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NASH-MOSER TECHNIQUES FOR NONLINEAR BOUNDARY-VALUE PROBLEMS

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Abstract. A new linearization method is introduced for smooth short-time solvability of initial boundary value problems for nonlinear evolution equations. The technique based on an inverse function theorem of Nash-Moser type is illustrated by an application in the parabolic case. The equation and the boundary conditions may depend fully nonlinearly on time and space variables. The necessary compatibility conditions are transformed using a Borel's theorem. A general trace theorem for normal boundary conditions is proved in spaces of smooth functions by applying tame splitting theory in Fréchet spaces. The linearized parabolic problem is treated using maximal regularity in analytic semigroup theory, higher order elliptic a priori estimates and simultaneous continuity in trace theorems in Sobolev spaces.

1. Introduction

The purpose of this paper is to introduce a new linearization method for smooth short-time solvability of initial boundary value problems for nonlinear evolution equations. The technique based on an inverse function theorem of Nash-Moser type is illustrated by an application in the parabolic case. The equation and the boundary conditions may depend fully nonlinearly on time and space variables. The general Theorem 4.3 applies to a nonlinear evolutionary boundary value problem provided that the linearized equation with linearized boundary conditions is well posed; here a loss of derivatives is allowed in the estimates of the linearized problem. An application in the parabolic case is given in Theorem 8.1.

We mention some points of the proof which might be of independent interest. A Borel's theorem is applied to transform the compatibility conditions. A trace theorem is proved for normal boundary operators in spaces of smooth functions using tame splitting theory in Fréchet spaces. Some results on simultaneous continuity in trace theorems in Sobolev spaces are proved. In the application, higher order Sobolev norm estimates including the dependence of the constants from the coefficients are derived for the linearized parabolic problem using analytic semigroup theory involving evolution operators and maximal regularity.

Inverse function theorems of Nash-Moser type [13, 15, 25, 34, 39] have been applied to partial differential equations in several papers, for instance, concerning

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small global solutions in [17], periodic solutions in [41, 18, 11], and local solutions in [13], III. 2.2 or [21, 35, 36, 37, 38]. Different from these articles we consider initial boundary value problems including compatibility conditions in this note. It seems that the technique introduced in this paper is the first general linearization method in the literature based on a Nash Moser type inverse function theorem which applies to smooth initial boundary value problems with loss of derivatives including compatibility conditions.

This paper continues and completes the work in [36] where the whole space case is considered. It turned out that the case of boundary value problems treated in this note is completely different from the whole space case and requires substantially other methods.

In the literature many results are known on linear and on nonlinear parabolic boundary value problems. It is beyond the scope of this paper to give a complete survey. We here only mention some articles which contain also additional references. For the classical linear theory of parabolic equations and systems we refer to [4, 12, 49, 20, 24]. Early results on short-time solvability of nonlinear second order equations can be found in the references [1] through [8] of the survey paper [28]. Since then nonlinear parabolic problems have been studied in many papers, for instance in [16, 12, 20, 19], or, more recently, in [23, 22, 47]. Semigroup theory has been applied by many authors to the solution of linear and of nonlinear parabolic problems, we refer to [48, 50, 29, 5, 1, 26, 6].

This paper is organized as follows. Section 2 contains notations which are used throughout this article. In section 3 a smoothing property for Fréchet spaces is recalled from [32] which is required as a formal assumption in the inverse function theorem of Nash-Moser type [34]. The spaces $C_0^\infty([0,T], H^\infty(\Omega))$ are shown to enjoy this property with uniform constants for all small $T > 0$; here C_0^∞ denotes the subspace of C^∞ containing functions vanishing with all derivatives at the origin.

In section 4 the inverse function theorem [34] is used to linearize the initial boundary value problem. Mainly due to compatibility conditions this approach is completely different from the whole space case [36]. A transformation based on a Borel's theorem gives a reduction to zero compatibility conditions. The smallness assumptions required by the inverse function theorem can be achieved by choosing a small time interval without supposing smallness assumptions on the initial values. This is based on a uniformity argument and on Borel's theorem.

Using results of section 5 the linear problem is reduced to a problem with homogeneous boundary conditions. The results of section 5 might be of independent interest. A trace theorem including estimates is proved for normal boundary operators in spaces of smooth functions by applying the tame splitting theorem [40] in Fréchet spaces. Note that classical right inverses for trace operators in Sobolev spaces constructed e.g. by the Fourier transform depend on the order of the Sobolev space and do not induce right inverses in spaces of smooth functions. In addition, results on simultaneous continuity are proved for trace theorems in Sobolev spaces.

Sections 6, 7, 8 contain an application in the parabolic case.

In section 6 the linearized parabolic initial boundary value problem of arbitrary order is considered. Under suitable parabolicity assumptions the necessary higher order Sobolev norm estimates are proved. In order to derive the appropriate dependence of the constants from the coefficients these estimates are formulated and proved by means of a symbolic calculus involving the weighted multiseminorms

$[\ ]_{m,k}$ introduced in [34]. The estimates are based on maximal regularity in Hölder spaces and on the results of section 5 on simultaneous continuity in trace theorems.

In section 7 we obtain sufficient conditions of elliptic type for the parabolicity assumptions of section 6. It is shown that the constants in the higher order elliptic a priori estimates due to Agmon, Douglis, Nirenberg [3] depend on the coefficients of the problem as required by the Nash-Moser technique; we note that this means more than only uniformity as stated in [3]. Furthermore, resolvent estimates due to Agmon [2] are used to establish the assumptions of section 6.

Finally, in section 8 the short-time solvability of the nonlinear parabolic problem is proved in Theorem 8.1 under general and natural assumptions. It is enough that the linearized problem together with the linearized boundary conditions is given by a regular elliptic problem in the usual sense (cf. Definition 7.5 or [24]).

The technical Theorems 4.3, 4.4, 5.5 provide a general framework for applications to evolutionary boundary value problems where a loss of derivatives appears in the estimates of the linearized problem. This might be interesting for further applications which are not accessible to standard methods due to a loss of derivatives, for instance to other evolution equations or to coupled systems involving Navier Stokes system and heat equation where a loss of derivatives appears due to the coupling.

2. Preliminaries

We shall consider Fréchet spaces $E, F, \dots$ equipped with a fixed sequence $\|\ \|_0 \le \|\ \|_1 \le \|\ \|_2 \le \dots$ of seminorms defining the topology. The product $E \times F$ is endowed with the seminorms $\|(x,y)\|_k = \max\{\|x\|_k, \|y\|_k\}$. A linear map $T : E \to F$ is called tame (cf. [13]) if there exist an integer b and constants c_k so that $\|Tx\|_k \le c_k \|x\|_{k+b}$ for all k and x. A linear bijection T is called a tame isomorphism if both T and T^{-1} are tame.

A continuous nonlinear map $\Phi : (U \subset E) \to F$ between Fréchet spaces, U open, is called a C^1-map if the derivative $\Phi'(x)y = \lim_{t\to 0} \frac{1}{t}(\Phi(x+ty) - \Phi(x))$ exists for all $x \in U$, $y \in E$ and is continuous as a map $\Phi' : U \times E \to F$. Φ is called a C^2-map if it is C^1 and the second derivative $\Phi''(x)\{y_1, y_2\} = \lim_{t\to 0} \frac{1}{t}(\Phi'(x+ty_2)y_1 - \Phi'(x)y_1)$ exists and is continuous as a map $\Phi'' : U \times E^2 \to F$. Similar definitions apply to higher derivatives $\Phi^{(n)}$; Φ is called C^∞ if it is C^n for all n. Given a function of two variables $\Phi = \Phi(x,y)$ we can also consider the partial derivatives Φ_x and Φ_y where e.g. $\Phi_x(x,y)z = \lim_{t\to 0} \frac{1}{t}(\Phi(x+tz,y) - \Phi(x,y))$. One-dimensional derivatives $\Phi_t, t \in \mathbb{R}$ are alternatively considered as a map $\Phi_t : U \times \mathbb{R} \to F$ or as a map $\Phi_t : U \to F$, respectively. For these notions we refer to [13], I.3.

Let $\Omega \subset \mathbb{R}^n$ be bounded and open with C^∞-boundary $\partial\Omega$. In this paper, we restrict ourselves to the case of bounded domains Ω; most results are formulated in a way such that a generalization to uniformly regular domains of class C^∞ in the sense of [9], section 1 or [5], Ch. III, p. 642 is obvious (cf. [36]). For any integer $k \ge 0$ the Sobolev space $H^k(\Omega)$ is equipped with its natural norms (where $|\alpha| = \alpha_1 + \dots + \alpha_n$ for $\alpha \in \mathbb{N}_0^n$)

$$\|u\|_k = (\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u(x)|^2\, dx)^{1/2}, \quad u \in H^k(\Omega). \tag{2.1}$$

The space $H^\infty(\Omega) = \bigcap_{k=0}^\infty H^k(\Omega)$ is a Fréchet space with the norms $(\| \ \|_k)_{k=0}^\infty$. On the algebra $H^\infty(\Omega)$ we can consider sup-norms

$$\|u\|_k^\infty = \sup_{|\alpha|\le k} \sup_{x\in\Omega} |\partial^\alpha u(x)|, \quad u \in H^\infty(\Omega) \tag{2.2}$$

since by Sobolev's imbedding theorem there are constants $c_k > 0$ such that

$$\|u\|_k^\infty \le c_k \|u\|_{k+b}, \quad u \in H^{k+b}(\Omega), \quad b := [n/2] + 1 > n/2. \tag{2.3}$$

The Sobolev space $(H^s(\partial\Omega), \| \ \|_s)$ is defined as usual for a real $s \ge 0$ using a partition of unity (cf. [54], I. 4.2.). In particular, for an integer $k \ge 1$ the space $H^{k-1/2}(\partial\Omega)$ is the class of functions ϕ which are the boundary values of functions $u \in H^k(\Omega)$; the space $H^{k-1/2}(\partial\Omega)$ can be equipped with the equivalent norm

$$\|\phi\|_{k-1/2} = \inf\{\|u\|_k : u \in H^k(\Omega), u = \phi \text{ on } \partial\Omega\}, \phi \in H^{k-1/2}(\partial\Omega). \tag{2.4}$$

The Fréchet space $H^\infty(\partial\Omega) = \bigcap_{k=0}^\infty H^k(\partial\Omega)$ is equipped with these norms.

The Fréchet space $C^\infty(\overline{\Omega})$ of all C^∞-functions on Ω such that all partial derivatives are uniformly continuous on Ω is equipped with the norms $(\| \ \|_k^\infty)_{k=0}^\infty$. The Fréchet space $C^\infty(\partial\Omega)$ of all smooth functions on the manifold $\partial\Omega$ is endowed with the norm system $(\| \ \|_k^\infty)_{k=0}^\infty$ defined as usual using cutoff functions and a partition of unity (cf. [33], 4.14.). It is well known that there exists a linear continuous extension operator $R_\Omega : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ such that $\|R_\Omega f\|_k \le c_k \|f\|_k$ for all k and constants $c_k > 0$; this follows e.g. from [46] using a partition of unity.

A vector valued function $u = (u_1, \dots, u_M)$ belongs to $H^\infty(\Omega, \mathbb{R}^M)$ if each coordinate u_j is in $H^\infty(\Omega)$; the same applies to $H^\infty(\partial\Omega, \mathbb{R}^M)$. We use of the following symbolic calculus introduced in [34]. Let $p, q \ge 0$ be integers, $p + q \ge 1$, and let $E_1, \dots, E_p, F_1, \dots, F_q$ be linear spaces each equipped with a sequence $| \ |_0 \le | \ |_1 \le | \ |_2 \le \dots$ of seminorms. For any integer $m, k \ge 0$ and $x_1 \in E_1, \dots, x_p \in E_p$, $y_1 \in F_1, \dots, y_q \in F_q$ we define

$$[x_1, ..., x_p; y_1, ..., y_q]_{m,k} = \sup\{|x_{k_1}|_{m+i_1} ... |x_{k_r}|_{m+i_r} |y_1|_{m+j_1} ... |y_q|_{m+j_q}\}$$

the 'sup' running over all $i_1, \dots, i_r, j_1, \dots, j_q \ge 0$ and $1 \le k_1, \dots, k_r \le p$ with $0 \le r \le k$ and $i_1 + \dots + i_r + j_1 + \dots + j_q \le k$ (for $r = 0$ the $|x|$-terms are omitted). For $q = 0$ we write $[x_1, \dots, x_p]_{m,k}$ (the $|y|$-terms are omitted) and for $p = 0$ we write $[; y_1, \dots, y_q]_{m,k}$. For $m = 0$ we write $[\dots]_k = [\dots]_{0,k}$. Observe that $[x_1, \dots, x_p; y_1, \dots, y_q]_{m,k}$ is a seminorm seperately in each component y_j while it is completely nonlinear in the x_i-components. The weighted multiseminorms $[\ ]_{m,k}$ are increasing in m and in k. For the purely nonlinear terms (i.e., $q = 0$) we have $[x_1, \dots, x_p]_{m,0} = 1$ and $[x_1, \dots, x_p]_{m,k} \ge 1$ for all m, k. For properties of the terms $[\ ]_{m,k}$ we refer to [34], 1.7.; we shall often apply rules like $[x]_{m,k} \cdot [x]_{m,i} \le [x]_{m,k+i}$ and $[x]_{m,i+k} \le \max\{1, |x|_{m+i}^{i+k}\}[x]_{m+i,k} \le C'[x]_{m+i,k}$ if $|x|_{m+i} \le C$. If Sobolev spaces $H^\infty(\Omega)$ are involved then the following applies. The expressions $[u]_{m,k}$ and $[u; v]_{m,k}$ are defined by the corresponding Sobolev norms $\|u\|_i, \|v\|_j$. The terms $\|u\|_{m,k}^\infty$ or $\|u, v\|_{m,k}^\infty$ (i.e., $p = 2, q = 0$) are defined by sup-norms $\|u\|_i^\infty, \|v\|_j^\infty$. The expression $[u; v]_{m,k}^\infty$ (i.e., $p = q = 1$) is defined by means of the sup-norms $\|u\|_i^\infty$ and Sobolev norms $\|v\|_j$. For a real number t let $[t]$ denote the largest integer j with $j \le t$.

3. A smoothing property for Fréchet spaces

In the inverse function theorem 3.4 the Fréchet spaces are assumed to satisfy smoothing property (S) introduced in [32], 3.4 and property (DN) of Vogt [53]. A Fréchet space E has property (DN) if there is b such that for any n there are k_n and $c_n > 0$ such that for all $x \in E$ we have

$$\|x\|_n^2 \le c_n \|x\|_b \|x\|_{k_n} \tag{3.1}$$

We say that E has smoothing property (S) if there exist $b, p \ge 0$ and constants $c_n > 0$ such that for any $\theta \ge 1$ and any $x \in E$ and for any sequence $(A_n)_n$ satisfying $\|x\|_n \le A_n \le A_{n+1}$ and $A_n^2 \le A_{n-1}A_{n+1}$ for all n there exists an element $S_\theta x \in E$ (which may depend on x and on the sequence (A_n)) such that

$$\begin{aligned} \|S_\theta x\|_n &\le c_n \theta^{n+p-k} A_k, \quad b \le k \le n+p \\ \|x - S_\theta x\|_n &\le c_k \theta^{n+p-k} A_k, \quad k \ge n+p. \end{aligned} \tag{3.2}$$

Smoothing property (S) generalizes (cf. [32]) the classical smoothing operators (cf. [13], [15], [25]). For a Fréchet space E and $T > 0$ we put

$$C_0^\infty([0,T],E) = \Big\{u \in C^\infty([0,T],E) : u^{(j)}(0) = 0, j = 0,1,2,\dots\Big\}. \tag{3.3}$$

In case E is one-dimensional we write $C_0^\infty[0,T]$ instead of $C_0^\infty([0,T],E)$.

