Electronic Journal of Differential Equations, Vol. 2003(2003), No. 50, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

EXISTENCE AND UNIQUENESS OF THE SOLUTION TO A 3D THERMOVISCOELASTIC SYSTEM

ELENA BONETTI & GIOVANNA BONFANTI

Abstract. This paper presents results on existence and uniqueness of solutions to a three-dimensional thermoviscoelastic system. The constitutive relations of the model are recovered by a free energy functional and a pseudo-potential of dissipation. Using a fixed point argument, combined with an a priori estimates-passage to the limit technique, we prove a local existence result for a related initial and boundary values problem. Hence, uniqueness of the solution is proved on the whole time interval, as well as positivity of the absolute temperature.

1. Introduction to the model

In this paper we deal with a three-dimensional model for thermoviscoelastic systems, which is derived by two energy functionals: the free energy and the pseudo-potential of dissipation. In particular, we refer to a modelling approach by Frémond which is fully detailed and justified in [8], where it is applied to different thermo-mechanics phenomena. Thus, we first introduce the state variables of the model, which are the absolute temperature θ and the linearized symmetric strain tensor $\varepsilon(\mathbf{u})$ ($\mathbf{u}$ is the vector of small displacements), specified by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i), \quad i,j = 1,2,3, \quad \mathbf{u} = (u_1, u_2, u_3). \tag{1.1}$$

Hence, the thermomechanical equilibrium of the system is described by a free energy functional Ψ, which depends on the non-dissipative variables. We make precise this functional as follows

$$\Psi(\theta, \varepsilon(\mathbf{u})) = -c_s\theta\log\theta + \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) + \alpha(\theta)\operatorname{tr}\varepsilon(\mathbf{u}), \tag{1.2}$$

where K denotes the elastic tensor, $c_s > 0$ the heat capacity of the system, and $\alpha(\theta)$ a thermal expansion coefficient acting only on the trace of the strain tensor. More precisely, we let

$$\alpha(\theta) = \alpha\theta, \quad \alpha \in \mathbb{R}. \tag{1.3}$$

Note that here and in the sequel, in regard of simplicity, we do not use a specific notation for the tensorial product, while in general by $\cdot$ we denote the scalar product

2000 *Mathematics Subject Classification.* 74D10, 35K60, 74A15.
Key words and phrases. 3D thermoviscoelastic system, thermomechanical modelling, nonlinear PDE's system, existence and uniqueness results.

Submitted December 9, 2002. Published April 29, 2003.

in $\mathbb{R}^n$. As usual in elasticity theory, we assume the material to be homogeneous and isotropic and we let

$$K\varepsilon(\mathbf{u}) = \lambda \operatorname{tr} \varepsilon(\mathbf{u})\mathbf{1} + 2\mu\varepsilon(\mathbf{u}), \tag{1.4}$$

where λ and μ stand for the Lamé constants and $\mathbf{1}$ for the identity matrix. Hence, we include dissipation in the model by following the approach proposed by Moreau (cf. [8] and references therein) and introduce a pseudo-potential of dissipation Φ depending on the dissipative variables $\nabla\theta$ and $\varepsilon(\mathbf{u}_t)$, related to the heat flux and the evolution of deformations, respectively. In particular, we set

$$\Phi(\nabla\theta, \varepsilon(\mathbf{u}_t)) = \frac{k_0}{2\theta}|\nabla\theta|^2 + \frac{1}{2}\varepsilon(\mathbf{u}_t)B\varepsilon(\mathbf{u}_t), \tag{1.5}$$

where $k_0 > 0$ and B is a positive definite (symmetric) matrix. Now, letting the system be located in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, during a finite time interval $[0,T]$, $T > 0$, we introduce the universal balance laws of thermomechanics, that are the energy balance

$$e_t + \operatorname{div} \mathbf{q} = r + \sigma\varepsilon(\mathbf{u}_t) \quad \text{in } \Omega \times (0,T), \tag{1.6}$$

and the momentum balance (in which we account for macroscopic accelerations)

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{G} \quad \text{in } \Omega \times (0,T), \tag{1.7}$$

whose ingredients will be specified in a moment. In advance, let us point out that by the subscript t we denote the partial derivative operator $\frac{\partial}{\partial t}$. In (1.6) e is the internal energy of the system, $\mathbf{q}$ the heat flux, r stands for an external heat source, and the term $\sigma\varepsilon(\mathbf{u}_t)$, σ being the stress tensor, accounts for mechanically induced heat sources; $\mathbf{G}$ in (1.7) denotes a volume force applied to the structure from the exterior. Next, (1.6) and (1.7) are complemented by suitable boundary conditions. Denoting by $\mathbf{n}$ the outward unit normal vector to the boundary $\Gamma := \partial\Omega$, we let

$$-\mathbf{q} \cdot \mathbf{n} = h \quad \text{on } \Gamma \times (0,T), \tag{1.8}$$

namely we assume that the heat flux through the boundary is known. Then, we prescribe homogeneous Dirichlet boundary conditions on the displacement $\mathbf{u}$ and the velocity $\mathbf{u}_t$.

$$\mathbf{u} = \mathbf{u}_t = \mathbf{0} \quad \text{on } \Gamma \times (0,T). \tag{1.9}$$

Now, we make precise the constitutive relations for dissipative and non dissipative quantities in the model, in terms of the functionals Ψ and Φ (cf. (1.2) and (1.5)). We have, as usual, the internal energy e related to the entropy

$$s = -\frac{\partial \Psi}{\partial \theta}, \tag{1.10}$$

by the Helmholtz relation $e = \Psi + s\theta$. Hence, concerning the stress tensor σ, we distinguish a non-dissipative contribution σ^{nd},

$$\sigma^{nd} = \frac{\partial \Psi}{\partial \varepsilon(\mathbf{u})}, \tag{1.11}$$

and a dissipative one

$$\sigma^{d} = \frac{\partial \Phi}{\partial \varepsilon(\mathbf{u}_t)}, \tag{1.12}$$

and let $\sigma = \sigma^{nd} + \sigma^d$. Finally, by following the approach by Frémond, we derive the usual Fourier heat flux law by the pseudo-potential of dissipation. To this aim, let us introduce the dissipative vector

$$\mathbf{Q}^d = -\frac{\partial \Phi}{\partial \nabla \theta}, \tag{1.13}$$

related to $\mathbf{q}$ by

$$\mathbf{q} = \theta \mathbf{Q}^d. \tag{1.14}$$

Note that, due to (1.5), from (1.13) and (1.14) it follows the Fourier law governing the heat flux

$$\mathbf{q} = -k_0 \nabla \theta. \tag{1.15}$$

Henceforth, if we substitute (1.10)–(1.14) in (1.6), applying the chain rule and after cancelling some terms, we can equivalently write the balance of the energy as follows

$$\theta(s_t + \operatorname{div} \mathbf{Q}^d - R) = \sigma^d \varepsilon(\mathbf{u}_t) - \mathbf{Q}^d \cdot \nabla\theta = \partial\Phi(\varepsilon(\mathbf{u}_t), \nabla\theta) \cdot (\varepsilon(\mathbf{u}_t), \nabla\theta), \tag{1.16}$$

