1>0 \quad in \quad D_1, \quad \forall R>R_1=\frac{h+1}{q-1}(|q-2|+q-1). \tag{4.13}$$

Proof. Indeed we have

$$\begin{aligned}
\Delta_q v_1 &= \operatorname{div}(|\nabla v_1|^{q-2}\nabla v_1)\\
&= \frac{\partial}{\partial x}\big(|\nabla v_1|^{q-2}\frac{\partial v_1}{\partial x}\big)+\frac{\partial}{\partial z}\big(|\nabla v_1|^{q-2}\frac{\partial v_1}{\partial z}\big)\\
&= (G'(f(x)-z))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\\
&\times\Big((q-1)G''(f(x)-z)(1+f'^2(x))+G'(f(x)-z)f''(x)\frac{1+(q-1)f'^2(x)}{1+f'^2(x)}\Big).
\end{aligned}$$

We distinguish two cases:
- If $x<-b$ or $x>x_R$, we have $f''(x)=0$ and then $\Delta_q v_1>0$ in D_1.
- If $-b<x\le x_R$, we have

$$\begin{aligned}
\Delta_q v_1 &= (2\lambda(1+f(x)-z))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(2\lambda\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\Big)\\
&\quad\times\Big((q-1)(R+z-1)+(q-2)(f(x)-z+1)\frac{(R^2-(x+b)^2)}{R^2}\Big)\\
&\ge (2\lambda(1+f(x)-z))^{q-2}\Big(1+f'^2(x)\Big)^{\frac{q-2}{2}}\Big(2\lambda\frac{(R^2-(x+b)^2)^{3/2}}{R^2}\Big)\\
&\quad\times\Big((q-1)(R-h-1)-|q-2|(h+1)\Big)
\end{aligned}$$

since $\big|(q-2)(f(x)-z+1)\frac{(R^2-(x+b)^2)}{R^2}\big|\le|q-2|(h+1)$. So for $R>R_1$ we get $\Delta_q v_1>0$ in D_1 $\square$

Lemma 11. *For all*

$$R>R_2(\alpha)=\max\Big(\frac{a}{\alpha},\frac{(q-1)+|q-2|(h+1)}{(q-1)\Big(\frac{2}{\sqrt{1+\alpha^2}}-1\Big)}\Big),$$

we have

$$\Delta_q v_2>0 \quad in \quad D_2, \quad \forall\alpha\in(0,1). \tag{4.14}$$

Proof. We have

$$\begin{aligned}
\Delta_q v_2 &= (K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(-(q-1)K''(z-f(x))(1+f'^2(x))\\
&\quad+K'(z-f(x))f''(x)\big(q-1-\frac{q-2}{1+f'^2(x)}\big)\Big).
\end{aligned}$$

We distinguish two cases:
- For $x_R<x<f_d^{-1}(-h)$, we have $f''(x)=0$ and then $\Delta_q v_2>0$ in D_2.

• If $-b < x \le x_R$, we have

$$
\begin{aligned}
&\Delta_q v_2 \\
&= \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\Big(\frac{q-1}{(1+z-f(x))^2}\frac{R^2}{R^2-(x+b)^2} \\
&\quad + \frac{1}{1+z-f(x)}\Big(\frac{-R^2}{(R^2-(x+b)^2)^{3/2}}\Big)\big((q-1)-(q-2)\frac{(R^2-(x+b)^2)}{R^2}\big)\Big) \\
&= \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\frac{R^2}{(R^2-(x+b)^2)^{3/2}}\frac{1}{(1+z-f(x))^2} \\
&\quad \times\Big((q-1)(2\sqrt{R^2-(x+b)^2}-R)-(q-1)(1+z) \\
&\quad + (q-2)(1+z-f(x))\frac{(R^2-(x+b)^2)}{R^2}\Big) \\
&\ge \mu(K'(z-f(x)))^{q-2}(1+f'^2(x))^{\frac{q-2}{2}}\frac{R^2}{(R^2-(x+b)^2)^{3/2}}\frac{1}{(1+z-f(x))^2} \\
&\quad \times\Big((q-1)\Big(\frac{2}{\sqrt{1+\alpha^2}}-1\Big)R-(q-1)-|q-2|(h+1)\Big)
\end{aligned}
$$