Lemma 3.1. *Let $T_1 > 0$. The spaces $C_0^\infty[0,T]$ have property* (S) *with $b = p = 0$ where c_n in* (3.2) *may be chosen uniformly for all $0 < T \le T_1$.*

Proof. The space $\mathcal{D}[0,2]$ of all smooth function with support in $[0,2]$ has property (S) with $b = p = 0$ (cf. [32], 5.1). The space $C_0^\infty[0,1]$ is a quotient space of $\mathcal{D}[0,2]$ by means of restriction and hence a direct summand of $\mathcal{D}[0,2]$ using an extension operator (cf. Seeley [46] or [33], 4.8). Therefore, $C_0^\infty[0,1]$ inherits property (S) from $\mathcal{D}[0,2]$ with $b = p = 0$. To prove uniformity we assume that $T_1 = 1$. We have

$$\|f\|_k^{[0,T]} = \sup_{j=0}^{k} \sup_{t\in[0,T]} |f^{(j)}(t)| = \sup_{t\in[0,T]} |f^{(k)}(t)| =: |f|_k^{[0,T]} \tag{3.4}$$

for $f \in C_0^\infty[0,T]$ and $0 < T \le 1$. Put $\Gamma_T : C_0^\infty[0,1] \to C_0^\infty[0,T], \Gamma_T f(x) = f(x/T)$. Notice that $|\Gamma_T f|_k^{[0,T]} = T^{-k}|f|_k^{[0,1]}$. If S_θ is induced by property (S) in $C_0^\infty[0,1]$ then $\Gamma_T \circ S_{T\theta} \circ \Gamma_T^{-1}$ gives property (S) for $C_0^\infty[0,T]$ with the same constants. □

The uniformity part of Lemma 3.1 does not work e.g. for $C^\infty[0,T]$. For a Fréchet space E and a sequence $0 \le \alpha_0 \le \alpha_1 \le \dots \nearrow +\infty$ we consider the power series space of E-valued sequences $x = (x_j)_{j=1}^\infty \subset E$ defined by

$$\Lambda_\infty^\infty(\alpha;E) = \{(x_j)_j \subset E : \|x\|_k = \sup_{i=0}^{k} \sup_j \|x_j\|_{k-i} e^{i\alpha_j} < \infty, k = 0,1,\dots\}.$$

In case $\dim E = 1$ we write $\Lambda_\infty^\infty(\alpha)$ instead of $\Lambda_\infty^\infty(\alpha;E)$. The corresponding space defined by l^2-norms instead of sup-norms is denoted by $\Lambda_\infty^2(\alpha)$.

Lemma 3.2. *If E has property* (S) *then $\Lambda_\infty^\infty(\alpha;E)$ has* (S) *as well.*

Proof. Let $0 \ne x \in \Lambda_\infty^\infty(\alpha;E)$ and $\|x\|_k \le A_k \le A_{k+1}, A_k^2 \le A_{k-1}A_{k+1}$. We may assume that $t \mapsto \log A_t$ is convex and increasing. We have

$$\|x_j\|_i \le \inf_{i\le k\in\mathbb{N}_0} e^{(i-k)\alpha_j} A_k =: B_i^j \le D_{i+1}^j := \inf_{i+1\le t\in\mathbb{R}} e^{(i+1-t)\alpha_j} A_t \le A_{i+1}$$

for any i, j. It is easy to see that $D^j_{i+1} \le D^j_{i+2}$ and $(D^j_{i+1})^2 \le D^j_i D^j_{i+2}$ for all i, j. We hence may choose $S_\theta x_j$ according to the sequence $(D^j_{i+1})_i$ such that

$$\begin{aligned} \|S_\theta x_j\|_n &\le c_n \theta^{n+p+1-k} D^j_k, \quad b+1 \le k \le n+p+1 \\ \|x_j - S_\theta x_j\|_n &\le c_k \theta^{n+p+1-k} D^j_k, \quad k \ge n+p+1. \end{aligned} \tag{3.5}$$

We define $T_\theta x$ for $\theta \ge 1$ by $(T_\theta x)_j = 0$ if $e^{\alpha_j} \ge \theta$ and $(T_\theta x)_j = S_\theta x_j$ if $e^{\alpha_j} < \theta$. For $e^{\alpha_j} \ge \theta$ we get for $k \ge n+p+1$ and $0 \le i \le n$ the estimate

$$\|x_j\|_{n-i} e^{i\alpha_j} \le e^{(n-k)\alpha_j} A_k \le \theta^{n-k} A_k. \tag{3.6}$$

For $e^{\alpha_j} < \theta$ we establish for $k \ge n+p+1$ and $0 \le i \le n$ the estimate

$$\|x_j - S_\theta x_j\|_{n-i} e^{i\alpha_j} \le c_k \theta^{n-i+p+1-k} e^{i\alpha_j} D^j_k \le c_k \theta^{n+p+1-k} A_k. \tag{3.7}$$

Let $e^{\alpha_j} < \theta$ and $b+1 \le k \le n+p+1$. In the case $0 \le i \le k-b-1$ we get

$$\|S_\theta x_j\|_{n-i} e^{i\alpha_j} \le c_{n-i} D^j_{k-i} \theta^{n+p+1-k} e^{i\alpha_j} \le c_{n-i} \theta^{n+p+1-k} A_k \tag{3.8}$$

and for $k-b-1 \le i \le n$ we obtain (where we may assume that $p \ge b$)

$$\|S_\theta x_j\|_{n-i} e^{i\alpha_j} \le c_{n-i} \theta^{n+p-i-b} D^j_{b+1} e^{i\alpha_j} \le c_{n-i} \theta^{n+p+1-k} A_k \tag{3.9}$$

since $D^j_{b+1} e^{i\alpha_j} \le e^{(i+b+1-k)\alpha_j} A_k \le \theta^{i+b+1-k} A_k$. This gives the result. □

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n, n \ge 2$, be open and bounded with C^∞-boundary. Let $T_1 > 0$ and an integer $m \ge 1$ be fixed. Then the spaces $C_0^\infty([0,T], H^\infty(\Omega))$ and $C_0^\infty([0,T], H^\infty(\partial\Omega))$ equipped with the norms*

$$\|u\|_k = \sup \Big\{ \|u^{(i)}(t)\|_{k-mi} : t \in [0,T], 0 \le i \le k/m \Big\} \tag{3.10}$$

have properties (S), (DN). *In addition, the constants c_n, k_n, b, p in the above definitions of* (S), (DN) *can be chosen uniformly for all $0 < T \le T_1$.*

Proof. Clearly the spaces have (DN); the uniformity statement holds since $C_0^\infty[0,T]$ is a subspace (by trivial extension) of $C^\infty[-1+T, T] \cong C^\infty[0,1]$ if $T \le 1$. It is enough to show property (S) for the spaces equipped with the new norm system $(\| \ \|_{mk})_{k=0}^\infty$ (cf. [31], 4.3). There are tame isomorphisms $H^\infty(\Omega) \cong \Lambda^\infty_\infty(\alpha)$ for $\alpha_j = (\log j)/n$ and $H^\infty(\partial\Omega) \cong \Lambda^\infty_\infty(\beta)$ for $\beta_j = (\log j)/(n-1)$; this is proved in [33], 4.10, 4.14. We put $\tilde\alpha_j = m\alpha_j$ and obtain a tame isomorphism

$$\Big(C_0^\infty([0,T], H^\infty(\Omega)), (\| \ \|_{mk})_{k=0}^\infty \Big) \cong \Lambda^\infty_\infty(\tilde\alpha; C_0^\infty[0,T]). \tag{3.11}$$

The same argument applies to $H^\infty(\partial\Omega)$. Now 3.1, 3.2 give the assertion. □

In section 4 we shall apply the following inverse function theorem of Nash-Moser type which is proved in [34], 4.1 (cf. [13], [15], [25]).

Theorem 3.4. *Let E, F be Fréchet spaces with smoothing property* (S) *and* (DN). *Let $U_0 = \{x \in E : |x|_b < \eta\}$ for some $b \ge 0, \eta > 0$. Let $\Phi : (U_0 \subset E) \to F$ be a C^2-map with $\Phi(0) = 0$ such that $\Phi'(x) : E \to F$ is bijective for all $x \in U_0$. Assume that there are an integer $d \ge 0$ such that*

$$\begin{aligned} \|\Phi'(x)v\|_k &\le c_k [x; v]_{d,k} \\ \|\Phi'(x)^{-1} y\|_k &\le c_k [x; y]_{d,k} \\ \|\Phi''(x)\{v, v\}\|_k &\le c_k [x; v, v]_{d,k} \end{aligned} \tag{3.12}$$

for all $x \in U_0, v \in E, y \in F$ and all $k = 0, 1, 2, \dots$ with constants $c_k > 0$. Then there exist open zero neighbourhoods $V = \{y \in F : \|y\|_s < \delta\} \subset F$ and $U \subset E$ such that $\Phi : U \to V$ is bijective and $\Phi^{-1} : (V \subset F) \to E$ is a C^2-map. If Φ is C^n then Φ^{-1} is C^n as well, $2 \le n \le \infty$. Moreover, the numbers $s \ge 0$ and $\delta > 0$ depend only on the constants in the assumption, i.e., on b, d, η, c_k and on the constants in properties (S), (DN).

4. Linearization of boundary-value problems

Let $\Omega \subset \mathbb{R}^n$ be bounded and open with C^∞-boundary. We fix a real number $T > 0$ and integers $M \ge 1, m \ge 2$. We write $H^\infty(\Omega) = H^\infty(\Omega, \mathbb{R}^M)$ and $H^\infty(\partial\Omega) = H^\infty(\partial\Omega, \mathbb{R}^M)$. We assume that m is even and put $I(m) = \{\alpha \in \mathbb{N}_0^n : |\alpha| \le m\}$. Let $A \subset (\mathbb{R}^M)^{I(m)}$ be open; then the set

$$U_0 = \{u \in H^\infty(\Omega) : \{\partial^\alpha u(x)\}_{|\alpha| \le m} \in A, x \in \overline{\Omega}\} \tag{4.1}$$

is open in $H^\infty(\Omega)$ as well. Let $F \in C^\infty([0,T] \times \overline{\Omega} \times \overline{A}, \mathbb{R}^M), F = F(t, x, u)$. We consider $\mathcal{F} : [0,T] \times (U_0 \subset H^\infty(\Omega)) \to H^\infty(\Omega)$ defined by

$$\mathcal{F}(t,u)(x) = F(t, x, \{\partial^\alpha u(x)\}_{|\alpha| \le m}), \quad u \in U_0, t \in [0,T], x \in \Omega. \tag{4.2}$$

It is proved in [36], section 2 (cf. [15]) that $\mathcal{F}$ is a nonlinear C^∞-map between Fréchet spaces where $\mathcal{F}' : ([0,T] \times U_0) \times (\mathbb{R} \times H^\infty(\Omega)) \to H^\infty(\Omega)$ is given by $\mathcal{F}'(t,u)(s,v) = \mathcal{F}_t(t,u)s + \mathcal{F}_u(t,u)v$ where

$$\mathcal{F}_u(t,u)v = \sum_{|\alpha| \le m} F_{\partial^\alpha u}(t, \cdot, \{\partial^\beta u(\cdot)\}_{|\beta| \le m}) \partial^\alpha v. \tag{4.3}$$

If A is bounded then [36], 2.3, 2.4 give with $b = [n/2] + 1$ the estimates

$$\begin{gathered} \|\mathcal{F}'(t,u)(s,v)\|_k \le c_k[(t,u); (s,v)]_{m+b,k} \\ \|\mathcal{F}''(t,u)\{(s,v),(s,v)\}\|_k \le c_k[(t,u); (s,v),(s,v)]_{m+b,k} \end{gathered} \tag{4.4}$$

for all $t \in [0,T], u \in U_0, s \in \mathbb{R}, v \in H^\infty(\Omega)$ where $c_k > 0$ are constants.

We define nonlinear boundary operators $\mathcal{B}_j$ and put $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_{m/2})$. For that we fix integers $m_j \ge 0$ and choose open sets $A_j \subset (\mathbb{R}^M)^{I(m_j)}$ and mappings $B_j \in C^\infty([0,T] \times \partial\Omega \times \overline{A}_j, \mathbb{R}^M), j = 1, \dots, m/2$. Then the sets

$$U_j = \{u \in H^\infty(\Omega) : \{\partial^\beta u(x)\}_{|\beta| \le m_j} \in A_j, x \in \partial\Omega\} \tag{4.5}$$

are open in $H^\infty(\Omega)$. We define $\mathcal{B}_j : [0,T] \times (U_j \subset H^\infty(\Omega)) \to H^\infty(\partial\Omega)$ by

$$\mathcal{B}_j(t,u)(x) = B_j(t, x, \{\partial^\beta u(x)\}_{|\beta| \le m_j}), \; u \in U_j, t \in [0,T], x \in \partial\Omega. \tag{4.6}$$

The arguments used in [36], section 2 or [15] show that $\mathcal{B}_j$ is a C^∞-map between Fréchet spaces where $\mathcal{B}_j' : ([0,T] \times U_j) \times (\mathbb{R} \times H^\infty(\Omega)) \to H^\infty(\partial\Omega)$ is given by $\mathcal{B}_j'(t,u)(s,v) = (\mathcal{B}_j)_t(t,u)s + (\mathcal{B}_j)_u(t,u)v$ where

$$(\mathcal{B}_j)_u(t,u)v = \sum_{|\alpha| \le m_j} (B_j)_{\partial^\alpha u}(t, \cdot, \{\partial^\beta u(\cdot)\}) \partial^\alpha v. \tag{4.7}$$

For a bounded set A_j the proof of [36], 2.3, 2.4 yields

$$\begin{gathered} \|\mathcal{B}_j'(t,u)(s,v)\|_{k-\frac12} \le c_k[(t,u); (s,v)]_{m_j+b,k} \\ \|\mathcal{B}_j''(t,u)\{(s,v),(s,v)\}\|_{k-\frac12} \le c_k[(t,u); (s,v),(s,v)]_{m_j+b,k} \end{gathered} \tag{4.8}$$

for all $t \in [0,T], u \in U_j, s \in \mathbb{R}, v \in H^\infty(\Omega)$ with some $c_k > 0$ and b as above. Our goal is to solve the nonlinear initial boundary-value problem

$$\begin{aligned} u_t &= \mathcal{F}(t,u) \quad \text{in } \Omega,\ t \in [0,T_0] \\ \mathcal{B}(t,u) &= h(t) \quad \text{on } \partial\Omega,\ t \in [0,T_0] \\ u(0) &= \phi\,. \end{aligned} \tag{4.9}$$

More precisely, for a given initial value $\phi \in U := \bigcap_{j=0}^{m/2} U_j \subset H^\infty(\Omega)$ and a given boundary value $h \in C^\infty([0,T], H^\infty(\partial\Omega)^{m/2})$ we are looking for a solution u of problem (4.9) for some suitable small $T_0 > 0$; by a solution we mean a function $u \in C^\infty([0,T_0], H^\infty(\Omega))$ such that $u(t) \in U$ for all $t \in [0,T_0]$ and (4.9) is satisfied.