where $R = r/\theta$ and $\partial\Phi$ denotes the subdifferential of Φ w.r.t. (with respect to) the dissipative variables $(\varepsilon(\mathbf{u}_t), \nabla\theta)$. Note that, since Φ in (1.5) is such that

$$\Phi(\mathbf{0}) = 0,\ \Phi \geq 0, \Phi \text{ is convex w.r.t. the dissipative variables,}$$

the right-hand side of (1.16) turns out to be non-negative. This fact as well as the positivity of the absolute temperature θ are sufficient to guarantee the thermodynamical consistence of the model. Now, we are in the position of recovering the PDE's system describing the model we have introduced above, written in terms of the unknowns θ and $\mathbf{u}$. After specifying (1.10)–(1.14) by (1.2) and (1.5) (cf. also (1.3)), we substitute in (1.6) and (1.7) the constitutive relations. We obtain in $\Omega \times (0, T)$

$$c_s\theta_t - k_0\Delta\theta - \alpha\theta \operatorname{div} \mathbf{u}_t = B\varepsilon(\mathbf{u}_t)\varepsilon(\mathbf{u}_t) + r, \tag{1.17}$$

$$\mathbf{u}_{tt} - \operatorname{div}(B\varepsilon(\mathbf{u}_t) + K\varepsilon(\mathbf{u}) + \alpha\theta\mathbf{1}) = \mathbf{G}, \tag{1.18}$$

and fix the Cauchy conditions in Ω

$$\theta(0) = \theta_0, \tag{1.19}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_t(0) = \mathbf{u}_1. \tag{1.20}$$

Finally, due to (1.8) and (1.9), we complete the above system by the following boundary conditions

$$k_0\partial_n\theta = h \quad \text{on } \Gamma \times (0, T), \tag{1.21}$$

$$\mathbf{u} = \mathbf{u}_t = \mathbf{0} \quad \text{on } \Gamma \times (0, T). \tag{1.22}$$

Before stating and proving our main existence and uniqueness theorems, let us recall some related results in the literature. In the one-dimensional setting, we recall the papers [4, 5] where the authors prove a global existence result for classical solutions to a thermoviscoelastic system for solidlike materials, in the case when one of the endpoints is stress-free. In [3], an existence result is given for pinned and thermally insulated endpoints in the special case of a one-dimensional model for shape-memory alloys. Other related existence and uniqueness results for this kind of models, and assuming a Landau-Devonshire or Landau-Ginzburg free-energy, can be found, e.g., in [10, 14, 11, 18]. Hence, in [15] the solid-solid phase transition problem in nonlinear thermoviscoelasticity is investigated in terms of the asymptotic behaviour of the solutions. Finally, in a recent contribution [19] the theory for solidlike and gaseous materials is treated in an unified manner. In particular, the main result of that paper is the global existence and uniqueness of the solution to an initial and boundary value problem corresponding to pinned endpoints held at a constant temperature. Notice that all the above results are given in the one-dimensional setting. Concerning the derivation of the thermoviscoelastic model, one can refer, e.g., to [9]. In the three-dimensional case, we recall the homogenization approach by [7] for a fairly simplified quasi-static model, but retaining the nonlinear dissipative contributions, and a related global existence result given in [2]. More precisely, exploiting the techniques of renormalized solutions for parabolic equations with L^1 data, and under suitable assumptions on the thermal stress with respect to the temperature, in [2] existence of small solutions is established. Then, this result is extended for arbitrary data under stronger assumptions on the thermal stress. Nonetheless, we point out that our setting is different not only for the techniques we use (mainly exploiting L^2 arguments). Indeed, in [7, 2] the state quantities are derived by a free energy functional obtained approximating the thermal energy contribution $-\theta \log \theta$ in (1.2) by a first order approximation of $\log \theta$. In particular, the energy equation (1.17) we deal with is not included in the framework of [2]. Other results concerning a thermoviscoelastic system in the multi-dimensional case (with the same approximation of the free energy), can be found in [16] where the author proves a global existence and uniqueness result of small solutions, assuming smooth and sufficiently small data.

The outline of the present work is as follows. In Section 2 we introduce a variational formulation of the problem and we state the main results (Theorems 2.1 and 2.2). Section 3 is devoted to the proof of the local existence result (Theorem 2.1) performed by a fixed point procedure. Section 4 is concerned with the proof of the uniqueness result (Theorem 2.2) established by some (global) contracting estimates. Finally, in Section 5 we derive some global estimates on the solution (Proposition 5.3) exploiting the positivity of the temperature (Theorem 5.1), which is proved by a maximum principle argument. Throughout the paper some comments and remarks are given.

2. Variational formulation and main results

The aim of this section is to introduce an abstract formulation of the initial and boundary values problem (1.17)–(1.22) and state related existence and uniqueness results. We set the problem in the domain Ω, which is a bounded smooth domain in $\mathbb{R}^3$, and investigate its evolution during a finite time interval $[0, T]$. We first introduce the Hilbert triplet (V, H, V'), where

$$H := L^2(\Omega), \quad V := H^1(\Omega),$$

and identify, as usual, H with its dual space H', so that $V \hookrightarrow H \hookrightarrow V'$, with dense and continuous embeddings. Besides, let the symbol $\|\cdot\|_X$ denote the norm either of some space X or X^3 and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V. Then, the associated Riesz isomorphism $J : V \to V'$ and the scalar product in V and in V' can be specified by

$$\langle J v_1, v_2 \rangle := ((v_1, v_2)), \quad ((u_1, u_2))_* := \langle u_1, J^{-1} u_2 \rangle, \tag{2.1}$$

for $v_i \in V$, $u_i \in V'$, $i = 1, 2$. Hence, we set the further Hilbert space

$$\mathbf{W} := H_0^1(\Omega)^3,$$

endowed with the usual norm. In addition, let introduce on $\mathbf{W} \times \mathbf{W}$ two bilinear continuous symmetric forms: $a(\cdot,\cdot)$ defined by

$$a(\mathbf{u},\mathbf{v}) = \lambda \int_\Omega \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} + 2\mu \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}),$$

and $b(\cdot,\cdot)$ defined by

$$b(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{3} \int_\Omega b_{ij}\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \quad B = (b_{ij}).$$

In the sequel, just for the sake of simplicity but without loss of generality, we let $b_{ij} = 1$, $i, j = 1, 2, 3$, so that

$$b(\mathbf{u},\mathbf{u}) = \|\varepsilon(\mathbf{u})\|_H^2, \tag{2.2}$$

and fix $c_s = \alpha = k_0 = 1$. Hence, recalling Korn's inequality, we are allowed to infer that there exists a positive constant $\widetilde{c}$, depending only on λ, μ, and Ω, such that

$$a(\mathbf{v},\mathbf{v}) \ge \widetilde{c}\, \|\mathbf{v}\|_{\mathbf{W}}^2, \tag{2.3}$$

for any $\mathbf{v}$ in $\mathbf{W}$. Note also that

$$b(\mathbf{v},\mathbf{v}) \ge \widehat{c}\, \|\mathbf{v}\|_{\mathbf{W}}^2, \quad \widehat{c} > 0. \tag{2.4}$$