since $2\sqrt{R^2-(x+b)^2}-R \ge 2\sqrt{R^2-(x_R+b)^2}-R = R\Big(\frac{2}{\sqrt{1+\alpha^2}}-1\Big)$.

If we choose $\alpha \in (0,1)$ such that $\frac{2}{\sqrt{1+\alpha^2}}-1>0$ and $R \ge \max(\frac{a}{\alpha}, R_2(\alpha))$, we get (4.14). □

Lemma 12. *There exists $\alpha_* > 0$ and $R_* > 0$ such that*

$$
\forall \alpha \in (0,\alpha_*), \quad \forall R > R_*, \ \text{one has } \Delta_q v + \theta_x \ge 0 \ \text{in } \mathcal{D}'(D). \tag{4.15}
$$

Proof. Let $\xi \in \mathcal{D}(D)$, $\xi \ge 0$. For $\alpha \in (0,1)$ and $R \ge \max\big(R_1, R_2(\alpha), \frac{a}{\alpha}\big)$, we have by (4.14)-(4.15)

$$
\begin{aligned}
I(\xi) &= \int_D (|\nabla v|^{q-2}\nabla v + \theta e_x)\nabla\xi \\
&= \int_{D_1}(|\nabla v_1|^{q-2}\nabla v_1 + \gamma_s e_x)\nabla\xi + \int_{D_2}(|\nabla v_2|^{q-2}\nabla v_2 + \gamma_f e_x)\nabla\xi \\
&= \langle -\Delta_q v_1, \xi > + < -\Delta_q v_2, \xi\rangle \\
&\quad + \int_{[z=f(x)]\cap D}\big((|\nabla v_1|^{q-2}\nabla v_1 - |\nabla v_2|^{q-2}\nabla v_2).\nu + (\gamma_s-\gamma_f)\nu_x\big)\xi \\
&\le \int_{[z=f(x)]\cap D}\big((|\nabla v_1|^{q-2}\nabla v_1 - |\nabla v_2|^{q-2}\nabla v_2).\nu + (\gamma_s-\gamma_f)\nu_x\big)\xi \\
&= \int_{[z=f(x)]\cap D} J_1\xi.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\nabla v_1(x,f(x)) &= \frac{2Q_s}{h^2+2h}(f'(x),-1) \\
|\nabla v_1|^{q-2}\nabla v_1 &= \Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1}\Big(1+f'^2(x)\Big)^{\frac{q-2}{2}}(f'(x),-1) \\
\nabla v_2(x,f(x)) &= \frac{Q_f}{\log(1+d)}(f'(x),-1)
\end{aligned}
$$

$$|\nabla v_1|^{q-2}\nabla v_1 = \Big(\frac{Q_f}{\log(1+d)}\Big)^{q-1}\Big(1+f'^2(x)\Big)^{\frac{q-2}{2}}(f'(x),-1)$$
$$\nu(x,f(x)) = \frac{1}{\sqrt{1+f'^2(x)}}(-f'(x),1).$$

Then

$$\begin{aligned}J_1 &= (1+f'^2(x))^{\frac{q-1}{2}}\Big(\big(\frac{Q_f}{\log(1+d)}\big)^{q-1} - \big(\frac{2Q_s}{h^2+2h}\big)^{q-1}\Big)\\ &\quad + (\gamma_s-\gamma_f)\frac{-f'(x)}{\sqrt{1+f'^2(x)}}\\ &= \Big(1+f'^2(x)\Big)^{\frac{q-1}{2}}\Big(\big(\frac{Q_f}{\log(1+d)}\big)^{q-1} - \frac{1}{2}\big(\frac{2Q_s}{h^2+2h}\big)^{q-1}\Big)\\ &\quad + \Big(1+f'^2(x)\Big)^{\frac{q-1}{2}}\Big((\gamma_s-\gamma_f)\frac{-f'(x)}{(1+f'^2(x))^{q/2}} - \frac{1}{2}\big(\frac{2Q_s}{h^2+h}\big)^{q-1}\Big).\end{aligned}$$