There are some natural necessary constraints on the given data h, ϕ. In order such that (4.9) can admit a smooth solution the data h, ϕ have to satisfy the following necessary compatibility conditions which are obtained by computing $h^j(0)$ as a differential operator acting on ϕ on the boundary of Ω by means of (4.9). For instance, we get $h(0) = \mathcal{B}(0,\phi) =: \Gamma_0(\phi)$ and

$$h'(0) = \mathcal{B}_t(0,\phi) + \mathcal{B}_u(0,\phi)\mathcal{F}(0,\phi) =: \Gamma_1(\phi). \tag{4.10}$$

In a similar way we obtain from (4.9) the necessary compatibility conditions

$$h^{(j)}(0) = \partial_t^j \mathcal{B}(t,u(t))_{|t=0} =: \Gamma_j(\phi) \quad \text{on } \partial\Omega, \quad j = 0,1,2,\dots \tag{4.11}$$

where differential operators Γ_j acting on ϕ on $\partial\Omega$ are obtained by first computing $\partial_t^j \mathcal{B}(t,u(t))$, then replacing all derivatives $\partial_t^i u$ by terms involving u using $u_t = \mathcal{F}(t,u(t))$ and finally evaluating at $t = 0$ using $u(0) = \phi$. Analogously, the values $u^j(0)$ are a priori determined in Ω by (4.9). For instance, we get $u(0) = \phi =: \Psi_0(\phi)$ and $u'(0) = \mathcal{F}(0,\phi) =: \Psi_1(\phi)$ and

$$u''(0) = \mathcal{F}_t(0,\phi) + \mathcal{F}_u(0,\phi)\mathcal{F}(0,\phi) =: \Psi_2(\phi). \tag{4.12}$$

Using the first and the third equation in (4.9) we see that solutions u satisfy

$$u^{(j)}(0) = \partial_t^{j-1} \mathcal{F}(t,u(t))_{|t=0} =: \Psi_j(\phi) \quad \text{in } \Omega,\ j = 1,2,3,\dots \tag{4.13}$$

with differential operators Ψ_j acting on ϕ in Ω where Ψ_j are defined using $u_t = \mathcal{F}(t,u)$ and $u(0) = \phi$. We note that (4.13) are by no means compatibility conditions like (4.11). However, the a priori knowledge of $u^{(j)}(0)$ can be used to transform problem (4.9) such that solutions v of the transformed problem satisfy $v^j(0) = 0$ for all j. This simplifies the compatibility conditions (4.11). We shall apply the following version of a theorem of E. Borel's [8].

Lemma 4.1. *Let E be a Fréchet space. Let $(a_j)_{j=0}^\infty \subset E$ be an arbitrary sequence. Then there is $\psi \in C^\infty([0,1], E)$ such that $\psi^{(j)}(0) = a_j$ for all j.*

The proof of this lemma follows the standard proof in (cf. [14], 1.2.6 or [30], 1.3).

We choose $\psi \in C^\infty([0,T], H^\infty(\Omega))$ such that $\psi(t) \in U$ for $t \in [0,T]$ and

$$\psi^{(j)}(0) = \Psi_j(\phi), \quad j = 0,1,2,\dots \tag{4.14}$$

We put $v = u - \psi$ and get from (4.9) the transformed problem

$$\begin{aligned} v_t &= \mathcal{F}(t, v+\psi) - \psi'(t) \quad \text{in } \Omega,\ t \in [0,T_0] \\ \mathcal{B}(t,v+\psi) - \mathcal{B}(t,\psi) &= h(t) - \mathcal{B}(t,\psi) \quad \text{on } \partial\Omega,\ t \in [0,T_0] \\ v(0) &= 0\,. \end{aligned} \tag{4.15}$$

Remark 4.2. (i) If u solves (4.9) then $v = u - \psi$ solves (4.15). On the other hand, if v solves (4.15) then $u = v + \psi$ solves (4.9). (ii) Solutions u of (4.9) satisfy $u^{(j)}(0) = \psi^{(j)}(0)$ for all j. On the other hand, solutions v of (4.15) automatically satisfy $v^{(j)}(0) = 0$ for all j. (iii) For $\gamma(t) = \mathcal{B}(t, \psi(t))$ we have $\gamma^{(j)}(0) = \Gamma_j(\phi)$ for all j. (iv) If h satisfys $h^{(j)}(0) = \Gamma_j(\phi)$ for all j then the right hand side $\tilde{h}(t) = h(t) - \mathcal{B}(t, \psi(t))$ in (4.15) satisfies $\tilde{h}^{(j)}(0) = 0$ for all j. (v) The left hand side $\tilde{\mathcal{B}}(t, v) = \mathcal{B}(t, v + \psi(t)) - \mathcal{B}(t, \psi(t))$ considered in (4.15) as an operator in v satisfies $(\partial_t^j \tilde{\mathcal{B}})(0, 0) = 0$ for all j. Note that

$$(\partial_t \tilde{\mathcal{B}})(t, v) = \mathcal{B}_t(t, v + \psi) + \mathcal{B}_u(t, v + \psi)\psi' - \mathcal{B}_t(t, \psi) - \mathcal{B}_u(t, \psi)\psi'.$$

(vi) In the case of linear boundary conditions we have $\tilde{\mathcal{B}}(t, v) = \mathcal{B}(t, v)$. (vii) The right hand side $\tilde{\mathcal{F}}(t, v) = \mathcal{F}(t, v + \psi(t)) - \psi'(t)$ in (4.15) considered as a nonlinear differential operator in v satisfies $(\partial_t^j \tilde{\mathcal{F}})(0, 0) = 0$ for all j. This follows since $\tilde{\mathcal{F}}(0, 0) = \mathcal{F}(0, \phi) - \psi'(0) = 0$ and

$$(\partial_t^j \tilde{\mathcal{F}})(0, 0) = \partial_t^j \{\mathcal{F}(t, \psi(t))\}_{t=0} - \psi^{(j+1)}(0) = \Psi_{j+1}(\phi) - \psi^{(j+1)}(0) = 0.$$

Using the above notation we hence may consider the normalized problem

$$\begin{gathered} u_t = \tilde{\mathcal{F}}(t, u) \quad \text{in } \Omega,\ t \in [0, T_0] \\ \tilde{\mathcal{B}}(t, u) = \tilde{h}(t) \quad \text{on } \partial\Omega,\ t \in [0, T_0] \\ u(0) = 0\,, \end{gathered} \tag{4.16}$$

where we may assume the normalized conditions

$$\begin{gathered} (\partial_t^j \tilde{\mathcal{F}})(0, 0) = 0, \quad j = 0, 1, 2, \dots \\ (\partial_t^j \tilde{\mathcal{B}})(0, 0) = 0, \quad j = 0, 1, 2, \dots \\ \tilde{h}^{(j)}(0) = 0, \quad j = 0, 1, 2, \dots \end{gathered} \tag{4.17}$$

where $\tilde{h}^{(j)}(0) = 0$ are the natural compatibility conditions for (4.16) if we assume the first two conditions in (4.17). Since solutions u of (4.16), (4.17) satisfy $u^{(j)}(0) = 0$ for all j we have to look for solutions u in the space $C_0^\infty([0, T_0], H^\infty(\Omega))$. We formulate problem (4.16) by a mapping. We fix $T > 0$ and put $J = [0, T]$. We get an open set $W \subset C_0^\infty(J, H^\infty(\Omega))$ by

$$W = \{u \in C_0^\infty(J, H^\infty(\Omega)) : u(t) \in U, t \in J\} \tag{4.18}$$

where we may assume that $0 \in W$. We define the nonlinear map

$$\Phi : (W \subset C_0^\infty(J, H^\infty(\Omega))) \to C_0^\infty(J, H^\infty(\Omega)) \times C_0^\infty(J, H^\infty(\partial\Omega)) \tag{4.19}$$

by

$$\Phi(u) = (\partial_t u - \tilde{\mathcal{F}}(t, u) + \tilde{\mathcal{F}}(t, 0), \tilde{\mathcal{B}}(t, u) - \tilde{\mathcal{B}}(t, 0)). \tag{4.20}$$

We note that Φ is well defined since $\partial_t^j \{u_t - \tilde{\mathcal{F}}(t, u(t)) + \tilde{\mathcal{F}}(t, 0)\}(0) = 0$ and $\partial_t^j \{\tilde{\mathcal{B}}(t, u(t)) - \tilde{\mathcal{B}}(t, 0)\}(0) = 0$ for all j and every $u \in W$ in view of (4.17). The map Φ is a C^2-map satisfying $\Phi(0, 0) = 0$ where the first derivative is

$$\Phi'(u)v = (\partial_t v - \tilde{\mathcal{F}}_u(t, u)v, \tilde{\mathcal{B}}_u(t, u)v) \tag{4.21}$$

For a fixed $T_1 > 0$ the first and third estimate in (3.12) hold with uniform constants for $0 < T \le T_1$ where the norms are defined by (3.10). This follows from the proof of [36], 4.3 using (4.4), (4.8). We consider the equation

$$\Phi(u) = (\tilde{\mathcal{F}}(t, 0), \tilde{h}(t) - \tilde{\mathcal{B}}(t, 0)). \tag{4.22}$$

The inverse function theorem 3.4 requires the smallness condition

$$\|\tilde{\mathcal{F}}(t,0)\|_s + \|\tilde{\mathcal{B}}(t,0)\|_s + \|\tilde{h(t)}\|_s < \delta. \tag{4.23}$$

By (4.17) condition (4.23) holds if $T > 0$ is chosen sufficiently small. We here shall have to observe that s, δ in Theorem 3.4 can be chosen uniformly for all $0 < T \leq T_1$. We consider the nonlinear problem (4.9) for some given initial value $\phi \in U$ and $h \in C^\infty([0,T_1], H^\infty(\partial\Omega))$. We assume that the compatibility conditions (4.11) hold.

Theorem 4.3. *Let $T_1 > 0, \phi \in H^\infty(\Omega)$ and $h \in C^\infty([0,T_1]), H^\infty(\partial\Omega))$ satisfy (4.11). Assume that there are $b \geq 0$ and $c_k > 0$ and an open neighbourhood U of ϕ in $H^\infty(\Omega)$ so that for any $0 < T \leq T_1$ and $u \in W = \{w \in C^\infty([0,T], H^\infty(\Omega)) : w(t) \in U, t \in [0,T]\}$ the linear problem*

$$\begin{aligned} z_t(t) = \mathcal{F}_u(t,u(t))z(t) + f(t) \quad &\text{in } \Omega,\ t \in [0,T] \\ \mathcal{B}_u(t,u(t))z(t) = g(t) \quad &\text{on } \partial\Omega,\ t \in [0,T] \\ z(0) = 0& \end{aligned} \tag{4.24}$$

admits for any $f \in C_0^\infty([0,T], H^\infty(\Omega))$ and $g \in C_0^\infty([0,T], H^\infty(\partial\Omega)^{m/2})$ a unique solution $z \in C_0^\infty([0,T], H^\infty(\Omega))$ satisfying the estimates

$$\|z\|_k \leq c_k[u;(f,g)]_{b,k}, \quad k = 0,1,2,\dots \tag{4.25}$$

Then (4.9) has a unique solution $u \in C^\infty([0,T_0], H^\infty(\Omega))$ for some $T_0 > 0$.

Proof. We choose $\psi \in C^\infty([0,T_1], H^\infty(\Omega))$ satisfying (4.14) such that $\psi(t) \in U$ for all $t \in [0,T_1]$. By remark 4.2 (i) it is enough to solve problem (4.15) for $v = u - \psi$. For that we define Φ by (4.22) where $\Phi(0,0) = 0$ and $\tilde{\mathcal{F}}, \tilde{\mathcal{B}}, \tilde{h}$ are defined as in Remark 4.2 satisfying (4.17). We have to solve equation (4.22). By our assumption on the linear problem (4.24) the operator $\Phi'(u)$ is bijective for all u in some zero neighbourhood in $C_0^\infty([0,T], H^\infty(\Omega)), 0 < T \leq T_1$. The inequalities (4.25) yield the second estimate in (3.12) while the first and third estimate in (3.12) hold as observed above. The assumptions of Theorem 3.4 on the spaces are satisfied by Proposition 3.3. Hence there exist numbers $s \geq 0$ and $\delta > 0$ as in Theorem 3.4 which can be chosen uniformly for all $0 < T \leq T_1$. We can choose $T_0 > 0$ so small such that the smallness condition (4.23) holds in $[0,T_0]$. Theorem 3.4 gives a solution $v \in C_0^\infty([0,T_0], H^\infty(\Omega))$ of problem (4.15) and thus a solution $u = v + \psi$ of problem (4.9). The uniqueness can be shown using Theorem 3.4 and a standard argument as in [36], Theorem 4.4. This gives the result. □

Problem (4.24) can be reduced to a problem with homogeneous boundary conditions provided that the boundary conditions can be solved. Let $T_1 > 0$ and choose U, W as in Theorem 4.3. We assume there exist $b \geq 0$ and $c_k > 0$ such that for any $u \in W$ and $0 < T \leq T_1$ there exists a map

$$R_u : C_0^\infty([0,T], H^\infty(\partial\Omega)^{\frac{m}{2}}) \to C_0^\infty([0,T], H^\infty(\Omega)), B_u(\cdot,u)R_u = \mathrm{Id} \tag{4.26}$$

which satisfies for all $g \in C_0^\infty([0,T], H^\infty(\partial\Omega)^{m/2})$ the estimates

$$\|R_u g\|_k \leq c_k[u;g]_{b,k}, \quad k = 0,1,2,\dots \tag{4.27}$$

In section 5 we show that such R_u exist for normal boundary conditions.

Theorem 4.4. *Let $T_1 > 0, \phi \in H^\infty(\Omega), h \in C^\infty([0,T_1]), H^\infty(\partial\Omega))$ satisfy* (4.11). *Assume that there are $b \geq 0$ and $c_k > 0$ and open sets U, W as in Theorem* 4.3 *so that for any $0 < T \leq T_1$ and $u \in W$ there exist R_u satisfying* (4.26), (4.27) *such that for any $f_1 \in C_0^\infty([0,T], H^\infty(\Omega))$ the problem*

$$\begin{gathered} w_t(t) = \mathcal{F}_u(t,u(t))w(t) + f_1(t) \quad \text{in } \Omega,\ t \in [0,T] \\ \mathcal{B}_u(t,u(t))w(t) = 0 \quad \text{on } \partial\Omega,\ t \in [0,T] \\ w(0) = 0 \end{gathered} \tag{4.28}$$

admits a unique solution $w \in C_0^\infty([0,T], H^\infty(\Omega))$ satisfying the estimates

$$\|w\|_k \leq c_k[u; f_1]_{b,k}, \quad k = 0,1,2,\dots \tag{4.29}$$

Then (4.9) *has a unique solution $u \in C^\infty([0,T_0], H^\infty(\Omega))$ for some $T_0 > 0$.*

Proof. Let $f \in C_0^\infty([0,T], H^\infty(\Omega))$ and $g \in C_0^\infty([0,T], H^\infty(\partial\Omega)^{m/2})$. We choose $v = R_u g$ satisfying (4.27). We put $f_1(t) = f(t) - v_t(t) + \mathcal{F}_u(t,u(t))v(t)$. Then $f_1 \in C_0^\infty([0,T], H^\infty(\Omega))$. By assumption we find a solution $w \in C_0^\infty([0,T], H^\infty(\Omega))$ of (4.28) satisfying (4.29). Then $z = v + w$ is a solution of (4.24) satisfying (4.25). The solution is unique by means of the unique solvability of (4.28). Hence Theorem 4.3 gives the result. □

5. Normal boundary conditions

In this section we are concerned with normal boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞-boundary. Let $\{B_j\}_{j=1}^p$ be a set of differential operators $B_j = B_j(x,\partial)$ of order m_j given by

$$B_j = B_j(x,\partial) = \sum_{|\beta| \leq m_j} b_{j,\beta}(x)\partial^\beta, \qquad j = 1,\dots,p \tag{5.1}$$

with $b_{j,\beta} \in C^\infty(\partial\Omega)$. There is a linear extension operator $S : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ satisfying $\|Sf\|_k \leq c_k\|f\|_k$ for all k, f with constants $c_k > 0$; this follows from Seeley [46] using a partition of unity (cf. [33]). We hence may assume that $b_{j,\beta} \in C^\infty(\overline{\Omega})$. The set $\{B_j\}_{j=1}^p$ is called normal (cf. [24], [44], [54]) if $m_j \neq m_i$ for $j \neq i$ and if for any $x \in \partial\Omega$ we have $B_j^P(x,\nu) \neq 0, j = 1,\dots,p$ where $\nu = \nu(x)$ denotes the inward normal vector to $\partial\Omega$ at x and B_j^P denotes the principal part of B_j. A normal set $\{B_j\}_{j=1}^p$ is called a Dirichlet system if $m_j = j-1, j = 1,\dots,p$. We can consider the Dirichlet boundary conditions $u \mapsto \frac{\partial^{j-1}u}{\partial\nu^{j-1}}{}_{|\partial\Omega}, j = 1,\dots,p$, which give for any $k \geq p$ a trace operator

$$T_k^p : H^k(\Omega) \to \prod_{i=1}^p H^{k-i+1/2}(\partial\Omega), \quad T_k^p u = \Big\{\Big(\frac{\partial^{j-1}u}{\partial\nu^{j-1}}\Big)_{|\partial\Omega}\Big\}_{j=1}^p. \tag{5.2}$$

The trace operators T_k^p are surjective admitting a continuous linear right inverse Z_k^p which depends on k (cf. [24], [54]). To construct a tame linear right inverse for the induced trace operator $T^p : H^\infty(\Omega) \to H^\infty(\partial\Omega)^p$ we apply tame splitting theory in Fréchet spaces developed by Vogt (cf. [52]).