Next, to set our problem in the abstract framework of the dual spaces V' and $\mathbf{W}'$, we introduce the operators

$$A : V \to V', \quad \langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V, \tag{2.5}$$

$$\mathcal{A} : \mathbf{W} \to \mathbf{W}', \quad {}_{\mathbf{W}'}\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle_{\mathbf{W}} = a(\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in \mathbf{W}, \tag{2.6}$$

$$\mathcal{B} : \mathbf{W} \to \mathbf{W}', \quad {}_{\mathbf{W}'}\langle \mathcal{B}\mathbf{u}, \mathbf{v}\rangle_{\mathbf{W}} = b(\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v} \in \mathbf{W}, \tag{2.7}$$

$$\mathcal{H} : H \to \mathbf{W}', \quad {}_{\mathbf{W}'}\langle \mathcal{H}u, \mathbf{v}\rangle_{\mathbf{W}} = \int_\Omega u \operatorname{div}\mathbf{v}, \quad u \in H, \mathbf{v} \in \mathbf{W}. \tag{2.8}$$

In particular, let us point out that $\mathcal{H}$ turns out to be a continuous and linear operator $V \subset H \to H^3 \subset \mathbf{W}'$, i.e. there exists a positive constant c such that

$$\|\mathcal{H}v\|_H \le c\|v\|_V, \quad \forall v \in V. \tag{2.9}$$

Now we set hypotheses on the data. Concerning the Cauchy conditions (1.19)–(1.20), we assume

$$\theta_0 \in H, \quad \mathbf{u}_0 \in \mathbf{W} \cap H^2(\Omega)^3, \quad \mathbf{u}_1 \in \mathbf{W}. \tag{2.10}$$

Then, dealing with the other functions, we prescribe that

$$r \in L^2(0,T;H), \quad h \in L^2(0,T;L^2(\Gamma)), \quad \mathbf{G} \in L^2(0,T;H^3) \tag{2.11}$$

and introduce the functions $\mathcal{R}$ and $\mathcal{G}$ specified by

$$\langle \mathcal{R}(t), v\rangle = \int_\Omega r(t)v + \int_\Gamma h(t)v_{|\Gamma}, \quad v \in V, \quad \text{for a.a. } t \in (0,T) \tag{2.12}$$

$${}_{\mathbf{W}'}\langle \mathcal{G}(t), \mathbf{v}\rangle_{\mathbf{W}} = \int_\Omega \mathbf{G}(t) \cdot \mathbf{v}, \quad \mathbf{v} \in \mathbf{W}, \quad \text{for a.a. } t \in (0,T) \tag{2.13}$$

so that, by (2.11), it is natural to postulate

$$\mathcal{R} \in L^2(0,T;V'), \quad \mathcal{G} \in L^2(0,T;H^3). \tag{2.14}$$

Here is a precise formulation of the abstract problem.

Problem P_a. Find $(\theta, \mathbf{u})$ satisfying (1.19)–(1.20) and such that, a.e. in $(0,T)$, (cf. also (2.2))

$$\theta_t + A\theta = \theta \operatorname{div} \mathbf{u}_t + \mathcal{R} + |\varepsilon(\mathbf{u}_t)|^2 \quad \text{in } V', \tag{2.15}$$

$$\mathbf{u}_{tt} + \mathcal{B}\mathbf{u}_t + \mathcal{A}\mathbf{u} + \mathcal{H}\theta = \mathcal{G} \quad \text{in } \mathbf{W}'. \tag{2.16}$$

Using a fixed point theorem, combined with contracting estimates, we prove the following local in time existence result.

Theorem 2.1. *Let* (2.10)–(2.11) *and* (2.14) *hold. Then, there exist* $T_0 \in (0,T]$ *and a pair of functions* $(\theta, \mathbf{u})$ *solving Problem* P_a *in* $(0,T_0)$ *and fulfilling*

$$\theta \in H^1(0,T_0;V') \cap C^0([0,T_0];H) \cap L^2(0,T_0;V), \tag{2.17}$$

$$\mathbf{u} \in H^2(0,T_0;H^3) \cap W^{1,\infty}(0,T_0;\mathbf{W}) \cap H^1(0,T_0;H^2(\Omega)^3). \tag{2.18}$$

The proof of this theorem is given in the following section, using the Schauder theorem. Next, in Section 4, performing local in time contracting estimates, which can be iterated on the whole time interval $[0,T]$, we show that this solution is unique. Indeed, the following theorem holds.

Theorem 2.2. *Let* $(\theta, \mathbf{u})$ *fulfilling* (2.17)–(2.18) *be a solution to Problem* P_a *during a time interval* $[0,T_0]$. *Then, this solution is unique on the whole time interval* $[0,T_0]$.

Remark 2.3. Note that, in spite of the fact that we are able to prove only a local in time existence result, uniqueness of a solution to Problem P_a with regularity (2.17)–(2.18) can be stated on the whole interval $[0,T]$. In particular, if we were able to extend Theorem 2.1 to a global existence result, Theorem 2.2 guarantees the (global) uniqueness of this solution.

3. Proof of the existence result

In this section, we detail the proof of the local existence result stated in Theorem 2.1. To this aim, we apply the Schauder theorem to a suitable operator $\mathcal{T}$ constructed as it will be specified in a moment. Now, for $D > 0$, consider the set

$$X := \{\mathbf{v} \in H^1(0,T_0;(W_0^{1,4}(\Omega))^3) : \|\mathbf{v}\|_{H^1(0,T_0;(W_0^{1,4}(\Omega))^3)} \le D\},$$

where $T_0 \in (0,T]$ will be fixed later, in such a way that

- $\mathcal{T}$ maps X into itself;
- $\mathcal{T}$ is compact;
- $\mathcal{T}$ is continuous.