Moreover we have

$$\frac{-f'(x)}{(1+f'^2(x))^{q/2}} \le -f'(x) \le \alpha \quad \forall x\in(-b, f_d^{-1}(-h))$$

and for

$$R > R_3(\alpha) = \frac{e^{\frac{Q_f(h^2+2h)}{2^{\frac{q-2}{q-1}}Q_s}} - 1 + a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}$$

one has

$$\Big(\frac{Q_f}{\log(1+d)}\Big)^{q-1} < \frac{1}{2}\Big(\frac{2Q_s}{h^2+2h}\Big)^{q-1} = \Big(\frac{Q_s}{2^{\frac{1}{q-1}}(h^2+2h)}\Big)^{q-1}.$$

Thus if

$$\alpha\in\Big(0,\alpha_* = \min\big(1, \frac{1}{2(\gamma_s-\gamma_f)}\big(\frac{2Q_s}{h^2+2h}\big)^{q-1}\big)\Big)$$

and $R\ge R_* = \max\big(R_1, R_2(\alpha), R_3(\alpha), \frac{a}{\alpha}\big)$, we get $J_1\le 0$ and then $I(\xi)\le 0$ for all $\xi\in\mathcal{D}(D)$, $\xi\ge 0$. Hence (4.15) holds. □

Proof of Theorem 4.3. Clearly for $R > \max\big(\frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}, \frac{a}{\alpha}\big)$. Then we have (see Lemma 8)

$$v\le\psi \quad \text{on } \partial D \quad \text{and} \quad \lim_{x\to-\infty} v(x,z) \le \lim_{x\to-\infty}\psi(x,z).$$

Using (4.15) and arguing as in the proof of Lemma 1, we can establish that for all $\alpha\in(0,\alpha_*)$ and $R>\max\big(R_*, \frac{a\alpha}{\sqrt{1+\alpha^2}(\sqrt{1+\alpha^2}-1)}\big)$,

$$(\Delta_q v - \Delta_q\psi_0) + (\gamma-\theta_0)_x = 0 \quad \text{in} \quad \mathcal{D}'(D).$$

Assume that $D_1\cap[\psi<0]\ne\emptyset$ and note that we have $\gamma=\gamma_f=\gamma_0$ and $\theta=\gamma_s$ in $D_1\cap[\psi<0]$. We also have $\psi=\psi_0$ in $D_1\cap[\psi<0]$ since $\psi<0<v$. So we deduce that

$$\Delta_q v = \Delta_q\psi_0 - (\gamma-\theta_0)_x = \Delta_q\psi - (\gamma_f-\gamma_s)_x = 0 \quad \text{in } \mathcal{D}'(D_1\cap[\psi<0]).$$

This leads to a contradiction since $\Delta_q v = \Delta_q v_1 > 0$ in D_1. Therefore $D_1 \cap [\psi < 0] = \emptyset$ and $\psi \geq 0$ in D_1. In particular we have $g(z) \geq f^{-1}(z) \quad \forall z \in (-h^*, 0)$ and then $g(0-) \geq f^{-1}(0) = -b > -\infty$. $\square$

4.4. **The Linear Homogeneous Case.** Under the assumption $\mathcal{B}(z,\xi) = \xi$ we show that $g(0-) < 0$. This result was announced in [1] without an explicit proof. In that paper the authors indicated that this result can be obtained by using hodograph techniques to get a semi-explicit expression for $\bar{\psi}$ which is the unique harmonic function satisfying the same boundary and limit conditions as ψ. However an explicit proof was not given. That is why we propose here a proof of this result.