Let $(F_k)_k, (G_k)_k$ be families of Hilbert spaces with injective linear continuous imbeddings $F_{k+1} \hookrightarrow F_k, G_{k+1} \hookrightarrow G_k$ for all k. Let $T_k : F_k \to G_k$ be surjective continuous linear maps such that $(T_k)_{|F_{k+1}} = T_{k+1}$ for all k. Let $E_k = N(T_k) \subset F_k$ denote the kernel of T_k; we have $E_{k+1} \hookrightarrow E_k$ and

$$0 \longrightarrow E_k \hookrightarrow F_k \xrightarrow{T_k} G_k \longrightarrow 0 \tag{5.3}$$

are exact sequences of Hilbert spaces. We equip the Fréchet spaces $E = \bigcap_k E_k, F = \bigcap_k F_k, G = \bigcap_k G_k$ with the induced norms. We then have a mapping $T : F \to G$ defined by $Tx = T_k x, x \in F$ where $N(T) = E$. The following splitting theorem is a simplified version of [40], 6.1, 6.2.

Lemma 5.1. *Let E_k, F_k, G_k, T_k and E, F, G, T be as above where* (5.3) *is an exact sequence of Hilbert spaces for every k. Assume that there are tame isomorphisms $E \cong \Lambda^2_\infty(\alpha)$ and $G \cong \Lambda^2_\infty(\beta)$ for some α, β. Then*

$$0 \longrightarrow E \hookrightarrow F \xrightarrow{T} G \longrightarrow 0 \tag{5.4}$$

is an exact sequence of Fréchet spaces which splits tamely, i.e., there is a tame linear map $Z : G \to F$ such that $T \circ Z = \mathrm{Id}_G$.

Lemma 5.2. *Let $p \geq 1$. The trace operator $T^p : H^\infty(\Omega) \to H^\infty(\partial\Omega)^p$ admits a tame linear right inverse $Z^p : H^\infty(\partial\Omega)^p \to H^\infty(\Omega)$, $T^p \circ Z^p = \mathrm{Id}$.*

Proof. The trace operator T^p_k induces for $k \geq p$ an exact sequences

$$0 \longrightarrow N(T^p_k) \hookrightarrow H^k(\Omega) \xrightarrow{T^p_k} \prod_{i=1}^{p} H^{k-i+1/2}(\partial\Omega) \longrightarrow 0 \tag{5.5}$$

of Hilbert spaces. We show using Lemma 5.1 that the sequence

$$0 \longrightarrow N(T^p) \hookrightarrow H^\infty(\Omega) \xrightarrow{T^p} H^\infty(\partial\Omega)^p \longrightarrow 0 \tag{5.6}$$

of Fréchet spaces splits tamely. Note that there are tame isomorphisms $H^\infty(\Omega) \cong \Lambda^2_\infty(\alpha)$ and $H^\infty(\partial\Omega)^p \cong \Lambda^2_\infty(\beta)$ (cf. the proof of Proposition 3.3). Let Δ denote the Laplacian. We consider Δ^p as an unbounded operator in $L^2(\Omega)$ under null Dirichlet boundary conditions, the domain given by $D_p = N(T^p_{2p}) = \{u \in H^{2p}(\Omega) : T^p_{2p} u = 0\}$. It is well known that the spectrum of Δ^p is discrete (cf. [9], Theorem 17, [2], Theorem 2.1). We thus can choose λ such that $\Delta^p - \lambda$ is an isomorphism $D_p \to L^2(\Omega)$. Therefore, $\Delta^p - \lambda : N(T^p) \to H^\infty(\Omega)$ is an isomorphism (cf. [54]) which is a tame isomorphism by means of classical elliptic a priori estimates (cf. [3], Theorem 15.2). Hence $N(T^p) \cong H^\infty(\Omega) \cong \Lambda^2_\infty(\alpha)$ tamely isomorphic. By Lemma 5.1 the sequence (5.6) splits tamely. This gives the result. ☐

For a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$ with $a_\alpha \in C^\infty$ we put

$$\|P\|_i = \textstyle\sum_{|\alpha| \leq m} \|a_\alpha\|^\infty_i, \quad i = 0, 1, \dots \tag{5.7}$$

For P as above and $Q = \sum_{|\beta| \leq n} b_\beta(x)\partial^\beta$ we get

$$\|PQ\|_i \leq C_i \sum_{j=0}^{i} \|P\|_{i-j} \|Q\|_{m+j} \tag{5.8}$$

with constants $C_i > 0$. For smooth nonvanishing functions f we get with $C_i > 0$ depending only on i, m, n and on $\|1/f\|^\infty_0$ the estimates

$$\|1/f\|^\infty_i \leq C_i [f]_i, \quad \|P/f\|_i \leq C_i [f; P]_i \tag{5.9}$$

for all i. Here the expressions $[f]_i$ and $[f;P]_i$ are defined by the norms $\|f\|_j^\infty$ and (5.7). To prove a generalization of Lemma 5.2 to normal boundary conditions we first consider the case of a half space. We consider the cubes

$$\Sigma = \{x \in \mathbb{R}^n : |x_i| < 1 \ (i = 1, \dots, n), \ x_n > 0\}. \tag{5.10}$$

$$\sigma = \{x \in \mathbb{R}^n : |x_i| < 1 \ (i = 1, \dots, n), \ x_n = 0\}. \tag{5.11}$$

The following lemma is well known and is due to [7] (see [24, 43, 44, 51, 54]); we prove additional estimates which are important for our purposes. In the following lemma we consider smooth function on $\overline{\Sigma}$.

Lemma 5.3. *Let $\{B_j\}_{j=1}^p$ and $\{B_j'\}_{j=1}^p$ be two Dirichlet systems on $\overline{\sigma}$. Then there exist smooth differential operators $\Lambda_{kj}, 1 \le j \le k \le p$, of order $k-j$ containing only tangential derivatives $\partial_1, \dots, \partial_{n-1}$ such that*

$$B_k' = \sum_{j=1}^k \Lambda_{kj} B_j, \quad k = 1, \dots, p, \quad on \quad \overline{\sigma} \tag{5.12}$$

where Λ_{kk} is a function which vanishes nowhere on $\overline{\sigma}$. In addition, we have

$$\|\Lambda_{kj}\|_i \le C[B_j, \dots, B_k; B_k']_{i+k-j}, \quad 1 \le j \le k, \ i = 0, 1, \dots \tag{5.13}$$

with some constant $C > 0$ depending only on i, n, k and on $\|1/\sigma_k\|_0^\infty$ where σ_k is the nonvanishing coefficient of the term ∂_n^{k-1} in B_k.

Proof. Let first $B_k' = \partial_n^{k-1}$. We assume that $B_k = \sum_{j=1}^k \Gamma_{kj} \partial_n^{j-1}$ where Γ_{kj} has order $k-j$ and Γ_{kk} is a function not vanishing on $\overline{\sigma}$. We have

$$\partial_n^{k-1} = \Gamma_{kk}^{-1} B_k - \Gamma_{kk}^{-1} \sum_{j=1}^{k-1} \Gamma_{kj} \partial_n^{j-1} = \sum_{j=1}^k \Lambda_{kj} B_j \tag{5.14}$$

where $\Lambda_{kk} = \Gamma_{kk}^{-1}$ and $\Lambda_{kj} = -\Gamma_{kk}^{-1} \sum_{l=j}^{k-1} \Gamma_{kl} \Lambda_{lj}, j < k$. For $j < k$ we get

$$\|\Lambda_{kj}\|_i \le C \sum_{l=j}^{k-1} \sum_{m=0}^{i} [B_k]_{i-m} \|\Gamma_{kl} \Lambda_{lj}\|_m \le C \sum_{l=j}^{k-1} [B_k; \Lambda_{lj}]_{i+k-l} \tag{5.15}$$

from (5.8), (5.9) and $\|\Lambda_{kk}\|_i \le C[B_k]_i$. By induction we see that

$$\|\Lambda_{kj}\|_i \le C[B_j, \dots, B_k]_{i+k-j} \tag{5.16}$$

for all k. In the general case we may write

$$B_l' = \sum_{k=1}^l \Psi_{lk} \partial_n^{k-1} = \sum_{k=1}^l \sum_{j=1}^k \Psi_{lk} \Lambda_{kj} B_j = \sum_{j=1}^l \Phi_{lj} B_j, \quad l = 1, \dots, p \tag{5.17}$$

where Ψ_{lk} are tangential operators of order $l-k$ and Ψ_{ll} does not vanish on $\overline{\sigma}$; here Λ_{kj} as above and $\Phi_{lj} = \sum_{k=j}^l \Psi_{lk} \Lambda_{kj}$. From (5.8), (5.16) we get

$$\|\Phi_{lj}\|_i \le C \sum_{k=j}^l \sum_{m=0}^i \|\Psi_{lk}\|_{i-m} \|\Lambda_{kj}\|_{l-k+m} \le C[B_j, \dots, B_l; B_l']_{i+l-j} \tag{5.18}$$

which proves the result. □

The assertion of Lemma 5.3 is invariant w.r.t. normal coordinate transformations (cf. [54]). In Theorem 5.4 we follow [54], Theorem 14.1.

Theorem 5.4. *Let $\{B_j\}_{j=1}^p$ be a smooth normal system. Then there exists a linear map $R : H^\infty(\partial\Omega)^p \to H^\infty(\Omega)$ such that $B_j Rg = g_j$ for $j = 1, \dots, p$ and any $g = \{g_j\}_{j=1}^p \in H^\infty(\partial\Omega)^p$. There is $b \geq 0$ such that*

$$\|Rg\|_k \leq C \sum_{j=1}^{p} [B_j, \dots, B_p; g_j]_{b,k}, \quad k = 0, 1, 2, \dots \tag{5.19}$$

for all $g \in H^\infty(\partial\Omega)^p$ where C depends only on k, p, n, Ω and on

$$\sup\{|B_j^P(x, \nu(x))| + |B_j^P(x, \nu(x))|^{-1} : x \in \partial\Omega,\ j = 1, \dots, p\}. \tag{5.20}$$

Proof. We may assume that $\{B_j\}_{j=1}^p$ is a Dirichlet system. We choose for $x \in \partial\Omega$ an open neighbourhood U_x in $\mathbb{R}^n$ and a normal diffeomorphism $U_x \leftrightarrow \{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \dots, n\}$ where $U_x \cap \Omega \leftrightarrow \Sigma$, $U_x \cap \partial\Omega \leftrightarrow \sigma$, $U_x \cap \overline{\Omega}^c \leftrightarrow -\Sigma$. We cover $\partial\Omega$ by finitely many open sets $U_i = U_{x_i}$ and choose a subordinate partition of unity α_i. We write $D_j = \partial^{j-1}/\partial\nu^{j-1}$. By Lemma 5.3 we get on $\overline{U_i} \cap \partial\Omega$ for $j = 1, \dots, p$ a representation

$$B_j = \sum_{l=1}^{j} \Lambda_{jl}^i D_l, \quad D_j = \sum_{l=1}^{j} \Phi_{jl}^i B_l, \quad \sum_{l=m}^{j} \Lambda_{jl}^i \Phi_{lm}^i = \delta_{jm} \tag{5.21}$$

where $\Lambda_{jl}^i, \Phi_{jl}^i$ are tangential differential operators of order $j - l$ and $\Lambda_{jj}^i, \Phi_{jj}^i$ do not vanish on $\overline{U_i} \cap \partial\Omega$. Note that (5.16) holds for Φ_{jl}. We choose smooth functions β_i such that $\beta_i = 1$ on an open neighbourhood V_i of $\operatorname{supp}\alpha_i$ and $\operatorname{supp}\beta_i \subset U_i$. Let Z^p be the extension operator from Lemma 5.2. We define

$$Rg = \sum_i \beta_i Z^p \{\sum_{l=1}^{j} \Phi_{jl}^i(\alpha_i g_l)\}_{j=1}^p. \tag{5.22}$$

Let $v_i = \beta_i Z^p(w_j^i)$ and $w_j^i = \sum \Phi_{jl}^i(\alpha_i g_l)$ as in (5.22). We claim that $D_j v_i = w_j^i$ on $\overline{U_i} \cap \partial\Omega$. This holds on $V_i \cap \partial\Omega$ by Lemma 5.2. On $(\overline{U_i} \setminus V_i) \cap \partial\Omega$ all derivatives of order $\leq p-1$ of $Z^p(w_j^i)$ vanish; the normal derivatives are $w_j^i = 0$ and the tangential derivatives vanish since $Z^p(w_j^i) = 0$ on $(\overline{U_i} \setminus V_i) \cap \partial\Omega$. Thus $D_j v_i = 0 = w_j^i$ on this set. We obtain

$$B_j Rg = \sum_i B_j(v_i) = \sum_i \sum_{l=1}^{j} \Lambda_{jl}^i w_l^i = \sum_i \sum_{m=1}^{j} \sum_{l=m}^{j} \Lambda_{jl}^i \Phi_{lm}^i(\alpha_i g_m) = g_j.$$

By Lemma 5.2 we have $\|Z^p\{g_j\}\|_k \leq \sum \|g_j\|_{k+a}$ for some $a \geq 0$. We get

$$\begin{aligned}
\|Rg\|_k &\leq C \sum_i \sum_{j=1}^{p} \sum_{l=1}^{j} \sum_{m=0}^{k+a} \|\Phi_{jl}^i\|_m \|\alpha_i g_l\|_{k+j+a-l-m} \\
&\leq C' \sum_i \sum_{j=1}^{p} \sum_{l=1}^{j} \sum_{m=0}^{k+a} [B_l, \dots, B_j]_{m+j-l} \|g_l\|_{k+j+a-l-m} \\
&\leq C'' \sum_{l=1}^{p} [B_l, \dots, B_p; g_l]_{p+a-1,k}
\end{aligned}$$

which gives the result where $b = p + a - 1$. □

Theorem 4.4 requires a parameter depending version of Theorem 5.4. Let $T_1 > 0, p \geq 1, m \geq 2$ and assume that $B_j = B_j(t) = B_j(t,x,\partial), j = 1,\dots,p$ have C^∞-coefficients $b_{j,\beta} \in C^\infty([0,T_1], C^\infty(\overline{\Omega}))$. We equip the space $C_0^\infty([0,T],E)$ for $E = H^\infty(\Omega)$ or $E = H^\infty(\partial\Omega)^p$ with the norms given by (3.10) (involving m). Since the proof of Theorem 5.4 is constructive we obtain the following result.