First step. Take an arbitrary $\widetilde{\mathbf{u}} \in X$ and substitute $\mathbf{u}$ in (2.15). In particular, notice that the right hand side of (2.15) turns out to be in $L^1(0,T;H)+L^2(0,T;V')$, as by construction of X $|\varepsilon(\widetilde{\mathbf{u}}_t)|^2 \in L^1(0,T;H)$ via the Hölder inequality (cf. also (2.14)). Thus, we can apply the result of [1, Theorem 3.2, p. 256] to infer that there exists a unique

$$\theta := \mathcal{T}_1(\widetilde{\mathbf{u}}) \in [W^{1,1}(0,T;H) + H^1(0,T;V')] \cap C^0([0,T];H) \cap L^2(0,T;V), \tag{3.1}$$

solving the corresponding equation with associated Cauchy condition (1.19). Moreover, if we test the equation by θ and integrate over $(0,t)$, with t arbitrary in $(0,T)$, after some integrations by parts, applying the Hölder inequality, owing to the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and trace theorems, we have (cf. (1.19) and (2.10))

$$\begin{aligned} &\frac{1}{2}\|\theta(t)\|_H^2 - \frac{1}{2}\|\theta_0\|_H^2 + \|\nabla\theta\|^2_{L^2(0,t;H)} \\ &\le c\Big(\int_0^t \|\theta\|_V \|\operatorname{div}\widetilde{\mathbf{u}}_t\|_{L^4(\Omega)}\|\theta\|_H + \int_0^t \big(\|\widetilde{\mathbf{u}}_t\|^2_{W_0^{1,4}(\Omega)} + \|r\|_H\big)\|\theta\|_H \\ &\quad + \int_0^t \|h\|_{L^2(\Gamma)}\|\theta\|_V\Big). \end{aligned} \tag{3.2}$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. The Young inequality

$$pq \le \frac{1}{2\delta}p^2 + \frac{\delta}{2}q^2, \quad \delta > 0, \quad p, q \in \mathbb{R}, \tag{3.3}$$

leads to

$$\begin{aligned} &\frac{1}{2}\|\theta(t)\|_H^2 + \frac{1}{2}\|\theta\|^2_{L^2(0,t;V)} \\ &\le \frac{1}{2}\|\theta_0\|_H^2 + c\Big(\|\theta\|^2_{L^2(0,t;H)} + \|h\|^2_{L^2(0,t;L^2(\Gamma))} \\ &\quad + \int_0^t (\|\widetilde{\mathbf{u}}_t\|^2_{W_0^{1,4}(\Omega)} + \|r\|_H)\|\theta\|_H + \int_0^t \|\operatorname{div}\widetilde{\mathbf{u}}_t\|^2_{L^4(\Omega)}\|\theta\|_H^2\Big). \end{aligned} \tag{3.4}$$

By the generalized version of the Gronwall lemma introduced in [1] and recalling the definition of X, due to (3.4) we deduce that there exists a constant C_1 depending on T, Ω, h, r, θ_0, and D, such that

$$\|\theta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C_1. \tag{3.5}$$

Second step. After fixing $\theta = \mathcal{T}_1(\widetilde{\mathbf{u}})$ in (2.16), standard results on parabolic equations (see e.g. [6]) ensure that there exists a unique corresponding solution $\mathbf{u} := \mathcal{T}_2(\theta)$ satisfying (1.20) (cf. (1.22)). Concerning the regularity and the boundedness of $\mathbf{u}$, let us first test equation (2.16) by $\mathbf{u}_t$ and integrate over $(0,t)$ (cf. (2.3) and (2.10))

$$\begin{aligned} &\|\mathbf{u}_t(t)\|_H^2 + \|\mathbf{u}_t\|^2_{L^2(0,t;\mathbf{W})} + \|\mathbf{u}(t)\|^2_{\mathbf{W}} \\ &\le c\Big(\|\mathbf{u}_1\|_H^2 + \|\mathbf{u}_0\|^2_{\mathbf{W}} + \int_0^t \|\theta\|_H\|\operatorname{div}\mathbf{u}_t\|_H + \int_0^t \|\mathcal{G}\|_{\mathbf{W}'}\|\mathbf{u}_t\|_{\mathbf{W}}\Big) \\ &\le \frac{1}{2}\|\mathbf{u}_t\|^2_{L^2(0,t;\mathbf{W})} + c\left(1 + \|\theta\|^2_{L^2(0,t;H)} + \|\mathcal{G}\|^2_{L^2(0,T;\mathbf{W}')}\right). \end{aligned} \tag{3.6}$$

Then, (3.5) yields

$$\|\mathbf{u}\|_{W^{1,\infty}(0,T;H^3)\cap H^1(0,T;\mathbf{W})} \le C_2, \tag{3.7}$$

for a suitable constant C_2, with the same dependence of C_1 and depending in addition on $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathcal{G}$ (cf. (3.5)). Next, we can formally test by $\mathbf{u}_{tt}$ and integrate over $(0,t)$. We have, after integrating by parts,

$$\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + \frac{1}{2}\|\varepsilon(\mathbf{u}_t)(t)\|_H^2 - \frac{1}{2}\|\varepsilon(\mathbf{u}_t)(0)\|_H^2 \le \sum_{i=1}^3 |I_i(t)|, \tag{3.8}$$

where (some notation is only formal)

$$I_1(t) = -a(\mathbf{u}(t), \mathbf{u}_t(t)) + a(\mathbf{u}_0, \mathbf{u}_1) + \int_0^t a(\mathbf{u}_t, \mathbf{u}_t), \tag{3.9}$$

$$I_2(t) = -\int_0^t \int_\Omega \mathcal{H}\theta \mathbf{u}_{tt}, \tag{3.10}$$

$$I_3(t) = \int_0^t \int_\Omega \mathbf{G} \cdot \mathbf{u}_{tt}. \tag{3.11}$$

Now, we can handle the above integrals as follows. First, by the definition of a and due to (2.10) and (3.7), we can infer

$$\begin{aligned} |I_1(t)| &\le c\left(\|\mathbf{u}_0\|^2_{\mathbf{W}} + \|\mathbf{u}_1\|^2_{\mathbf{W}}\right) + \delta\|\mathbf{u}_t(t)\|^2_{\mathbf{W}} + C_\delta\|\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{W})} + c\int_0^t \|\mathbf{u}_t\|^2_{\mathbf{W}} \\ &\le c + \delta\|\mathbf{u}_t(t)\|^2_{\mathbf{W}}, \end{aligned} \tag{3.12}$$

where $\delta > 0$ will be chosen later. Analogously, by definition of $\mathcal{H}$, (2.9), and (3.3), we get

$$|I_2(t)| \le \delta\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + C_\delta\|\theta\|^2_{L^2(0,t;V)} \le \delta\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + c. \tag{3.13}$$

Finally, we can infer

$$|I_3(t)| \le \delta\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + C_\delta\|\mathbf{G}\|^2_{L^2(0,T;H^3)} \le \delta\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + c. \tag{3.14}$$

Thus, for a suitable choice of δ and combining (3.12)–(3.14) with (3.8) one obtains

$$\|\mathbf{u}_{tt}\|^2_{L^2(0,t;H^3)} + \|\mathbf{u}_t(t)\|^2_{\mathbf{W}} \le c + \frac{1}{2}\|\mathbf{u}_t\|_{L^\infty(0,t;\mathbf{W})}, \tag{3.15}$$

and consequently

$$\|\mathbf{u}\|_{H^2(0,T;H^3)\cap W^{1,\infty}(0,T;\mathbf{W})} \le C_3, \tag{3.16}$$

where the dependence of C_3 easily follows by the previous estimates. Now, a comparison in (2.16) leads to

$$\|\mathcal{B}\mathbf{u}_t + \mathcal{A}\mathbf{u}\|_{L^2(0,T;H^3)} \le c, \tag{3.17}$$

so that, recalling (1.22), (2.10), and the definition of the operators $\mathcal{B}$, $\mathcal{A}$, we deduce

$$\|\mathbf{u}\|_{H^1(0,T;H^2(\Omega)^3)} \le C_4, \tag{3.18}$$

for a constant C_4 with the same dependence of the previous constants.