Theorem 4.4.

$$\lim_{z\to 0^-} g(z) = g(0-) \leq -\frac{h}{\pi}\log\Big(\frac{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}}-1}{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}}-e^{\frac{a\pi}{h}}}\Big) < 0. \tag{4.16}$$

Proof. Since we have $\Delta\psi \geq 0$ and $\Delta\bar{\psi} = 0$ in Ω, we get by taking into account the boundary conditions and the limit behavior at infinity that $\psi \leq \bar{\psi}$ in Ω. Moreover if we denote by $\bar{g}$ the function defined by $\bar{\psi}(\bar{g}(z), z) = 0 \ \forall z \in (-h^*, 0)$, we get $g(z) \leq \bar{g}(z)$ for all $z \in (-h^*, 0)$ which leads to

$$g(0-) \leq \bar{g}(0-). \tag{4.17}$$

Next, we will prove that

$$\bar{g}(0-) = -\frac{h}{\pi}\log\left(\frac{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}}-1}{e^{\frac{a\pi(Q_s+Q_f)}{hQ_f}}-e^{\frac{a\pi}{h}}}\right) < 0. \tag{4.18}$$

For simplicity we introduce the function

$$\psi_0(x,z) = \bar{\psi}(hx, h(z-1)) \quad \text{for } (x,z) \in \mathbb{R}\times[0,1]$$

which is harmonic in $\mathbb{R}\times(0,1)$ and satisfies $\psi_0(x,0) = Q_s$, $\psi_0(x,1) = \phi_0(x)$ and $\lim_{x\to\pm\infty}\psi_0(x,z) = v_{\pm\infty}(h(z-1))$. In [12], one can find that

$$\begin{aligned}
\psi_0(x,z) &= \operatorname{Re}\left(\frac{i}{2}\int_{-\infty}^{+\infty} Q_s \coth\frac{\pi(t-x-iz)}{2} + \phi_0(ht)th\frac{\pi(t-x-iz)}{2}dt\right)\\
&= -\frac{Q_s}{2}\int_{-\infty}^{+\infty}\frac{ch\pi(t-x)+\cos\pi z}{sh^2\pi(t-x)+\sin^2\pi z}\sin\pi z dt\\
&\quad+\frac{1}{2}\int_{-\infty}^{+\infty}\phi_0(ht)\frac{\sin\pi z}{ch\pi(t-x)+\cos\pi z}dt\\
&= -\frac{Q_s}{2}\int_{-\infty}^{+\infty}\frac{ch\pi s+\cos\pi z}{sh^2\pi s+\sin^2\pi z}\sin\pi z ds + \frac{1}{2}I(x,z)
\end{aligned}$$

where

$$\begin{aligned}
I(x,z) &= \int_0^{\frac{a}{h}} -\frac{Q_f}{a}ht\frac{\sin\pi z}{ch\pi(t-x)+\cos\pi z}dt + \int_{\frac{a}{h}}^{+\infty} -Q_f\frac{\sin\pi z}{ch\pi(t-x)+\cos\pi z}dt\\
&= -\frac{Q_f}{a}h\int_{-x}^{\frac{a}{h}-x} t\frac{\sin\pi z}{ch\pi t+\cos\pi z}dt - \frac{Q_f}{a}hx\int_{-x}^{\frac{a}{h}-x}\frac{\sin\pi z}{ch\pi t+\cos\pi z}dt\\
&\quad - Q_f\int_{\frac{a}{h}-x}^{+\infty}\frac{\sin\pi z}{ch\pi t+\cos\pi z}dt.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}\frac{\partial\psi_0}{\partial x} &= \frac{1}{2}\frac{\partial I}{\partial x} = -\frac{Q_f}{2a}h\int_{-x}^{\frac{a}{h}-x}\frac{\sin\pi z}{ch\pi t+\cos\pi z}dt\\ &= -\frac{hQ_f}{a\pi}\Big[\operatorname{Arctan}\Big(\frac{e^{(\frac{a}{h}-x)\pi}+\cos\pi z}{\sin\pi z}\Big)-\operatorname{Arctan}\Big(\frac{e^{-\pi x}+\cos\pi z}{\sin\pi z}\Big)\Big]\end{aligned}$$