Theorem 5.5. *Let $T_1 > 0, m \geq 2$. Assume that $\{B_j(t)\}_{j=1}^p$ is normal for each $t \in [0,T_1]$. Then there exists for any $0 < T \leq T_1$ a linear map*

$$R : C_0^\infty([0,T], H^\infty(\partial\Omega)^p) \to C_0^\infty([0,T], H^\infty(\Omega)) \tag{5.23}$$

such that $B_j(t)Rg(t) = g_j(t)$ for any $t \in [0,T], g = (g_j)_j$. There is $b \geq 0$ such that (5.19) *holds for any k and $0 < T \leq T_1$ where the norms in* (5.19) *are given by* (3.10) *and where C in* (5.19) *depends only on k, p, n, m, Ω, T_1 and on*

$$\sup\{|B_j^P(t,x,\nu(x))|^{\pm 1} : x \in \partial\Omega, 1 \leq j \leq p, t \in [0,T_1]\}. \tag{5.24}$$

Proof. On $C^\infty([0,T], C^\infty(\overline{\Omega}))$ we consider the norms defined by

$$\|u\|_k^\infty = \sup\{\|u^{(i)}(t)\|_{k-mi}^\infty : t \in [0,T], 0 \leq i \leq k/m\}. \tag{5.25}$$

These norms satisfy $\|uv\|_k^\infty \leq C_k \sum \|u\|_{k-i}^\infty \|v\|_i^\infty$. The rules (5.8), (5.9) can easily be established for the norms (5.25) as well where the definition (5.7) uses the norms (5.25) on the right hand side in (5.7). Therefore, Lemma 5.3 holds with the same proof also for t-depending differential operators where the estimate (5.13) is formulated using the norms defined by (5.25). The proof of Theorem 5.4 gives the result since R maps C_0^∞ into C_0^∞. □

In the situation of Theorem 4.4 we have the linearized boundary operators

$$\mathcal{B}_u(t,u) = \{ \sum_{|\alpha| \leq m_j} (B_j)_{\partial^\alpha u}(t, \cdot, \{\partial^\beta u(\cdot)\}_{|\beta| \leq m_j}) \partial^\alpha \}_{j=1}^{m/2}. \tag{5.26}$$

Let $\phi \in H^\infty(\Omega)$ be an initial value such that $\mathcal{B}_u(0,\phi)$ is a normal system. By continuity, we can choose $T_1 > 0$ and an open neighbourhood U of ϕ in $H^\infty(\Omega)$ such that $\mathcal{B}_u(t,u)$ is normal for $t \in [0,T_1], u \in U$ such that

$$| \sum_{|\alpha| = m_j} (B_j)_{\partial^\alpha u}(t,x,\{\partial^\beta u(x)\}) \nu(x)^\alpha | \geq \mu > 0 \tag{5.27}$$

uniformly for all $t \in [0,T_1], x \in \overline{\Omega}, u \in U, j = 1,\dots,m/2$. We get

$$\|\mathcal{B}_u(t,u)\|_k^\infty \leq C[u]_{p,k}^\infty, \quad t \in [0,T_1],\ u \in U \tag{5.28}$$

where $p = \max\{m_j\}$ and the norm on the left hand side in (5.28) is defined by (5.7), (5.25). We choose a as in 5.4 and put $b = \max\{a,p\} + p + [n/2] + 1$.

Corollary 5.6. *Let $\phi \in H^\infty(\Omega)$ and let $\mathcal{B}_u(0,\phi)$ be normal. Then there exist $T_1 > 0$ and a neighbourhood W of ϕ in $C^\infty([0,T], H^\infty(\Omega))$ and constants $c_k > 0$ and b (as above) such that for any $0 < T \leq T_1$ and $u \in W$ there exists a mapping R_u satisfying* (4.26), (4.27)*, as required by* Theorem 4.4.

The proof follows from Theorems 5.4 and 5.5 using (5.28).

Corollary 5.6 shows that the assumption of Theorem 4.4 on the existence of right inverses R_u is satisfied for normal boundary conditions.

We prove that the trace operators T_k^p in (5.2) admit right inverses which are continuous simultaneously for different values of k. Generalizing techniques of [24] we obtain additional continuity estimates for lower order derivatives which will be important in the sequel. Let X and Y be separable Hilbert spaces such that there is a continuous injection $X \hookrightarrow Y$ with a dense range. As in [24], we shall make use of a representation

$$\|v\|^2_{[X,Y]_\theta} = \int_{\lambda_0}^{\infty} \lambda^{2(1-\theta)} \|v(\lambda)\|^2_{h(\lambda)}\, d\lambda, \quad 0 \le \theta \le 1 \tag{5.29}$$

based on a spectral decomposition of $X \hookrightarrow Y$ where $(h(\lambda), \|\ \ \|_{h(\lambda)})$ is a scale of Hilbert spaces and $\lambda_0 > 0$ (cf. [24], Ch. 1, 2.3). The interpolation space $[X,Y]_\theta = h(1-\theta)$ coincides with X, Y if $\theta = 0, 1$. We shall identify elements of $[X,Y]_\theta$ with functions $v = v(\lambda)$ satisfying (5.29). For $s > 0$ put

$$W(\mathbb{R}, s, X, Y) = \{u : \mathbb{R} \to X |\ \hat{u} \in L^2(\mathbb{R}, X), \tau^s \hat{u} \in L^2(\mathbb{R}, Y)\} \tag{5.30}$$

where $\hat{u} = \mathcal{F}_{t\to\tau}(u)$ denotes the Fourier transform of u (cf. [24], Ch.1, 4.1). The space $W(\mathbb{R}, s, X, Y)$ is a Hilbert space equipped with the norm

$$\|u\|^2_{W(\mathbb{R},s,X,Y)} = \|\hat{u}\|^2_{L^2(\mathbb{R},X)} + \|\tau^s \hat{u}\|^2_{L^2(\mathbb{R},Y)}. \tag{5.31}$$

The following Lemma improves [24], Ch. 1, Theorem 4.2.

Lemma 5.7. *Let q be an integer and $0 \le q < s - \frac{1}{2}$. Then the map*

$$W(\mathbb{R}, s, X, Y) \to \prod_{j=0}^{q} [X,Y]_{(j+1/2)/s}, \quad u \mapsto \{u^{(j)}(0)\}_{j=0}^{q} \tag{5.32}$$

is continuous, linear, surjective and admits a continuous linear right inverse R where $R : \prod_j [X,Y]_{(j+1/2)/s} \to W(\mathbb{R}, s, X, Y)$. There is $C > 0$ only depending on s such that

$$\|Rg\|_{W(\mathbb{R},s(1-\mu),[X,Y]_\mu,Y)} \le C \sum_{j=0}^{q} \|g_j\|_{[X,Y]_{\mu+(j+1/2)/s}} \tag{5.33}$$

for any $0 \le \mu < 1 - (q+1/2)/s$ and $g = (g_j) \in \prod_j [X,Y]_{(j+1/2)/s}$.

Proof. The case $\mu = 0$ is proved in [24], Ch. 1, Theorem 4.2. It remains to show (5.33) for $0 \le \mu < 1 - (q+1/2)/s$ and R constructed in [24]. As in [24] (cf. Ch. 1, Theorem 3.2) it is enough to consider the map $u \mapsto u^{(j)}(0)$ for a fixed j. We fix j and write $g = g_j, g = g(\lambda)$ as above. We choose $\phi \in C^\infty(\mathbb{R})$ with compact support such that $\phi^{(j)}(0) = 1$. Then the function

$$w(\lambda, t) = \lambda^{-j/s} g(\lambda) \phi(\lambda^{1/s} t), \quad \hat{w}(\lambda, \tau) = \lambda^{-(j+1)/s} g(\lambda) \hat{\phi}(\lambda^{-1/s} \tau) \tag{5.34}$$

satisfies $w \in W(\mathbb{R}, s, x, y)$ and $w^{(j)}(0) = g$. We get the estimates (cf. [24])

$$\begin{aligned} &\|\hat{w}\|^2_{L^2(\mathbb{R},[X,Y]_\mu)} + \|\tau^{s(1-\mu)} \hat{w}\|^2_{L^2(\mathbb{R},Y)} \\ &\le \int_{-\infty}^{\infty} \int_{\lambda_0}^{\infty} |\lambda|^{2(1-\mu-(j+\frac{1}{2})/s)} |g(\lambda)|^2 |\hat{\phi}(\tau)|^2 (1 + |\tau|^{2s(1-\mu)}) d\lambda d\tau \\ &\le C \|g\|_{[X,Y]_{\mu+(j+1/2)/s}}. \end{aligned}$$

This proves the assertion. □

The following Lemma follows from the proof of [24], Ch. 1, Theorems 8.3, 9.4 using Lemma 5.7 instead of [24], Ch. 1, Theorem 4.2.

Lemma 5.8. *Let $k \geq p \geq 1$. The trace operator T_k^p in (5.2) admits a continuous linear right inverse which is simultaneously continuous*

$$Z_k^p : \prod_{j=0}^{p-1} H^{l-j-1/2}(\partial\Omega) \to H^l(\Omega), \quad p - \frac{1}{2} < l \leq k. \tag{5.35}$$

Proof. By usual methods (cf. [24], Ch. 1, Thms. 8.3, 9.4) we are brought back to a half space $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}, \partial\Omega = \mathbb{R}^{n-1}_{x'}, x = (x', x_n)$. We put $X_l = H^l(\mathbb{R}^{n-1}_{x'}), Y = H^0(\mathbb{R}^{n-1}_{x'})$. For an integer $l \geq 0$ we get

$$H^l(\Omega) = W(\mathbb{R}_+, l, X_l, Y) = \{u \in L^2(\mathbb{R}_+, X_l) : \frac{\partial^l u}{\partial x_n^l} \in L^2(\mathbb{R}_+, Y)\}$$

(cf. [24], Ch. 1, Theorem 7.4). For noninteger values of l we consider analogously restrictions of functions belonging to the space defined in (5.30) (cf. [24], Ch. 1, Theorem 9.4). By Lemma 5.7 the map

$$W(\mathbb{R}, s, X_k, Y) \to \prod_{j=0}^{p-1} [X_k, Y]_{(j+1/2)/k}, \quad u \mapsto \{u^{(j)}(0)\}_{j=0}^{p-1} \tag{5.36}$$

admits a right inverse R which is simultaneously continuous as a map

$$R : \prod_{j=0}^{p-1} [X_k, Y]_{\mu+(j+1/2)/k} \to W(\mathbb{R}, (1-\mu)k, [X_k, Y]_\mu, Y) \tag{5.37}$$

for $0 \leq \mu < 1 - (p - 1/2)/k$. Since $[H^l(\mathbb{R}^i), H^0(\mathbb{R}^i)]_\theta = H^{(1-\theta)l}(\mathbb{R}^i)$ we get

$$R : \prod_{j=0}^{p-1} H^{(1-\mu)k-j-\frac{1}{2}}(\mathbb{R}^{n-1}_{x'}) \to W(\mathbb{R}, (1-\mu)k, H^{(1-\mu)k}(\mathbb{R}^{n-1}_{x'}), H^0(\mathbb{R}^{n-1}_{x'}))$$

By restriction $\mathbb{R} \to \mathbb{R}_+$ and putting $l = (1-\mu)k$ we get a right inverse Z_k^p for T_k^p satisfying (5.35) for all l with $p - \frac{1}{2} < l \leq k$. The lemma is proved. □

The following lemma follows from the proof of [54], Lemma 3.2.

Lemma 5.9. *For any real number $l \geq 0$ there is a constant $C_l > 0$ such that for any $\phi \in C^{[l]+1}(\overline{\Omega})$ and $u \in H^l(\Omega)$ we have $\phi u \in H^l(\Omega)$ and*

$$\|\phi u\|_l \leq C_l \sum_{i=0}^{[l]} \Big(\|\phi\|_i^\infty \|u\|_{l-i} + \|\phi\|_{i+1}^\infty \|u\|_{[l]-i} \Big) \leq 2C_l \sum_{i=0}^{[l]} \|\phi\|_{i+1}^\infty \|u\|_{l-i}.$$

The assertion of Lemma 5.9 holds analogously for $u \in H^l(\partial\Omega)$. In the next section we shall apply results on maximal regularity for parabolic problems. These results require Hölder estimates in the time variable. Let X be a Banach space, let $0 \leq \delta < 1$ and $T > 0$, let $i \geq 0$ be an integer. By $C^{i+\delta}([0,T], X)$ we denote the set of functions u in $C^i([0,T], X)$ having a Hölder continuous derivative $u^{(i)}$ with exponent δ, equipped with the norm

$$|u|_{i+\delta,X} = \sum_{j=0}^{i} \|u^{(j)}\|_{C([0,T],X)} + \sum_{j=0}^{i} \sup_{s,t\in[0,T], s\neq t} \frac{\|u^{(j)}(t) - u^{(j)}(s)\|_X}{|t-s|^\delta}. \tag{5.38}$$

Writing $H^r = H^r(\Omega)$ or $H^r = H^r(\partial\Omega)$, respectively, we put in particular

$$|u|_{i+\delta,r} = |u|_{i+\delta,H^r}, \quad |u|_{i+\delta,k}^\infty = |u|_{i+\delta,C^k(\overline{\Omega})}. \tag{5.39}$$

Lemma 5.10. *For $f \in C^\infty([0,T], C^\infty(\overline{\Omega})), g \in C^\infty([0,T], H^\infty)$ we have*

$$|fg|_{i+\delta,r} \le C \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{q=0}^{[r]} |f|^\infty_{l+\delta,q+\epsilon} |g|_{j-l+\delta,r-q} \tag{5.40}$$

with some constant $C > 0$ where $\epsilon = 0$ if r is an integer and $\epsilon = 1$ otherwise.

Proof. The case $\delta = 0$ follows from Lemma 5.9. For $0 < \delta < 1$ we write

$$\frac{f^l(t)g^{j-l}(t) - f^l(s)g^{j-l}(s)}{|t-s|^\delta} = \frac{f^l(t) - f^l(s)}{|t-s|^\delta} g^{j-l}(t) + f^l(s) \frac{g^{j-l}(t) - g^{j-l}(s)}{|t-s|^\delta}$$

An application of Lemma 5.9 gives the result. □

For a differential operator $P = \sum_{|\alpha| \le m} a_\alpha(t,x)\partial_x^\alpha$ we put

$$|P|_{i+\delta,r} = \sum_{|\alpha| \le m} |a_\alpha|^\infty_{i+\delta,r}. \tag{5.41}$$

We note that (5.8) holds for the norms $|\quad|_{\delta,i}$ (replacing $\|\quad\|_i$ in (5.8)) as well. From Lemma 5.10 we get for $u \in C^\infty([0,T], H^\infty)$ with ϵ as in 5.10 that

$$|Pu|_{i+\delta,r} \le C \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{q=0}^{[r]} |P|_{l+\delta,q+\epsilon} |u|_{j-l+\delta,r-q+m}. \tag{5.42}$$

Let $\{B_j\}_{j=1}^p$ be a normal system with smooth (t,x)-dependent coefficients as in Theorem 5.5 where $B_j = B_j(t,x,\partial)$ has order m_j.

Lemma 5.11. *Let $1 \le j \le p, r > m_j + \frac{1}{2}$, let $s \ge 0$ be an integer. Then*

$$|(\partial_t^s B_j)u|_{\delta,r-m_j-1/2} \le C \sum_{q=0}^{[r-m_j-1/2]} |B_j|_{s+\delta,q+1} |u|_{\delta,r-q} \tag{5.43}$$

for $u \in C^\infty([0,T], H^r(\Omega))$ with a constant $C > 0$ only depending on p, s, r.

Proof. We may assume that $m_j = j-1$. As in the proof of Theorem 5.4 (cf. formula (5.21)) we can choose local representations $B_j = \sum_{l=1}^j \Lambda_{jl} D_l$ where $D_l = \partial^{l-1}/\partial\nu^{l-1}$. Then, locally, $\partial_t^s B_j = \sum_{l=1}^j (\partial_t^s \Lambda_{jl}) D_l$ and $|\Lambda_{jl}|_{s+\delta,q} \le C|B_j|_{s+\delta,q}$ for any q. Applying Lemma 5.10 and observing that Λ_{jl} has order $j-l$ we get locally

$$|(\partial_t^s B_j)u|_{\delta,r-j+1/2} \le C \sum_{l=1}^{j} \sum_{q=0}^{[r-j+1/2]} |\Lambda_{jl}|_{s+\delta,q+1} |D_l u|_{\delta,r-l+1/2-q}. \tag{5.44}$$

This gives the result since $D_l : H^k(\Omega) \to H^{k-l+1/2}(\partial\Omega)$ for $k > l - 1/2$. □

Let $\{B_j\}_{j=1}^p$ be as above and $M = \max\{m_j : j = 1, \dots, p\}$. Let $m \ge M + 1$. The proofs of Theorem 5.4 and Lemma 5.8 give a linear right inverse R for $\{B_j\}$ satisfying $B_j(t)Rg(t) = g_j(t)$ for every t, j and all $g = \{g_j\}$ such that R is simultaneously for $M + \frac{1}{2} < k \le m$ defined as a map

$$R : C_0^\infty([0,T], \prod_{j=1}^p H^{k-m_j-\frac{1}{2}}(\partial\Omega)) \to C_0^\infty([0,T], H^k(\Omega)), B_j R g = g_j \tag{5.45}$$

For Dirichlet systems $\{B_j\}_{j=1}^p$ the map R is locally given by (5.22) using Z_m^p from (5.35) instead of Z^p. This gives R for normal systems $\{B_j\}_{j=1}^p$ as well.