Third step. Now, our aim is to find T_0 such that the operator

$$\mathcal{T} : X \to X, \quad \mathcal{T}(\widetilde{\mathbf{u}}) := \mathcal{T}_2(\mathcal{T}_1(\widetilde{\mathbf{u}})), \tag{3.19}$$

turns out to be well-defined. Note that thanks to a Gagliardo-Nirenberg estimate (cf. [13]), due to (3.16) and (3.18), there exists C_5 such that

$$\|\mathcal{T}(\widetilde{\mathbf{u}})\|_{W^{1,8/3}(0,T;W_0^{1,4}(\Omega)^3)} \le C_5. \tag{3.20}$$

By the Hölder inequality we have

$$\|\mathcal{T}(\widetilde{\mathbf{u}})\|_{H^1(0,T;W_0^{1,4}(\Omega)^3)} \le cT^{1/8}\|\mathcal{T}(\widetilde{\mathbf{u}})\|_{W^{1,8/3}(0,T;W_0^{1,4}(\Omega)^3)}. \tag{3.21}$$

Thus, to ensure that $\mathcal{T}(\widetilde{\mathbf{u}}) \in X$, i.e.

$$\|\mathcal{T}(\widetilde{\mathbf{u}})\|_{H^1(0,T_0;W_0^{1,4}(\Omega)^3)} \le T_0^{1/8}C_6 \le D, \tag{3.22}$$

we can find $T_0 \in (0,T]$ such that, e.g., $T_0 \le D^8/C_6^8$. Next, we observe that the above argument leads to (cf. (3.16)–(3.18))

$$\|\mathcal{T}(\widetilde{\mathbf{u}})\|_{H^2(0,T_0;H^3)\cap W^{1,\infty}(0,T_0;\mathbf{W})\cap H^1(0,T_0;H^2(\Omega)^3)} \le c, \tag{3.23}$$

for a constant c independent of the choice of $\widetilde{\mathbf{u}} \in X$, which ensures that $\mathcal{T}$ is a compact operator. From now on, for the sake of simplicity, we directly refer to T instead of T_0. Hence, to achieve the proof of the Schauder theorem, it remains to show that $\mathcal{T}$ is continuous with respect to the natural strong topology induced in X by $H^1(0,T;W_0^{1,4}(\Omega)^3)$. Towards this goal, we consider a sequence in X such that

$$\widetilde{\mathbf{u}}_n \to \widetilde{\mathbf{u}} \quad \text{in } H^1(0,T;W_0^{1,4}(\Omega)^3), \tag{3.24}$$

and let

$$\theta_n = \mathcal{T}_1(\widetilde{\mathbf{u}}_n), \quad \mathbf{u}_n = \mathcal{T}_2(\theta_n). \tag{3.25}$$

Proceeding as for the previous estimates, we can find a positive constant c not depending on n such that

$$\|\theta_n\|_{[W^{1,1}(0,T;H)+H^1(0,T;V')]\cap L^\infty(0,T;H)\cap L^2(0,T;V)} \le c, \tag{3.26}$$

$$\|\mathbf{u}_n\|_{H^2(0,T;H^3)\cap W^{1,\infty}(0,T;\mathbf{W})\cap H^1(0,T;H^2(\Omega)^3)} \le c. \tag{3.27}$$

By well-known weak and weak star compactness results the following convergences are satisfied, at least for some suitable subsequences,

$$\theta_n \rightharpoonup^* \theta \quad \text{in } L^\infty(0,T;H)\cap L^2(0,T;V), \tag{3.28}$$

$$\mathbf{u}_n \rightharpoonup^* \mathbf{u} \quad \text{in } H^2(0,T;H^3)\cap W^{1,\infty}(0,T;\mathbf{W})\cap H^1(0,T;H^2(\Omega)^3). \tag{3.29}$$

In particular, by compactness we have (cf. [12, 17])

$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } H^1(0,T;W_0^{1,4}(\Omega)^3), \tag{3.30}$$

so that to conclude the proof it remains to verify that (cf. (3.24))

$$\mathbf{u} = \mathcal{T}(\widetilde{\mathbf{u}}). \tag{3.31}$$

We first show that $\theta = \mathcal{T}_1(\widetilde{\mathbf{u}})$ in (3.28). To this aim, we write (2.15) for $\widetilde{\mathbf{u}}_n$ and $\widetilde{\mathbf{u}}$, take the difference and test by $\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})$. After an integration over $(0,t)$ and performing an analogous estimate as (3.4), we can infer

$$\begin{aligned}
&\frac{1}{2}\|(\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}}))(t)\|_H^2 + \|\nabla(\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}}))\|_{L^2(0,t;H)}^2 \\
&\le c\int_0^t \|\theta_n\|_H \|\operatorname{div}(\widetilde{\mathbf{u}}_n - \widetilde{\mathbf{u}})_t\|_{L^4(\Omega)} \|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_V \\
&\quad + c\int_0^t \|\operatorname{div}\widetilde{\mathbf{u}}_t\|_{L^4(\Omega)} \|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_V \|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H \\
&\quad + \int_0^t \big(\|\varepsilon(\widetilde{\mathbf{u}}_{n_t})\|_{L^4(\Omega)}^2 - \|\varepsilon(\widetilde{\mathbf{u}}_t)\|_{L^4(\Omega)}^2\big)\|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H \le \delta\|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_{L^2(0,t;V)}^2 \\
&\quad + C_\delta\Big(\int_0^t \|\theta_n\|_H^2 \|\operatorname{div}(\widetilde{\mathbf{u}}_n - \widetilde{\mathbf{u}})_t\|_{L^4(\Omega)}^2 + \int_0^t \|\operatorname{div}\widetilde{\mathbf{u}}_t\|_{L^4(\Omega)}^2 \|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H^2\Big) \\
&\quad + \int_0^t \big(\|\varepsilon(\widetilde{\mathbf{u}}_{n_t})\|_{L^4(\Omega)}^2 - \|\varepsilon(\widetilde{\mathbf{u}}_t)\|_{L^4(\Omega)}^2\big)\|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H.
\end{aligned} \tag{3.32}$$

Thus, for a suitable choice of δ, we can get the following estimate

$$\begin{aligned} &\|(\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}}))(t)\|_H^2 + \|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_{L^2(0,t;V)}^2 \\ &\le c\Big(\|\theta_n\|_{L^\infty(0,t;H)}^2 \|\operatorname{div}(\widetilde{\mathbf{u}}_n - \widetilde{\mathbf{u}})_t\|_{L^2(0,t;L^4(\Omega))}^2 \\ &\quad + \int_0^t \big(1 + \|\operatorname{div}\widetilde{\mathbf{u}}_t\|_{L^4(\Omega)}^2\big)\|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H^2 \\ &\quad + \int_0^t \big(\|\varepsilon(\widetilde{\mathbf{u}}_{n_t})\|_{L^4(\Omega)}^2 - \|\varepsilon(\widetilde{\mathbf{u}}_t)\|_{L^4(\Omega)}^2\big)\|\theta_n - \mathcal{T}_1(\widetilde{\mathbf{u}})\|_H\Big). \end{aligned} \tag{3.33}$$