and for $(x,z)\in\Omega$

$$\begin{aligned}&\bar\psi(x,z) = v_{-\infty}(z)\\ &\quad -\frac{hQ_f}{a\pi}\int_{-\infty}^{\frac{x}{h}}\Big(\operatorname{Arctan}\Big(\frac{e^{-\pi t}-\cos\frac{\pi z}{h}}{\sin\frac{\pi z}{h}}\Big)-\operatorname{Arctan}\Big(\frac{e^{\left(\frac{a}{h}-t\right)\pi}-\cos\frac{\pi z}{h}}{\sin\frac{\pi z}{h}}\Big)\Big)dt.\end{aligned}$$

Using this formula we give an asymptotic behavior of $\bar\psi$ near $z=0$. Note that for $x\le 0$ and $t<\frac{x}{h}$, we have $e^{-\pi t}>1$ and $e^{\left(\frac{a}{h}-t\right)\pi}>1$. Therefore

$$\begin{aligned}\operatorname{Arctan}\Big(\frac{e^{-\pi t}-\cos\frac{\pi z}{h}}{\sin\frac{\pi z}{h}}\Big) &= -\frac{\pi}{2}+\frac{\pi}{h}\frac{1}{1-e^{-\pi t}}z+\frac{z^2}{2}p(z,t)\\ \operatorname{Arctan}\Big(\frac{e^{\left(\frac{a}{h}-t\right)\pi}-\cos\frac{\pi z}{h}}{\sin\frac{\pi z}{h}}\Big) &= -\frac{\pi}{2}+\frac{\pi}{h}\frac{1}{1-e^{\left(\frac{a}{h}-t\right)\pi}}z+\frac{z^2}{2}q(z,t)\end{aligned}$$

with $p(z,t),q(z,t)\approx e^{\pi t}$ as $t\to-\infty$. It follows that

$$\begin{aligned}0=\bar\psi(\bar g(z),z) &= -\frac{Q_s}{h}z-\frac{Q_f}{a}z\int_{-\infty}^{\bar g(z)}\Big(\frac{1}{1-e^{-\pi t}}-\frac{1}{1-e^{\left(\frac{a}{h}-t\right)\pi}}\Big)dt\\ &\quad -\frac{hQ_f}{2\pi a}z^2\int_{-\infty}^{\bar g(z)}(p(z,t)-q(z,t))dt-\frac{Q_s}{h}\\ &\quad -\frac{Q_f}{a}\int_{-\infty}^{\bar g(z)}\left(\frac{1}{1-e^{-\pi t}}-\frac{1}{1-e^{\left(\frac{a}{h}-t\right)\pi}}\right)dt\\ &= z\int_{-\infty}^{\bar g(z)}(p(z,t)-q(z,t))dt.\end{aligned}$$

Letting $z\to 0$, we obtain

$$\frac{Q_f}{a}\int_{-\infty}^{\bar g(0)}\big(\frac{1}{1-e^{-\pi t}}-\frac{1}{1-e^{\left(\frac{a}{h}-t\right)\pi}}\big)dt=\frac{Q_s}{h}$$

and by evaluating the last integral we get (4.17). Taking into account (4.17) and (4.18), we obtain (4.16). □

Remark 3. As a consequence of the above theorem, we have also $g(0-)<0$ when

$$\mathcal{B}(z,\xi)=\left(\frac{C^2}{B(z)}\xi_1,B(z)\xi_2\right)\quad\text{with } C=\frac{-h}{\int_{-h}^0\frac{ds}{B(s)}}.$$