We define the expressions $[\ \ ]_{k;\delta}$ by the norms $|\ \ |_{\delta,i}$ (which are for differential operators given by (5.41)). Analogously to (5.9) we then have

$$|1/f|^{\infty}_{\delta,i} \le C[f]_{i;\delta}, \quad |P/f|_{\delta,i} \le C[f;P]_{i;\delta} \tag{5.46}$$

where C depends on i, m, n and on a bound for $|1/f|^{\infty}_{0,0} + |f|^{\infty}_{\delta,0}$; we here have observed that $|1/f|^{\infty}_{\delta,0} \le (|1/f|^{\infty}_{0,0})^2 |f|^{\infty}_{\delta,0}$.

Lemma 5.12. *For $M + \frac{1}{2} < k \le m$ the map R in* (5.45) *satisfies*

$$|Rg|_{\delta,k} \le C_1 \sum_{j=1}^{p} \sum_{q=0}^{[k-m_j-1/2]} [B_1,\dots,B_p]_{M+1+q;\delta} |g_j|_{\delta,k-m_j-q-1/2} \tag{5.47}$$

where $C_1 > 0$ depends on $k, m, \sum |B_j|_{\delta,0}$ and on the constant in (5.20) *and*

$$|Rg|_{1+\delta,k} \le C_2 \sum_{j=1}^{p} |g_j|_{1+\delta,k-m_j-1/2} \tag{5.48}$$

where $C_2 > 0$ depends on the same data as C_1 and on $\sum |B_j|_{1+\delta,[k+1/2]}$.

Proof. We may assume that $m_j = j-1, M = p-1$. Using (5.22) with Z^p_m in place of Z^p and omitting the index i we get from 5.8 and (5.42) that

$$|Rg|_{\delta,k} \le C \sum_{1\le l\le j\le p} \sum_{q=0}^{[k-j+\frac{1}{2}]} |\Phi_{jl}|_{\delta,q+1} |g_l|_{\delta,k-l-q+\frac{1}{2}}. \tag{5.49}$$

Using (5.46) instead of (5.9) in the proof of 5.3 (cf. (5.16)) we get

$$|\Phi_{jl}|_{\delta,i} \le C[B_l,\dots,B_j]_{i+j-l;\delta}. \tag{5.50}$$

The inequalities (5.49), (5.50) yield (5.47). Analogously, we get the estimate

$$|\partial_t \Phi_{jl}|_{\delta,i} \le C \sum_{s=l}^{j} [B_l,\dots,B_j;\partial_t B_s]_{i+j-l;\delta}. \tag{5.51}$$

Together with (5.49) this proves the assertion. □

6. The linear parabolic problem

Let $T_1 > 0$. We consider for $0 < T \le T_1$ the linear evolution equation

$$\begin{gathered} \partial_t z(t) = A(t)z(t) + f(t), \quad t \in [0,T] \\ z(0) = 0\,. \end{gathered} \tag{6.1}$$

We assume that $A(t), t \in [0,T_1]$, is a closed linear operator in a Banach space X with a (not necessarily dense) domain $\mathcal{D}(A(t))$ (which may depend on t). We assume that there is a Banach space $Z \hookrightarrow X$ continuously imbedded into X such that $\mathcal{D}(A(t)) \subset Z$ for all t. In applications we put $X = L^2(\Omega), Z = H^m(\Omega)$ where $\mathcal{D}(A(t))$ is given by boundary conditions. We shall suppose the following conditions (P_0), …, (P_3) (cf. [51], [6], [50]).

(P_0) There is a constant $M_0 > 0$ such that

$$\|z\|_Z \le M_0(\|A(t)z\|_X + \|z\|_X), \quad z \in \mathcal{D}(A(t)), \quad t \in [0,T_1]. \tag{6.2}$$

(P_1) There is $\theta_0 \in (\pi/2, \pi)$ so that $\rho(A(t)) \supset \Sigma := \{\lambda : |\arg \lambda| < \theta_0\} \cup \{0\}$ and there is $M_1 > 0$ such that $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ satisfies

$$\|R(\lambda, A(t))\|_{L(X)} \leq M_1/|\lambda|, \quad \lambda \in \Sigma\backslash\{0\}, \quad t \in [0, T_1]. \tag{6.3}$$

(P_2) For each $\lambda \in \Sigma$ the operator valued function $t \mapsto R(\lambda, A(t))$ belongs to the space $C^1([0, T_1], L(X))$. There is a constant $M_2 > 0$ such that

$$\|(d/dt)R(\lambda, A(t))\|_{L(X)} \leq M_2/|\lambda|, \quad \lambda \in \Sigma\backslash\{0\}, \quad t \in [0, T_1]. \tag{6.4}$$

(P_3) There is a constant $M_3 > 0$ such that

$$\|(d/dt)A(t)^{-1} - (d/dt)A(\tau)^{-1}\|_{L(X)} \leq M_3|t - \tau|, \quad t, \tau \in [0, T_1]. \tag{6.5}$$

We take advantage of the following result on maximal regularity from [51].

Theorem 6.1. *Assume* (P_1), (P_2), (P_3). *Let* $0 < \delta < 1$. *Then there is* $C_0 > 0$ *depending only on* $M_1, M_2, M_3, \theta_0, T_1$ *such that for any* $f \in C^\delta([0, T], X)$ *with* $f(0) = 0$ *and* $0 < T \leq T_1$ *any solution* $z \in C^1([0, T], X)$ *of* (6.1) *with* $z(t) \in \mathcal{D}(A(t))$ *for all* t *satisfies* $z \in C^{1+\delta}([0, T], X)$ *and*

$$|z|^{[0,T]}_{1+\delta,X} \leq C_0|f|^{[0,T]}_{\delta,X}. \tag{6.6}$$

Assuming also (P_0) *there is* $C_1 > 0$ *depending only on* C_0, M_0 *such that*

$$|z|^{[0,T]}_{0,Z} \leq C_1|f|^{[0,T]}_{\delta,X}. \tag{6.7}$$

Proof. We have $z \in C^{1+\delta}([0, T], X)$ by [51], Theorem 6.4 since $z(0) = f(0) = 0$ and since z is a strict solution in the sense of [51]. The estimate (6.6) follows from the proof of [51], Theorem 6.4. We apply (P_0) and (6.6) and use equation (6.1) to get

$$\|z(t)\|_Z \leq M_0(\|A(t)z(t)\|_X + \|z(t)\|_X) \leq M_0(1 + 2C_0)|f|_{\delta,X}. \tag{6.8}$$

This gives the result. □

Let $\Omega \subset \mathbb{R}^n$ be bounded with C^∞-boundary, let $m \geq 2$ be even and let

$$A = A(t) = A(t, x, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(t, x)\partial_x^\alpha \tag{6.9}$$

be a differential operator with coefficients $a_\alpha \in C^\infty([0, T_1], C^\infty(\overline{\Omega}))$ and let

$$B_j = B_j(t) = B_j(t, x, \partial_x) = \sum_{|\beta| \leq m_j} b_{j,\beta}(t, x)\partial_x^\beta, \quad j = 1, \dots, m/2 \tag{6.10}$$

be boundary operators with coefficients $b_{j,\beta} \in C^\infty([0, T_1], C^\infty(\overline{\Omega}))$ where

$$0 \leq m_1 < \dots < m_{m/2} < m. \tag{6.11}$$

We suppose that $\{B_j(t)\}_{j=1}^{m/2}$ is normal for $t \in [0, T_1]$. Then the constant in (5.24) is bounded. We put $X = L^2(\Omega), Z = H^m(\Omega), Z_j = H^{m-m_j-1/2}(\partial\Omega)$. Then $A(t) : Z \to X$ and $B_j(t) : Z \to Z_j$ are continuous. We put

$$\mathcal{D}(A(t)) = \{z \in H^m(\Omega) : B_j(t)z(t) = 0 \text{ on } \partial\Omega, \quad j = 1, \dots, m/2\}. \tag{6.12}$$

We consider for $0 < T \leq T_1$ the boundary-value problem

$$\begin{gathered} \partial_t z(t) = A(t)z(t) + f(t) \quad \text{in } \Omega,\ t \in [0, T] \\ B_j(t)z(t) = 0 \quad \text{on } \partial\Omega,\ t \in [0, T], 1 \leq j \leq \frac{m}{2} \\ z(0) = 0\,. \end{gathered} \tag{6.13}$$

Solutions of (6.13) correspond to solutions of (6.1) where $z(t) \in \mathcal{D}(A(t)), t \in [0,T]$. In addition, we assume the following conditions.

(P_4) There is $M_4 > 0$ such that for all $z \in Z$ and $t \in [0,T_1]$ we have

$$\|z\|_Z \le M_4(\|A(t)z\|_X + \|z\|_X + \sum_{j=1}^{m/2} \|B_j(t)z\|_{Z_j}). \tag{6.14}$$

(P_5) There exist $0 < \delta \le 1, M_5 > 0$ such that $A \in C^\delta([0,T_1], L(Z,X))$ and $B_j \in C^\delta([0,T_1], L(Z,Z_j))$ satisfy

$$|A|_{\delta,L(Z,X)} + |B_j|_{\delta,L(Z,Z_j)} \le M_5, \quad j = 1,\dots,m/2. \tag{6.15}$$

We note that A and B_j as above enjoy this condition (P_5). In the following condition we use the notation $[A,B]_{m,k} = [A,B_1,\dots,B_{m/2}]_{m,k}$.

(P_6) For $k \ge m$ there is M_k so that for all $z \in H^k(\Omega), t \in [0,T_1]$ we have

$$\begin{aligned}\|z\|_k \le & M_k\Big\{\sum_{i=m}^{k}[A,B]_{m,k-i}\big(\|A(t)z\|_{i-m} + \sum_{j=1}^{m/2}\|B_j(t)z\|_{i-m_j-\frac12}\big)\\ &+ [A,B]_{m,k-m}\|z\|_0\Big\}.\end{aligned}$$

Lemma 6.2. *Assume* (P_0), …, (P_5) *Let* $f, z, w_0 \in C_0^\infty([0,T], H^\infty(\Omega))$ *where* $0 < T \le T_1$. *Assume that z is a solution of problem* (6.1) *and that*

$$B_j(t)w_0(t) = B_j(t)z(t) \quad \text{on} \quad \partial\Omega, \quad j = 1,\dots,m/2, \quad t \in [0,T]. \tag{6.16}$$

There is a constant $C_2 > 0$ *depending only on* $M_0,\dots,M_5,\theta_0,T_1$ *such that*

$$|z|_{1+\delta,0} + |z|_{\delta,m} \le C_2(|f|_{\delta,0} + |w_0|_{1+\delta,0} + |w_0|_{\delta,m}). \tag{6.17}$$

Proof. We note that $v(t) = z(t) - w_0(t) \in \mathcal{D}(A(t))$ for all t and

$$v_t(t) = A(t)v(t) + f(t) + A(t)w_0(t) - (w_0)_t(t). \tag{6.18}$$

From Lemma 6.1 we obtain the estimate

$$|v|_{1+\delta,0} + |v|_{0,m} \le C(|f|_{\delta,0} + |A(t)w_0(t)|_{\delta,0} + |w_0|_{1+\delta,0}). \tag{6.19}$$

Using (P_5) we get $|A(t)w_0(t)|_{\delta,0} \le CM_5|w_0|_{\delta,m}$ and thus

$$|z|_{1+\delta,0} + |z|_{0,m} \le C(|f|_{\delta,0} + |w_0|_{1+\delta,0} + |w_0|_{\delta,m}). \tag{6.20}$$

To estimate $|z|_{\delta,m}$ we apply (P_4) and obtain

$$\frac{\|z(t)-z(s)\|_Z}{|t-s|^\delta} \le M_4(\frac{\|A(t)(z(t)-z(s))\|_X}{|t-s|^\delta} + |z|_{\delta,0} + \sum_{j=1}^{m/2}\frac{\|B_j(t)(z(t)-z(s))\|_{Z_j}}{|t-s|^\delta}).$$

We further get

$$\frac{\|A(t)(z(t)-z(s))\|_X}{|t-s|^\delta} \le |z|_{1+\delta,0} + |f|_{\delta,0} + M_5|z|_{0,m} \tag{6.21}$$

which gives the desired estimate by means of (6.20). Finally we obtain

$$\frac{\|B_j(t)(z(t)-z(s))\|_{Z_j}}{|t-s|^\delta} \le M_5(|w_0|_{\delta,m} + |w_0|_{0,m} + |z|_{0,m}). \tag{6.22}$$

We proved estimate (6.17) and thus the result. □

In the sequel, the symbol H^∞ is used simultaneously to denote $H^\infty(\Omega)$ or $H^\infty(\partial\Omega)$. For $f \in C^\infty([0,T], H^\infty)$ and an integer k we put

$$\|f\|_{k;\delta} = \sup\{|f|_{i+\delta,k-mi} : i = 0,1,\dots,[k/m]\}. \tag{6.23}$$

For $f \in C^\infty([0,T], C^\infty(\overline{\Omega}))$ we define $\|f\|^\infty_{k;\delta}$ analogously using $|f|^\infty_{i+\delta,k-mi}$ in (6.23). For a differential operator P then $\|P\|_{k;\delta}$ is given by (6.23) replacing f by P while $[P]_{k;\delta}$ and $[P]_{m,k;\delta}$ are defined using the norms $\|P\|_{i;\delta}$. We write

$$[B]_{m,k;\delta} = [B_1,\dots,B_{m/2}]_{m,k;\delta}, [A,B]_{m,k;\delta} = [A,B_1,\dots,B_{m/2}]_{m,k;\delta}.$$

Lemma 6.3. *Assume conditions* (P_0), …, (P_6). *Let* $k \ge 1, 0 < T \le T_1$ *and* $f, z, w_i \in C_0^\infty([0,T], H^\infty(\Omega)), 0 \le i \le k-1$. *Let* z *be a solution of* (6.13) *and*

$$B_r(t)w_i(t) = B_r(t)\partial_t^i z(t) \quad \text{on} \quad \partial\Omega \tag{6.24}$$

for $r = 1,\dots,m/2, i = 0,\dots,k-1, t \in [0,T]$. *Then there is* $C > 0$ *depending only on* $k, M_0,\dots,M_5, M_{(k+1)m}, \theta_0, T_1, \|A\|_{2m;\delta}, \sum_r \|B_r\|_{2m;\delta}$ *such that*

$$\sum_{i=0}^{k} |z|_{i+\delta,(k-i)m} \le C \sum_{i=0}^{k-1} [A,B]_{2m,m(k-1-i);\delta} \Big\{ \|f\|_{mi;\delta} \tag{6.25}$$

$$+ |w_i|_{1+\delta,0} + \sum_{j=0}^{i} |w_j|_{\delta,m(i+1-j)} \Big\}. \tag{6.26}$$

Proof. The case $k = 1$ follows from Lemma 6.2. We fix $k \ge 1$. We assume that (6.25) is proved for k and that (6.24) holds for $i = 0,\dots,k$. We have to show (6.25) for $k+1$. Differentiating (6.1) we obtain

$$(\partial_t^k z)_t(t) = A(t)\partial_t^k z(t) + \sum_{i=0}^{k-1} \binom{k}{i} (\partial_t^{k-i} A(t))\partial_t^i z(t) + \partial_t^k f(t). \tag{6.27}$$

Applying Lemma 6.2 to $\partial_t^k z$ we get

$$|\partial_t^k z|_{1+\delta,0} + |\partial_t^k z|_{\delta,m} \le C\Big\{ \sum_{i=0}^{k-1} |A|_{k-i+\delta,0}|z|_{i+\delta,m} + \|f\|_{mk;\delta} + |w_k|_{1+\delta,0} + |w_k|_{\delta,m} \Big\}.$$