We can apply the Gronwall lemma to (3.33) and, thanks to (3.30) and (3.26), we get

$$\lim_{n\to+\infty} \|\theta_n - \mathcal{T}_1(\widetilde{u})\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} = 0, \tag{3.34}$$

and eventually θ in (3.28) can be identified with $\mathcal{T}_1(\widetilde{\mathbf{u}})$. Notice in particular that the following strong convergence holds

$$\theta_n \to \theta \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V). \tag{3.35}$$

Now, it is a standard matter to observe that (3.30) and (3.35) allow us to pass to the limit as $n \to +\infty$ in (2.16) and, thanks to the uniqueness result for the limit equation, once θ is fixed, eventually identify $\mathbf{u}$ with $\mathcal{T}_2(\theta)$, from which (3.31) easily follows. Finally, we have to complete the proof of (2.17). To this aim, after adding to both sides of (2.15) θ, we test by $J^{-1}\theta_t$ (cf. (2.1)) and integrate over $(0,t)$. By definition of J and using the Hölder inequality, we get (cf. (2.18) and (3.35))

$$\begin{aligned} &\|\theta_t\|_{L^2(0,t;V')}^2 + \|\theta(t)\|_H^2 \\ &\le c\Big(\|\theta_0\|_H^2 + \int_0^t \|\theta\|_H \|\operatorname{div}\mathbf{u}_t\|_{L^4(\Omega)} \|J^{-1}\theta_t\|_V \\ &\quad + \int_0^t \|\varepsilon(\mathbf{u}_t)\|_{L^4(\Omega)} \|\varepsilon(\mathbf{u}_t)\|_H \|J^{-1}\theta_t\|_V \\ &\quad + \int_0^t \|\mathcal{R}\|_{V'} \|J^{-1}\theta_t\|_V + \int_0^t \|\theta\|_H \|J^{-1}\theta_t\|_H\Big) \\ &\le \frac{1}{2}\|\theta_t\|_{L^2(0,t;V')}^2 + c\Big(1 + \|\operatorname{div}\mathbf{u}_t\|_{L^2(0,T;L^4(\Omega))}^2 \|\theta\|_{L^\infty(0,t;H)}^2 \\ &\quad + \|\varepsilon(\mathbf{u}_t)\|_{L^2(0,T;L^4(\Omega)^3)}^2 \|\varepsilon(\mathbf{u}_t)\|_{L^\infty(0,T;H^3)}^2 + \|\theta\|_{L^2(0,T;H)}^2\Big), \end{aligned} \tag{3.36}$$

from which (2.17) is easily deduced.

Remark 3.1. Due to the regularity of θ and $\mathbf{u}$ and (2.10), we could iterate the above argument, starting from T_0 and so on. Nonetheless, even if we can extend the existence result to some interval $[0, T_0^*]$, with $T_0 < T_0^* \le T$, we are not allowed to infer that a solution exists on the whole interval $[0,T]$.

4. Proof of the uniqueness result

This section is devoted to the proof of the uniqueness result in Theorem 2.2, i.e. we show uniqueness of the solution to the system (2.15)–(2.16), (1.19)–(1.20), during $[0,T_0]$, with the regularity specified by (2.17)–(2.18). Let us point out that this uniqueness result is stated in any interval $[0,T_0]$, with $T_0 \in [0,T]$ (cf. Remark 2.3).

First, we make some remarks about the notation to be used. We assume that (2.15)–(2.16), (1.19)–(1.20) admit two solutions

$$\mathcal{S}_1 = \{\theta_1, \mathbf{u}_1\}, \quad \mathcal{S}_2 = \{\theta_2, \mathbf{u}_2\}, \tag{4.1}$$

whose regularity is given by (2.17)–(2.18). We denote by $\widehat{f}$ the difference of two functions f_1 and f_2, i.e.

$$\widehat{f} = f_1 - f_2, \tag{4.2}$$

and make use of two trivial identities

$$p_1 q_1 - p_2 q_2 = \widehat{pq} = \hat{p} q_2 + p_1 \hat{q} = \hat{p} q_1 + p_2 \hat{q}, \tag{4.3}$$

so that, without loss of generality, in the sequel let us rewrite (4.3) and subsequent computations omitting subscripts, i.e.

$$\widehat{pq} = \hat{p} q + p \hat{q}. \tag{4.4}$$

Now, we take the difference of (2.15), written for $\mathcal{S}_1$ and $\mathcal{S}_2$, and integrate it in time

$$\widehat{\theta} + 1 * A\widehat{\theta} = 1 * \widehat{\theta \operatorname{div} \mathbf{u}_t} + 1 * \widehat{|\varepsilon(\mathbf{u}_t)|^2}, \tag{4.5}$$

where by $*$ we denote the usual convolution product over $(0,t)$, namely

$$(p * q)(t) = \int_0^t p(t-s) q(s)\, ds. \tag{4.6}$$

Then, we can test (4.5) by $\widehat{\theta}$ and integrate over $(0,t)$. After an integration by parts in time and eploiting (4.4), we can infer

$$\|\widehat{\theta}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|(1 * \nabla\widehat{\theta})(t)\|^2_H \le \sum_{i=4}^{6} I_i(t), \tag{4.7}$$

where the integrals $I_i(t)$, $i = 4,5,6$, are specified as follows

$$I_4(t) = \int_0^t \int_\Omega (1 * \theta \operatorname{div} \widehat{\mathbf{u}}_t)\widehat{\theta}, \tag{4.8}$$

$$I_5(t) = \int_0^t \int_\Omega (1 * \widehat{\theta} \operatorname{div} \mathbf{u}_t)\widehat{\theta}, \tag{4.9}$$

$$I_6(t) = 2\int_0^t \int_\Omega (1 * \varepsilon(\mathbf{u}_t)\widehat{\varepsilon(\mathbf{u}_t)})\widehat{\theta}\,. \tag{4.10}$$