Let $v(x,z)=\bar\psi(x,\omega(z))$ with $\omega(z)=C\int_z^0\frac{ds}{B(s)}$. We remark that

$$\operatorname{div}(\mathcal{B}(z,\nabla v))=\frac{C^2}{B(z)}\Delta\bar\psi(x,\omega(z))=0.$$

Moreover $v = \psi$ on $\partial\Omega$ and $\lim_{x\to\pm\infty} v = \lim_{x\to\pm\infty} \psi$. Then we deduce that $\psi \le v$ in Ω which leads to

$$\psi(\bar{g}(\omega(z)), z) \le v(\bar{g}(\omega(z)), z) = \bar{\psi}(\bar{g}(\omega(z)), \omega(z)) = 0 \quad \forall z \in (-h^*, 0).$$

Therefore $g(z) \le \bar{g}(\omega(z))$ for all $z \in (-h^*, 0)$ and then $g(0-) \le \bar{g}(0-) < 0$.

Appendix

This section is devoted to some technical Lemmas.

Lemma 13 (Strong maximum principle). *Let Ω be a domain of $\mathbb{R}^2$ and let $\mathcal{B}$ defined on $\Omega \times \mathbb{R}^2$ by $\mathcal{B}(X,\xi) = |B(X)\xi|^{q-2}B^2(X)\xi$, where $B(X)$ is a locally Lipschitz continuous symmetric and strictly elliptic matrix. Assume that u_1, $u_2 \in W^{1,q}_{\rm loc}(\Omega)$ are such that*

$$\operatorname{div}(\mathcal{B}(X,\nabla u_1)) = \operatorname{div}(\mathcal{B}(X,\nabla u_2)) = 0 \quad in\ \mathcal{D}'(\Omega), \tag{4.19}$$

$$u_1 \le u_2 \quad in\ \Omega. \tag{4.20}$$

Then we have either $u_1 = u_2$ in Ω or $u_1 < u_2$ in Ω.

For the proof of this lemma see [8].

Lemma 14. *Under the same hypothesis of Lemma 13 we assume that there exists $\Gamma_0 \subset \partial\Omega$ of class $C^{1,\alpha}$ such that*

$$u_1 = u_2 \quad on\ \Gamma_0, \quad u_1, u_2 \in C^1(\Omega \cup \Gamma_0), \tag{4.21}$$

$$\mathcal{B}(X,\nabla u_1).\nu = \mathcal{B}(X,\nabla u_2).\nu \quad on\ \Gamma_0, \tag{4.22}$$

$$\nabla u_1(X) \ne 0 \quad \forall X \in \Gamma_0 \quad or \quad \nabla u_2(X) \ne 0 \quad \forall X \in \Gamma_0. \tag{4.23}$$

Then $u_1 = u_2$ in Ω.

Proof. We first prove that $u_1 = u_2$ in a small open set near Γ_0. Then we conclude by Lemma 13. Assume for example that we have $\nabla u_1(X) \ne 0$ for all $X \in \Gamma_0$. Since $u_1 \in C^1(\Omega \cup \Gamma_0)$, there exists a small ball $B(X_0, \epsilon_0)$ centered at some point X_0 of Γ_0 such that

$$\nabla u_1(X) \ne 0 \quad \forall X \in B(X_0, \epsilon_0) \cap (\Omega \cup \Gamma_0)$$

and then there exists two positive constants c_0 and c_1 such that

$$c_0 \le |\nabla u_1(X)| \le c_1 \quad \forall X \in K_1 = \overline{B(X_0, \epsilon_0) \cap (\Omega \cup \Gamma_0)}. \tag{4.24}$$