The hypothesis of induction gives for $0 \le i \le k-1$ the estimates

$$|A|_{k-i+\delta,0}|z|_{i+\delta,m} \le C \sum_{l=0}^{i} [A,B]_{2m,m(k-l);\delta} \Big\{ \|f\|_{m;l} |w_l|_{1+\delta,0} + \sum_{j=0}^{l} |w_j|_{\delta,m(l+1-j)} \Big\}$$

and thus the desired estimate for $|z|_{k+1+\delta,0} + |z|_{k+\delta,m}$. Next we fix $0 \le i \le k-1$ and assume that the estimate in (6.25) is proved in the case $k+1$ for all terms $|z|_{l+\delta,(k+1-l)m}$ with $i+1 \le l \le k+1$; we have to show the estimate in (6.25) in the case $k+1$ for the term $|z|_{i+\delta,(k+1-i)m}$. We fix $0 \le j \le i$. Using (P_6) we first get

$$\|\partial_t^j z(t)\|_{(k+1-i)m}$$

$$\le M\{ \sum_{l=m}^{(k+1-i)m} [A,B]_{m,(k+1-i)m-l} (\|A(t)\partial_t^j z(t)\|_{l-m} \tag{6.28}$$

$$+ \sum_{r=1}^{m/2} \|B_r(t)w_j(t)\|_{l-m_r-\frac12}) + [A,B]_{m,(k-i)m} \|\partial_t^j z(t)\|_0 \} \tag{6.29}$$

The last term enjoys the desired estimate by induction. Using (6.27) we get

$$\|A(t)\partial_t^j z(t)\|_{l-m} \le |z|_{i+1,l-m} + |f|_{i,l-m} + C\sum_{q=0}^{j-1} \|(\partial_t^{j-q} A(t))\partial_t^q z(t)\|_{l-m}.$$

To estimate the term $[A,B]_{m,(k+1-i)m-l}|f|_{i,l-m}$ we consider two cases. If $l \ge (k-i)m$ then this term is $\le C\|f\|_{mk;\delta}$ since $[A,B]_{m,m} \le C$; in the case $l \le (k-i)m$ we can use the estimate $[A,B]_{m,(k+1-i)m-l} \le C[A,B]_{2m,(k-i)m-l}$ to estimate this term appropriately. Analogously, we can apply in the case $l \ge (k-i)m$ the hypothesis of the induction (case $k+1, i+1$) to the term $|z|_{i+1,l-m} \le |z|_{i+1,(k-i)m}$. In the case $l \le (k-i)m$ and $am < l \le (a+1)m$, thus $i+a+1 \le k+1$, we obtain

$$[A,B]_{2m,(k-i)m-l}|z|_{i+1,l-m} \le [A,B]_{2m,(k-i-a)m}|z|_{i+1,(i+1+a-i-1)m}$$

and induction gives the desired estimate. We further get

$$\|(\partial_t^{j-q} A(t))\partial_t^q z(t)\|_{l-m} \le C\sum_{r=0}^{l-m} |A|_{j-q,l-m-r}|z|_{q,r+m}. \tag{6.30}$$

For $am < r \le (a+1)m$ we have $|z|_{q,r+m} \le |z|_{q,(a+2)m}$ where $q+a+2 \le k$. If $(k+1-i)m-l \le m$ and $(i-q-a-1)m+l \le m$ then $l=(k-i)m, k = q+a+2$ and $|z|_{q,(a+2)m} = |z|_{q,(k-q)m}$ can be estimated by induction where $[A,B]_{m,(k+1-i)m-l}[A,B]_{(i-q-a-1)m+l} \le C$; otherwise we observe

$$[A,B]_{m,(k+1-i)m-l}[A,B]_{(i-q-a-1)m+l} \le C[A,B]_{2m,(k-q-a-1)m} \tag{6.31}$$

and can apply induction to $|z|_{q,(a+2)m}$. This yields the necessary estimate for the term on the right hand side in (6.28). By Lemma 5.11 we have

$$\|B_r(t)w_j(t)\|_{l-m_r-\frac12} \le C\sum_{q=0}^{l-m_r-1} |B_r|_{0,q+1}|w_j|_{0,l-q} \tag{6.32}$$

which gives the desired estimate for the first term in (6.29) since for $am < l-q \le (a+1)m$ we have $i+a \le k$ and $|B_r|_{0,q+1} \le [B_r]_{m,q}$ and thus

$$[A,B]_{m,(k+1-i)m-l}|B_r|_{0,q+1}|w_j|_{0,l-q} \le C[A,B]_{2m,(k-j-a)m}|w_j|_{0,(a+1)m}.$$

It remains to prove Hölder estimates for $\delta > 0$ for the term $|z|_{i+\delta,(k+1-i)m}$. For that we replace in (6.28), (6.29) the term $\partial_t^j z(t)$ by the term $(\partial_t^j z(t) - \partial_t^j z(s))/|t-s|^\delta$. The last term $[A,B]_{m,(k-i)m}|z|_{j+\delta,0}$ in this inequality satisfied the desired estimate by induction. For the first term we write

$$A(t)\frac{\partial_t^j z(t) - \partial_t^j z(s)}{|t-s|^\delta} = \frac{A(t)\partial_t^j z(t) - A(s)\partial_t^j z(s)}{|t-s|^\delta} + \frac{(A(s)-A(t))}{|t-s|^\delta}\partial_t^j z(s). \tag{6.33}$$

The first term $|A(\cdot)\partial_t^j z(\cdot)|_{\delta,l-m}$ resulting from (6.33) is estimated like $\|A(t)\partial_t^j z(t)\|_{l-m}$ using Hölder norms in the above estimates and observing the estimates $|A|_{j-q+\delta,l-m-r} \le [A]_{m,m(j-q)+l-m-r}$. For the other term we have

$$\Big\|\frac{(A(s)-A(t))}{|t-s|^\delta}\partial_t^j z(s)\Big\|_{l-m} \le C\sum_{r=0}^{l-m}[A]_{m,l-m-r}|z|_{i,r+m} \tag{6.34}$$

since $j \le i, |A|_{\delta,l-m-r} \le [A]_{m,l-m-r}$. The proved case (for $|z|_{i,(k+1-i)m}$) gives the required estimate for the terms appearing on the right hand side in (6.34). Finally we use for the term $B_r(t)(\partial_t^j z(t) - \partial_t^j z(s))/|t-s|^\delta$ a decomposition as in (6.33).

Lemma 5.11 gives for $|B_r(\cdot)\partial_t^j z(\cdot)|_{\delta,l-m_r-\frac12} = |B_r(\cdot)w_j(\cdot)|_{\delta,l-m_r-\frac12}$ an estimate as in (6.32) involving δ on both sides; using $|B|_{\delta,q+1} \le [B]_{m,q+1}$ we get the necessary estimate for this term as above. On the other hand, we have

$$\Big\|\frac{B_r(s)-B_r(t)}{|t-s|^\delta}\partial_t^j z(t)\Big\|_{l-m_r-\frac12} \le C\sum_{q=0}^{l-m_r-1}|B_r|_{\delta,q+1}|z|_{i,l-q}. \tag{6.35}$$

Since $|B_r|_{\delta,q+1} \le [B]_{m,q+1}$ we must estimate $[A,B]_{m,(k+1-i)m-l+q+1}|z|_{i,l-q}$ which is $\le [A,B]_{m,(k+1-i-a)m}|z|_{i,(a+1)m}$ if $am < l-q \le (a+1)m$. Since $i+a+1 \le k+1$ we can apply the above proved estimate for $|z|_{i,(a+1)m}$. This gives the result. □

It remains to choose and estimate the terms w_i in (6.24). We put $w_0 = 0$. For $i \ge 1$ we use the linear right inverse R for $\{B_r\}_{r=1}^{m/2}$ from Lemma 5.12 and (5.45). Since $\partial_t^i(B_j(t)z(t)) = 0$ on $\partial\Omega$ we may define

$$w_i = R\Big\{\Big(-\sum_{r=0}^{i-1}\binom{i}{r}(\partial_t^{i-r}B_j)\partial_t^r z\Big)_{j=1}^{m/2}\Big\}, \quad i \ge 1. \tag{6.36}$$

Theorem 6.4. *In the situation of* Lemma 6.3 *there is* $C > 0$ *depending on the same data as* C *in* Lemma 6.3 *and on the constant in* (5.20) *such that*

$$|w_i|_{1+\delta,0} + \sum_{j=0}^{i}|w_j|_{\delta,m(i+1-j)} \le C\sum_{l=0}^{i}[A,B]_{2m,(i-l)m;\delta}\|f\|_{ml;\delta} \tag{6.37}$$

for $i = 0,\dots,k-1$. *In addition, we have the inequality*

$$\sum_{i=0}^{k}|z|_{i+\delta,(k-i)m} \le C\sum_{i=0}^{k-1}[A,B]_{2m,(k-1-i)m;\delta}\|f\|_{mi;\delta}. \tag{6.38}$$

Proof. The case $k = 1$ follows from Lemma 6.2. If (6.37) is already proved for $i = 0,\dots,k-1$ then (6.38) follows from Lemma 6.3 since

$$[A,B]_{2m,(k-1-i)m;\delta}[A,B]_{2m,(i-l)m;\delta} \le [A,B]_{2m,(k-1-l)m;\delta}. \tag{6.39}$$

Let $k \ge 1$ and assume that (6.38) is proved for k; we show that this implies (6.37) for $i = k$. We choose R in (5.45) depending on k so that (5.47), (5.48) hold for $M + \frac12 < K \le km$ (replacing k by K in (5.47), (5.48)) where $M = \max\{m_j\}$; note that R depends on m in Lemma 5.12. From (5.47) we get

$$\begin{aligned}\sum_{i=1}^{k}|w_i|_{\delta,m(k+1-i)} \le C\sum_{i=1}^{k}\sum_{r=0}^{i-1}\sum_{j=1}^{m/2}\sum_{q=0}^{(k+1-i)m-m_j-1}&\Big\{[B]_{m,q;\delta}\\ &\times|(\partial_t^{i-r}B_j)\partial_t^r z|_{\delta,(k+1-i)m-m_j-q-\frac12}\Big\}.\end{aligned}$$

Using Lemma 5.11 we obtain

$$\begin{aligned}&|\partial_t^{i-r}B_j\partial_t^r z|_{\delta,(k+1-i)m-m_j-q-\frac12}\\ &\le C\sum_{l=0}^{(k+1-i)m-m_j-1}|B_j|_{i-r+\delta,l+1}|z|_{r+\delta,(k+1-i)m-l-q}\end{aligned}$$

For $am \le l+q < (a+1)m$ we have $k+r+1-i-a \le k$ and (6.38) shows

$$|z|_{r+\delta,(k+1-i-a)m} \le C \sum_{s=0}^{(k+r-i-a)m} [A,B]_{2m,(k+r-i-a-s)m;\delta} \|f\|_{ms;\delta}.$$

We have $|B_j|_{i-r+\delta,l+1} \le C[B]_{2m,m(i-1-r)+l;\delta}$ and thus

$$[B]_{m,q;\delta}|B_j|_{i-r+\delta,l+1}[A,B]_{2m,(k+r-i-a-s)m} \le C[A,B]_{2m,(k-s)m;\delta}.$$

We proved (6.37) for the second term in (6.37) for $i=k$. For the other term we fix a real number k_0 with $M+\frac{1}{2} < k_0 < m$. Applying (5.48) to k_0 we get

$$|w_k|_{1+\delta,0} \le C \sum_{j=1}^{m/2} \sum_{r=0}^{k-1} |(\partial_t^{k-r} B_j)\partial_t^r z|_{1+\delta,k_0-m_j-1/2}$$

where $|B_j|_{1+\delta,[k-\frac{1}{2}]} \le |B_j|_{1+\delta,m} \le \|B_j\|_{2m;\delta} \le C$. We have to estimate

$$|(\partial_t^{k+1-r} B_j)\partial_t^r z|_{\delta,k_0-m_j-1/2} + |(\partial_t^{k-r} B_j)\partial_t^{r+1} z|_{\delta,k_0-m_j-1/2}. \tag{6.40}$$

The above yields the required estimate for the first term in (6.40) since

$$|B_j|_{k+1-r+\delta,m} \le [B_j]_{2m,(k-r-1)m+q+1;\delta} \le C[B_j]_{2m,(k-r)m;\delta} \tag{6.41}$$

with C depending on $\|B_j\|_{2m;\delta}$. For the second term in (6.40) we get

$$|(\partial_t^{k-r} B_j)\partial_t^{r+1} z|_{\delta,k_0-m_j-1/2} \le C|B_j|_{k-r+\delta,m}|z|_{r+1+\delta,k_0}. \tag{6.42}$$

Induction does not apply. Since $k_0 < m$ we get for $\epsilon > 0$ by interpolation

$$|z|_{r+1+\delta,k_0} \le \epsilon |z|_{r+1+\delta,m} + C(\epsilon)|z|_{r+1+\delta,0} \tag{6.43}$$

where the constant $C(\epsilon)$ depends on ϵ. Since $r \le k-1$ we get from (6.38)

$$|B_j|_{k-r+\delta,m}|z|_{r+1+\delta,0} \le C \sum_{i=0}^{k-1} [A,B]_{2m,(k-i)m;\delta} \|f\|_{mi;\delta} \tag{6.44}$$

observing $|B_j|_{k-r+\delta,m} \le \|B_j\|_{(k+1-r)m;\delta} \le [B]_{m,(k-r)m;\delta}$. Note that (6.38) does not apply to the other term. We thus apply Lemma 6.3 and get

$$\begin{aligned} &\epsilon |B_j|_{k-r+\delta,m}|z|_{r+1+\delta,m} \\ &\le \epsilon C \sum_{i=0}^{k} [A]_{m,(k-i)m;\delta} \big\{ \|f\|_{mi;\delta} + |w_i|_{1+\delta,0} + \sum_{j=0}^{i} |w_j|_{\delta,(i+1-j)m} \big\} \end{aligned}$$

since $|B_j|_{k-r+\delta,m} \le [B]_{2m,(k-r-1;\delta}$. We here can estimate all terms appropriately except $|w_k|_{1+\delta,0}$. However, the proved cases give

$$|w_k|_{1+\delta,0} \le C \sum_{i=0}^{k} [A,B]_{2m,(k-i)m;\delta} \|f\|_{mi;\delta} + \epsilon C |w_k|_{1+\delta,0}. \tag{6.45}$$

Choosing $\epsilon > 0$ small enough we get (6.37) for $i=k$ and thus the result. $\square$

7. Elliptic a priori estimates

We formulate sufficient conditions of elliptic type for (P_0), …, (P_6). The classical elliptic a priori estimates due to Agmon, Douglis, Nirenberg [3] are well known. We accomplish these estimates including the dependence of the constants from the coefficients, as required by the Nash-Moser technique. Uniform dependence as stated in [3], Theorem 15.2 is not sufficient for (P_6).

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞-boundary and $n \geq 2$. Let

$$L = L(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha \tag{7.1}$$

and let $B_j = B_j(x, \partial), j = 1, \dots, m/2$ be given by (5.1) where $a_\alpha, b_{j,\beta} \in C^\infty(\overline{\Omega})$ are $\mathbb{C}$-valued. Let $m \geq 2$ be and assume (6.11). Write $L = L^P + L^R, B_j = B_j^R + B_j^R$ where L^P, B_j^P denote the principal parts.

Definitiion 7.1. The pair (L, B_j) is called *elliptic* if the following holds:

(i) *Ellipticity:* L is uniformly elliptic on $\overline{\Omega}$, i.e., there is $\mu > 0$ so that

$$|L^P(x, \xi)| \geq \mu |\xi|^m, \quad x \in \overline{\Omega}, \xi \in \mathbb{R}^n. \tag{7.2}$$

(ii) *Root Condition:* For every $x \in \partial\Omega$ and $\xi \neq 0$ tangential to $\partial\Omega$ at x the polynomial $\tau \mapsto L^P(x, \xi + \tau\nu)$ has exactly $m/2$ roots with positive imaginary part denoted by $\tau_1^+(x, \xi), \dots, \tau_{m/2}^+(x, \xi)$ ($\nu = \nu(x) =$ inner normal vector).