We first treat the integral I_4 recalling (2.17), integrating by parts in time, and applying the Young inequality. We have

$$\begin{aligned}
|I_4(t)| &\le \int_\Omega |(1 * \widehat{\theta})(t)| \int_0^t |\theta \operatorname{div} \widehat{\mathbf{u}}_t| + \int_0^t \int_\Omega |1 * \widehat{\theta}||\theta \operatorname{div} \widehat{\mathbf{u}}_t| \\
&\le \|(1 * \widehat{\theta})(t)\|_{L^4(\Omega)} \int_0^t \|\operatorname{div} \widehat{\mathbf{u}}_t\|_H \|\theta\|_{L^4(\Omega)} \\
&\quad + \int_0^t \|1 * \widehat{\theta}\|_{L^4(\Omega)} \|\operatorname{div} \widehat{\mathbf{u}}_t\|_H \|\theta\|_{L^4(\Omega)} \\
&\le \delta \|1 * \widehat{\theta}\|^2_{L^\infty(0,t;V)} + c_5 \int_0^t \|\theta\|^2_V \|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})},
\end{aligned} \tag{4.11}$$

for a suitable positive constant c_5 depending only on the data of the problem and $\delta > 0$. Analogously, by (2.18) we have

$$
\begin{aligned}
|I_5(t)| &\le \int_\Omega |(1*\widehat{\theta})(t)| \int_0^t |\operatorname{div}\mathbf{u}_t\widehat{\theta}| + \int_0^t \int_\Omega |1*\widehat{\theta}|\,|\operatorname{div}\mathbf{u}_t\widehat{\theta}| \\
&\le \|(1*\widehat{\theta})(t)\|_V \int_0^t \|\operatorname{div}\mathbf{u}_t\|_{L^4(\Omega)}\|\widehat{\theta}\|_H + \int_0^t \|1*\widehat{\theta}\|_V\|\operatorname{div}\mathbf{u}_t\|_{L^4(\Omega)}\|\widehat{\theta}\|_H \\
&\le \delta\|1*\widehat{\theta}\|^2_{L^\infty(0,t;V)} + c_6\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}\|\widehat{\theta}\|^2_{L^2(0,t;H)}.
\end{aligned}
\tag{4.12}
$$

Finally, by analogous arguments we obtain

$$
\begin{aligned}
|I_6(t)| &\le 2\int_\Omega |(1*\widehat{\theta})(t)| \int_0^t |\varepsilon(\mathbf{u}_t)\varepsilon(\widehat{\mathbf{u}}_t)| + 2\int_0^t |1*\widehat{\theta}|\,|\varepsilon(\mathbf{u}_t)\varepsilon(\widehat{\mathbf{u}}_t)| \\
&\le \delta\|1*\widehat{\theta}\|^2_{L^\infty(0,t;V)} + c_7\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}\|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})}.
\end{aligned}
\tag{4.13}
$$

Now, we test the difference of (2.16) by $\widehat{\mathbf{u}}_t$, integrate over $(0,t)$, and infer, using fairly standard arguments,

$$
\|\widehat{\mathbf{u}}_t(t)\|^2_H + \|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})} + \|\widehat{\mathbf{u}}(t)\|^2_{\mathbf{W}} \le c_9\|\widehat{\theta}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})}, \tag{4.14}
$$

for a suitable positive constant c_9. Hence, we combine (4.11)–(4.13) with (4.7) and add the resulting inequality to (4.14) multiplied by $1/(2c_9)$. In particular, after recalling that

$$
\|(1*\widehat{\theta})(t)\|_H \le T^{1/2}\|\widehat{\theta}\|_{L^2(0,t;H)},
$$

we can take e.g. $c_8 = \min\left\{\frac{1}{2}, \frac{1}{4T}\right\}$, fix $\delta = c_8/6$, and write

$$
\begin{aligned}
&\frac{1}{4}\|\widehat{\theta}\|^2_{L^2(0,t;H)} + c_8\|(1*\widehat{\theta})(t)\|^2_V + \frac{1}{2c_9}\|\widehat{\mathbf{u}}_t(t)\|^2_H + \frac{1}{4c_9}\|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})} \\
&\le \frac{c_8}{2}\|1*\widehat{\theta}\|^2_{L^\infty(0,t;V)} + \Big(c_5\int_0^t \|\theta\|^2_V + c_7\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}\Big)\|\widehat{\mathbf{u}}_t\|^2_{L^2(0,t;\mathbf{W})} \\
&+ c_6\int_0^t \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}\|\widehat{\theta}\|^2_{L^2(0,t;H)}.
\end{aligned}
\tag{4.15}
$$

Note that because of (2.17)–(2.18), $\|\mathbf{u}_t\|^2_{H^2(\Omega)^3}$ and $\|\theta\|^2_V$ belong to $L^1(0,T)$. Thus, we can find τ sufficiently small such that, e.g.

$$
\begin{aligned}
&\max\Big\{c_5\int_{\widehat{t}}^{\widehat{t}+\tau} \|\theta\|^2_V, c_6\int_{\widehat{t}}^{\widehat{t}+\tau} \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}, c_7\int_{\widehat{t}}^{\widehat{t}+\tau} \|\mathbf{u}_t\|^2_{H^2(\Omega)^3}\Big\} \\
&\le \min\Big\{\frac{1}{8}, \frac{1}{8c_9}\Big\},
\end{aligned}
\tag{4.16}
$$

for any $\widehat{t} \in [0, T-\tau]$. Thus, at the end, by combining (4.15) with (4.16), we eventually get

$$
\frac{1}{8}\|\widehat{\theta}\|^2_{L^2(0,t;H)} + c_8\|(1*\widehat{\theta})(t)\|^2_V + \frac{1}{2c_9}\|\widehat{\mathbf{u}}_t(t)\|^2_H \le \frac{c_8}{2}\|1*\widehat{\theta}\|^2_{L^\infty(0,t;V)}, \tag{4.17}
$$

for any $t \in (0,\tau)$. Now, from (4.17) it easily follows

$$
\widehat{\theta} = \widehat{\mathbf{u}} = 0 \quad \text{a.e. in } \Omega \times (0,\tau). \tag{4.18}
$$

Finally, as we can repeat the same estimates for any interval $(\widehat{t}, \tau + \widehat{t})$, by iterating the above procedure we are allowed to extend (4.18) to the whole interval $(0, T)$, which concludes the proof of the uniqueness result.

5. Positivity of the temperature and global estimates

In this section, we aim to prove positivity of the temperature, ensuring that the second principle of thermodynamics is satisfied, as we have discussed in the introduction. Hence, in the second part of the section, mainly exploiting the positivity of the temperature, we are able to prove some global estimates on the local solution of Problem P_a, which do not depend on the choice of T_0. In particular, these estimates could be extended to the whole initial interval $(0, T)$. We first detail the proof of the positivity of θ.

Theorem 5.1. *Let the hypotheses of Theorem 2.1 hold and suppose in addition*

$$r \geq 0 \quad a.e.\ in\ \Omega \times (0, T), \tag{5.1}$$

$$h \geq 0 \quad a.e.\ in\ \Gamma \times (0, T), \tag{5.2}$$

$$\theta_0 \geq 0 \quad a.e.\ in\ \Omega. \tag{5.3}$$

Then, the solution $(\theta, \mathbf{u})$ *to Problem* P_a *is such that*

$$\theta(x, t) \geq 0 \quad for\ a.a.\ (x, t) \in \Omega \times (0, T_0). \tag{5.4}$$

Remark 5.2. By construction (5.1)–(5.2) imply that $\mathcal{R}$ defined in (2.12) is nonnegative in the sense of distributions, i.e. for any positive test function ϕ one has $\langle \mathcal{R}(t), \phi \rangle \geq 0$, for a.a. t.