Let $u = u_1 - u_2$ and $u_t = tu_2 + (1-t)u_1$ for any $t \in [0,1]$. Then we deduce from (4.19) and (4.22)

$$\int_{B(X_0,\epsilon_0)\cap\Omega} (\mathcal{B}(X,\nabla u_2) - \mathcal{B}(X,\nabla u_1)).\nabla\zeta = 0 \quad \forall \zeta \in \mathcal{D}(B(X_0,\epsilon_0)). \tag{4.25}$$

Remark that

$$\mathcal{B}(X,\nabla u_2) - \mathcal{B}(X,\nabla u_1) = \int_0^1 \frac{d}{dt}(\mathcal{B}(X,\nabla u_t))dt,$$

then (4.25) becomes

$$\int_{B(X_0,\epsilon_0)\cap\Omega} \mathcal{C}(X)\nabla u.\nabla\zeta = 0 \quad \forall \zeta \in \mathcal{D}(B(X_0,\epsilon_0)) \tag{4.26}$$

where

$$\begin{aligned}\mathcal{C}(X) = \int_0^1 &\langle \mathcal{B}(X,\nabla u_t),\nabla u_t\rangle^{\frac{q-2}{2}} \\ &\times \left(\mathcal{B}(X,\nabla u_t) + (q-2)\frac{\mathcal{B}(X,\nabla u_t).^t(\mathcal{B}(X,\nabla u_t))}{\langle \mathcal{B}(X,\nabla u_t),\nabla u_t\rangle}\right) dt\end{aligned}$$

from which we deduce

$$c_2|Y|^2\lambda(X) \le \mathcal{C}(X).Y.Y \le c_3|Y|^2\lambda(X) \quad \forall (X,Y) \in (B(X_0,\epsilon_0)\cap\Omega)\times\mathbb{R}^2 \tag{4.27}$$

where c_2, c_3 are two positive constants and $\lambda(X) = \int_0^1 |\nabla u_t|^{q-2}dt$. Using (4.24) and arguing as in [11], we can prove that $\lambda(X)$ is bounded from both sides by two positive constants λ_0, λ_1 in $B(X_0,\epsilon_0')\cap\Omega$ for some $\epsilon_0' \in (0,\epsilon_0)$. So $\mathcal{C}(X)$ is strictly elliptic in $B(X_0,\epsilon_0')\cap\Omega$. By (4.21) we can extend u by 0 to $B(X_0,\epsilon_0')\setminus\Omega$ so that $u \in W^{1,q}(B(X_0,\epsilon_0'))$. One can also extend $\mathcal{C}(X)$ by $c_2\lambda_0 I_2$ to $B(X_0,\epsilon_0')\setminus\Omega$ so that it remains strictly elliptic in $B(X_0,\epsilon_0')$. Then from (4.26),

$$\int_{B(X_0,\epsilon_0')} \mathcal{C}(X)\nabla u.\nabla\zeta = 0 \quad \forall \zeta \in \mathcal{D}(B(X_0,\epsilon_0')). \tag{4.28}$$

Now since $u \ge 0$ and $u = 0$ in $B(X_0,\epsilon_0')\setminus\Omega$, we deduce from (4.28) and the strong maximum principle for linear elliptic equations that $u = 0$ in $B(X_0,\epsilon_0')$ which means that $u_1 = u_2$ in $B(X_0,\epsilon_0')\cap\Omega$. □

Lemma 15 (Non-oscillation Lemma). *Let $z_0 \in (-h^*,0)$, $x_0 \in \mathbb{R}$, $r > 0$ and assume that $S_{eg} = \{(x,z_0)/\,|x-x_0| \le r\} \subset \Gamma$, then we cannot have*

$$\forall (x,z) \in B_r(x_0,z_0)\setminus S_{eg} \quad \psi(x,z) \ne 0$$

where $B_r(x_0,z_0)$ is the open ball of center (x_0,z_0) and radius r.

The proof follows as in [4, Lemma 5.1] and uses Lemma 13.

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