Proof of Theorem 5.1. We use a maximum principle argument. Thus, we test (2.15) by $-\theta^-$, f^- denoting the so-called negative part of a function f, i.e. $f^- := \max\{0, -f\}$, and integrate over $(0, t)$. After an integration by parts in time, exploiting (5.1)–(5.3), and owing to the Hölder inequality, we can infer that

$$\begin{aligned} &\frac{1}{2}\|\theta^-(t)\|_H^2 + \|\nabla\theta^-\|_{L^2(0,t;H)}^2 \\ &\leq \int_0^t \|\theta^-\|_H \|\theta^-\|_{L^4(\Omega)} \|\operatorname{div} \mathbf{u}_t\|_{L^4(\Omega)} - \int_0^t\int_\Omega r\theta^- - \int_0^t\int_\Gamma h\theta^-_{|\Gamma} - \int_0^t\int_\Omega |\varepsilon(\mathbf{u}_t)|^2\theta^- \\ &\leq c\int_0^t \|\theta^-\|_H \|\theta^-\|_V \|\operatorname{div} \mathbf{u}_t\|_{L^4(\Omega)}. \end{aligned} \tag{5.5}$$

Hence, to handle the right hand side of (5.5) we apply the Young inequality and get, by definition of the V-norm,

$$\frac{1}{2}\|\theta^-(t)\|_H^2 + \frac{1}{2}\|\theta^-\|_{L^2(0,t;V)}^2 \leq c\Big(\|\theta^-\|_{L^2(0,t;H)}^2 + \int_0^t \|\operatorname{div} \mathbf{u}_t\|_{L^4(\Omega)}^2 \|\theta^-\|_H^2\Big). \tag{5.6}$$

Then, since $\|\operatorname{div} \mathbf{u}_t\|_{L^4(\Omega)}^2$ belongs to $L^1(0, T)$ (cf. (2.18)), the generalized version of the Gronwall lemma introduced in [1] allows us to deduce

$$\|\theta^-\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq 0, \tag{5.7}$$

which eventually gives (5.4). □

In the last part of this section, we show some uniform estimates on the local solution $(\theta, \mathbf{u})$ of Problem P_a, which actually could be extended on the whole interval $(0, T)$. Indeed, the following Proposition holds.

Proposition 5.3. *Under the assumptions of Theorem 5.1, there exists a positive constant c depending only on Ω, T, and the data of the problem, such that the following bound is fulfilled by the solution $(\theta, \mathbf{u})$ to Problem P_a*

$$\|\theta\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{W^{1,\infty}(0,T;H^3)\cap L^\infty(0,T;\mathbf{W})} \leq c. \tag{5.8}$$

Proof. The estimating process, used for proving (5.8), mainly exploits the positivity of the temperature stated by Theorem 5.1. We formally proceed by testing (2.15) by the constant function 1 and (2.16) by $\mathbf{u}_t$. Then, we add the resulting equations and finally integrate over $(0, t)$. As some terms cancel, after an integration by parts in time, we eventually write

$$\begin{aligned}
&\int_\Omega \theta(t) + \|\mathbf{u}_t(t)\|_H^2 + \|\mathbf{u}(t)\|_{\mathbf{W}}^2 \\
&\leq c\Big(\int_\Omega \theta_0 + \|\mathbf{u}_1\|_H^2 + \|\mathbf{u}_0\|_{\mathbf{W}}^2 + \|\mathcal{R}\|_{L^2(0,T;V')} + \int_0^t {}_{\mathbf{W}'}\langle \mathcal{G}, \mathbf{u}_t\rangle_{\mathbf{W}}\Big) \\
&\leq c\Big(1 + \|\mathcal{R}\|_{L^2(0,T;V')} + \int_0^t \|\mathbf{G}\|_H \|\mathbf{u}_t\|_H\Big).
\end{aligned} \tag{5.9}$$

Thus, using the Gronwall lemma, one can easily obtain (5.8), after observing that by (5.4) there holds

$$\int_\Omega \theta(t) = \|\theta(t)\|_{L^1(\Omega)}.$$

□

Acknowledgement. We would like to thank Professor Michel Frémond for having proposed us this interesting problem and for several suggestions especially concerning the modelling aspects.

References

[1] Baiocchi C.: *Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert*, Ann. Mat. Pura Appl. (IV), **76** (1967), 233–304.
[2] Blanchard D. and Guibé O.: *Existence of a solution for a nonlinear system in thermoviscoelasticity*, Adv. Differential Equations, **5** (2000), 1221–1252.
[3] Chen Z. and Hoffmann K.H.: *On a one-dimensional nonlinear thermoviscoelastic model for structural phase transitions in shape memory alloys*, J. Differential Equations, **112** (1994), 325–350.
[4] Dafermos C.M.: *Global smooth solutions to the initial boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity*, SIAM J. Math. Anal., **13** (1982), 397–408.
[5] Dafermos C.M. and Hsiao L.: *Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity*, Nonlinear Anal., **6** (1982), 435–454.
[6] Duvaut G. and Lions J.L.: *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976
[7] Francfort G.A. and Suquet P.: *Homogenization and mechanical dissipation in thermoviscoelasticity*, Arch. Rational Mech. Anal., **96** (1986), 265–293.
[8] Frémond M.: *Non-smooth Thermomechanics*, Springer-Verlag, Berlin, 2001
[9] Germain P.: *Cours de mecanique des milieux continus*, Masson et Cie, Paris, 1973
[10] Hoffmann K.H. and Zheng S.: *Uniqueness for structural phase transitions in shape memory alloys*, Math. Meth. Appl. Sci., **10** (1988), 145–151.

[11] Hoffmann K.H. and Zochowski A.: *Existence of solutions to some nonlinear thermoelastic system with viscosity*, Math. Mech. Appl. Sci., **15** (1992), 187–204.
[12] Lions J.L.: *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Gauthier-Villars Paris, 1969.
[13] Nirenberg L.: *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3), **13** (1959), 115–162.
[14] Niezgodka M., Sprekels J., and Zheng S.: *Global solutions to a model of structural phase transitions in shape memory alloys*, J. Math. Anal. Appl., **130** (1988), 39–54.
[15] Racke R. and Zheng S.: *Global existence and asymptotic behavior in nonlinear thermoviscoelasticity*, J. Differential Equations, **134** (1997), 46–67.
[16] Shibata Y.: *Global in time existence of small solutions of nonlinear thermoviscoelastic equations*, Math. Methods Appl. Sci., **18** (1995), 871–895.
[17] Simon J.: *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.
[18] Sprekels J. and Zheng S.: *Global solutions to the equations of Ginzburg-Landau theory for structural phase transitions in shape memory alloys*, Physica D, **39** (1989), 59–76.
[19] Watson S.J.: *Unique global solvability for initial-boundary value problems in one-dimensional nonlinear thermoviscoelasticity*, Arch. Rational Mech. Anal., **153** (2000), 1–37.

Elena Bonetti
Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy
E-mail address: bonetti@dimat.unipv.it

Giovanna Bonfanti
Dipartimento di Matematica, Università di Brescia, Via Branze 38, 25123 Brescia, Italy
E-mail address: bonfanti@ing.unibs.it