(iii) *Complementing Condition:* For every $x \in \partial\Omega$ and $\xi \neq 0$ as in (ii) the polynomials $\{B_j^P(x, \xi + \tau\nu)\}_{j=1}^{m/2}$ in τ are linearly independent modulo $\prod_{j=1}^{m/2}(\tau - \tau_j^+(x, \xi))$.

For $n \geq 3$ all elliptic operators satisfy the root condition. We consider in the half space $H_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ the problem

$$\begin{aligned} Lu &= F, \quad x_n > 0 \\ B_j u &= \Phi_j, \quad x_n = 0,\ j = 1, \dots m/2\,. \end{aligned} \tag{7.3}$$

We first assume that the elliptic pair (L, B_j) has constant coefficients. As in [3], (1.9) we define a determinant constant $\Delta = \min\{|\det(b_{jk}(\xi))| : |\xi| = 1\} > 0$. Here $\sum_{k=1}^{m/2} b_{jk}(\xi)\tau^{k-1} = B_j^P(\xi, \tau) \bmod \prod_{j=1}^{m/2}(\tau - \tau_j^+(\xi))$ and thus $\Delta > 0$ by means of the complementing condition. If L, B_j have variable coefficients then Δ means a lower bound for the determinant constants of the frozen operators. If L, B_j depend continuously on additional parameters then Δ depends continuously on these parameters as well. As in [3] (2.12) we introduce the characteristic constant

$$E = \mu^{-1} + \Delta^{-1} + \|A\|_m + \sum_j \|B_j\|_m + n + m + \sum_j m_j. \tag{7.4}$$

Lemma 7.2 (cf. [3], Thm. 14.1)**.** *Let the elliptic pair $(L, B_j) = (L^P, B_j^P)$ have constant coefficients. Let $u \in H^k(H_+), k \geq m$, satisfy $u(x) = 0$ for $|x| \geq 1$. Then*

$$\|u\|_k \leq C\Big(\|Lu\|_{k-m} + \sum_{j=1}^{m/2} \|B_j u\|_{k-m_j-1/2}\Big) \tag{7.5}$$

where C depends only on k and on the characteristic constant E.

Lemma 7.3 (cf. [3], Theorem 15.1)**.** *Let (L, B_j) be an elliptic pair. Let $k \geq m$. Then there exist $C > 0, r > 0$ depending only on k, E such that*

$$\begin{aligned} \|u\|_k \leq C\Big\{ \sum_{i=m}^{k} [L,B]_{m,k-i}\big(\|Lu\|_{i-m} + \sum_{j=1}^{m/2} \|B_j u\|_{i-m_j-1/2}\big) \\ + [L,B]_{m,k-m}\|u\|_0 \Big\}. \end{aligned} \tag{7.6}$$

for all $u \in H^k(H^+)$ satisfying $u(x) = 0$ for $|x| \geq r$.

Proof. We put $x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}$, and write (7.3) as

$$\begin{gathered} L^P(0,\partial)u(x) = F(x) + (L^P(0,\partial) - L^P(x,\partial))u(x) - L^R(x,\partial)u(x) \\ B_j^P(0,\partial)u(x',0) = \Phi_j(x') + (B_j^P(0,\partial) - B_j^P(x',\partial))u(x',0) - B_j^R(x',\partial)u(x',0). \end{gathered}$$

For L defined by (7.1) we obtain the estimates

$$\begin{aligned} &\|(L^P(0,\partial) - L^P(x,\partial))u\|_{k-m} + \|L^R(x,\partial)u\|_{k-m} \\ &+ C\Big\{ r\|u\|_k + \sum_{|\alpha|=m} \sum_{i=m}^{k-1} \|a_\alpha\|_{k-i}^\infty \|u\|_i + \sum_{|\alpha|<m} \sum_{i=m-1}^{k-1} \|a_\alpha\|_{k-1-i}^\infty \|u\|_i \Big\} \end{aligned}$$

and for $B_j = B_j(x', \partial)$ defined by (5.1) the definition of the norms imply

$$\begin{aligned} &\|(B_j^P(0,\partial) - B_j^P(x',\partial) - B_j^R(x',\partial))u(x',0)\|_{k-m_j-1/2} \\ &\leq C\Big\{ r\|u\|_k + \sum_{|\beta|=m_j} \sum_{i=m_j}^{k-1} \|b_{j,\beta}\|_{k-i}^\infty \|u\|_i + \sum_{|\beta|<m_j} \sum_{i=m_j-1}^{k-1} \|b_{j,\beta}\|_{k-1-i}^\infty \|u\|_i \Big\}. \end{aligned}$$

Applying Lemma 7.2 we get the estimates

$$\|u\|_k \leq C\Big\{ r\|u\|_k + \|F\|_{k-m} + \sum_{j=1}^{m/2} \|\Phi_j\|_{k-m_j-1/2} + \sum_{i=0}^{k-1} [L,B]_{k-i}\|u\|_i \Big\}$$

and hence, choosing r sufficiently small, the inequalitity

$$\|u\|_k \leq C\Big\{ \|F\|_{k-m} + \sum_{j=1}^{m/2} \|\Phi_j\|_{k-m_j-1/2} + \sum_{i=0}^{k-1} [L,B]_{k-i}\|u\|_i \Big\}. \tag{7.7}$$

This gives the case $k = m$ by interpolation. If the assertion is proved for $k \geq m$ then we can apply (7.7) with $k+1$ in place of k. For $m \leq i \leq k$ the terms $[L,B]_{k+1-i}\|u\|_i$ satisfy the desired estimate by induction observing that $[L,B]_{k+1-i}[L,B]_{m,i-l} \leq [L,B]_{m,k+1-l}$. For $0 \leq i < m$ we get

$$[L,B]_{k+1-i}\|u\|_i \leq C[L,B]_{m,k+1-m}\|u\|_m \tag{7.8}$$

and the proved case $k = m$ gives (7.6) for $k + 1$ and thus the result. □

Now let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞-boundary (we note that the following Theorem 7.4 holds for uniformly regular sets of class C^k as well, cf. [51], Theorem 4.10). We consider the boundary value problem

$$\begin{gathered} Lu = F \quad \text{in } \Omega \\ B_j u = \Phi_j \quad \text{on } \partial\Omega,\ j = 1, \dots m/2\,. \end{gathered} \tag{7.9}$$

Theorem 7.4 (cf. [3], Theorem 15.2)**.** *Let* (L, B_j) *be an elliptic pair. Let* $k \geq m, u \in H^k(\Omega)$*. Then* (7.6) *holds with* $C > 0$ *depending on* Ω, k, E*.*

Proof. We use the notation of [3], Theorem 15.2. Let (U_i) be a finite open covering of $\partial\Omega$ and let $T_i : \overline{U_i} \cap \overline{\Omega} \to \Sigma_{R_i}$ be bijective C^∞-maps onto the hemisphere $\Sigma_{R_i} = \{x \in \mathbb{R}^n : x_n \geq 0, |x| \leq R_i\}$ such that $S_i = T_i^{-1}$ is C^∞ and $\overline{U_i} \cap \partial\Omega$ is mapped onto the part $x_n = 0$ of Σ_{R_i}. We may assume that $R_i < r$ where r is the constant from Lemma 7.3. We choose a finite C^∞-partition of unity (ω_σ) in $\overline{\Omega}$ such that the support of each ω_σ is either contained in Ω or in one of the sets U_i denoted by $U_{i(\sigma)}$. We want to estimate $u = \sum \omega_\sigma u$. As in [3] we consider here the case that the support of ω_σ is not contained in Ω; the other case follows analogously using [3], Theorem 14.1' instead of Lemma 7.2. Let ω_σ be such an element and $T = T_{i(\sigma)}, S = T^{-1}$. We put $v = u \circ S$ and $\omega = \omega_\sigma \circ S$ where the support of ω is contained in Σ_R with $R < r$. The transformed operators L, B_j are denoted by $\tilde{L}, \tilde{B}_j$. For $Lu = L(x, \partial_x)u$ given by (7.1) we obtain from the chain rule

$$\tilde{L}(y, \partial_y)v = \sum_{1 \leq |\beta| \leq |\alpha| \leq m} a_\alpha(S(y)) A_{\beta,\alpha}(S(y)) \partial_y^\beta v(y) + a_0(S(y))v(y) \tag{7.10}$$

with smooth $A_{\beta,\alpha}$ depending on T. This gives for $i \geq m$ the estimates

$$\|\tilde{L}(\omega v)\|_{i-m} \leq C\Big\{\|Lu\|_{i-m} + \sum_\alpha \sum_{l=m-1}^{i-1} \|a_\alpha\|_{i-l-1}^\infty \|u\|_l\Big\}. \tag{7.11}$$

Analogously we obtain the estimates (cf. the proof of [3], Theorem 15.2)

$$\|\tilde{B}_j(\omega v)\|_{i-m_j-\frac{1}{2}} \leq C\Big\{\|B_j u\|_{i-m_j-\frac{1}{2}} + \sum_\beta \sum_{l=m_j-1}^{i-1} \|b_{j,\beta}\|_{i-1-l}^\infty \|u\|_l\Big\}.$$

Since $\|u\|_k \leq \sum \|\omega_\sigma u\|_k \leq C \sum \|\omega v\|_k$ we get from Lemma 7.3 that

$$\begin{aligned} \|u\|_k \leq C\Big\{ &\sum_{i=m}^{k} [L,B]_{m,k-i}\Big(\|Lu\|_{i-m} + \sum_{j=1}^{m/2} \|B_j\|_{i-m_j-\frac{1}{2}} \\ &+ \sum_{l=0}^{i-1} [L,B]_{i-1-l}\|u\|_l\Big) + [L,B]_{m,k-m}\|u\|_0\Big\} \end{aligned} \tag{7.12}$$

Inequality (7.12) gives the case $k = m$. The general case follows from (7.12) by induction on k as in the proof of Lemma 7.3. The theorem is proved. □

Theorem 7.4 gives (P_0,) (P_4,) (P_6) of section 6. The resolvent estimates (P_1), (P_2), (P_3) require stronger ellipticity assumptions due to Agmon [2].

Definitiion 7.5. (cf. [2], [24], Ch. 4, [26], 3.2, [50], 3.8, [51], 5.2). The pair (L, B_j) is called a *regular elliptic pair* if the following holds.

(i) *Smoothness:* L, B_j are given by (7.1), (5.1) with $a_\alpha, b_{j,\beta} \in C^\infty(\overline{\Omega})$.

(ii) *Normality:* The set $\{B_j\}_{j=1}^{m/2}$ is normal and m_j satisfy (6.11).

(iii) *Strong ellipticity:* The order $m \geq 2$ of L is even and there exists $\mu > 0$ such that for each $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and any $x \in \overline{\Omega}, \xi \in \mathbb{R}^n, r \geq 0$ we have

$$|L^P(x,\xi) - (-1)^{m/2} r^m e^{i\theta}| \geq \mu(|\xi|^m + r^m). \tag{7.13}$$

(iv) *Root and Complementing Condition:* For each $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], r \geq 0, x \in \partial\Omega$ and for any $\xi \in \mathbb{R}^n$ tangential to $\partial\Omega$ at x with $(\xi, r) \neq 0$ the polynomial $\tau \mapsto L^P(x, \xi + \tau\nu(x)) - (-1)^{m/2} r^m e^{i\theta}$ has exactly $m/2$ roots with positive imaginary part $\{\tau_j^+(x, \xi, r, \theta)\}_{j=1}^{m/2}$, and the polynomials $\{B_j^P(x, \xi + \tau\nu(x))\}_{j=1}^{m/2}$ are linearly independent modulo $\prod_{j=1}^{m/2}(\tau - \tau_j^+(x, \xi, r, \theta))$.

Taking $r = 0$ we get back Definition 7.1. The above assumptions are made such that for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the operator $L_\theta = L - (-1)^{m/2} e^{i\theta} \partial_t^m$ in $(n+1)$ variables is elliptic in $\overline{\Omega} \times \mathbb{R}$ and satisfies together with (B_j) the root and complementing condition in 7.1. Let E_0 denote the maximum of the characteristic constants of the frozen operator $L_\theta(x, t, \partial), t \in [-1, 1], x \in \overline{\Omega}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Condition (iii) holds iff L is strongly elliptic, i.e., if

$$-(-1)^{m/2} \mathrm{Re}\, L^P(x, \xi) > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n \backslash \{0\}. \tag{7.14}$$

Any strongly elliptic operator satisfies the root condition (cf. [51], Theorem 5.4) and together with the Dirichlet boundary conditions the complementing condition (cf. [24], Ch. 4). For instance, $-(-\Delta)^{m/2}$ is strongly elliptic.

L is a closed operator in $L^2(\Omega)$ with

$$\mathcal{D}(L) = \{u \in H^m(\Omega) : B_j u = 0 \text{ on } \partial\Omega, j = 1, \dots, m/2\}. \tag{7.15}$$

We state the following result of Agmon [2], Theorem 2.1. (cf. [26], Theorem 3.1.3, [29], 7.3.2, [50], 3.8, [51], Theorem 5.5).

Theorem 7.6. *Let (L, B_j) be a regular elliptic pair. Then there exist $C > 0, \gamma > 0$ depending on E_0, Ω such that for any $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have*

$$\rho(L) \supset \Gamma_{\theta,\gamma} = \{\lambda \in \mathbb{C} : \arg \lambda = \theta, |\lambda| \geq \gamma\}. \tag{7.16}$$

For $\lambda \in \Gamma_{\theta,\gamma}, u \in H^m(\Omega), g_j \in H^{m-m_j}(\Omega)$ with $B_j u = g_j$ on $\partial\Omega$ we have

$$\sum_{j=0}^{m} |\lambda|^{\frac{m-j}{m}} \|u\|_j \leq C\Big\{\|\lambda u - Lu\|_0 + \sum_{j=1}^{m/2} \big(|\lambda|^{\frac{m-m_j}{m}} \|g_j\|_0 + \|g_j\|_{m-m_j}\big)\Big\}.$$

In particular, we have for any $\lambda \in \Gamma_{\theta,\gamma}$ and $u \in \mathcal{D}(L)$ the estimate

$$|\lambda| \|u\|_0 + \|u\|_m \leq C\|\lambda u - Lu\|_0. \tag{7.17}$$

Proof. The estimates in Theorem 7.6 are proved in [50], Lemma 3.8.1 by applying (7.6) with L_θ and $k = m$ only to functions $u = u(x, t)$ which vanish for $|t| \geq 1$. This and the proof of [50], Lemma 3.8.1 give the statement on the constants C, γ. These estimates imply (7.16) (cf. [2], [50]). □

Corollary 7.7. *Let (L, B_j) be a regular elliptic pair. Then there exist $\omega > 0, \theta_0 \in (\pi/2, \pi), M_1 > 0$ depending on E_0, Ω such that $A = L - \omega I$ satisfies $\rho(A) \supset \Sigma_{\theta_0} \cup \{0\}$ where $\Sigma_{\theta_0} = \{\lambda : -\theta_0 < \arg \lambda < \theta_0\}$ and*

$$\|(\lambda I - A)^{-1}\|_{L(H^0)} \leq M_1/|\lambda|, \quad \lambda \in \Sigma_{\theta_0}. \tag{7.18}$$

The proof of this corollary follows immediately from Theorem 7.6 choosing $\omega = 2\gamma$.

We assume that L, B_j depend on t. Let $a_\alpha, b_{j,\beta} \in C^\infty([0, T_1] \times \overline{\Omega})$ where $T_1 > 0$ is fixed. Let $(L(t), B_j(t))$ be a regular elliptic pair for each $t \in [0, T_1]$ where m, m_j do not depend on t. Let $E_0(t)$ denote the corresponding characteristic constant defined as E_0 above. We then have $E_0 = \max\{E_0(t) : t \in [0, T_1]\} < +\infty$ by continuity. Hence the constants entering in Theorem 7.6 and Corollary 7.7 can be chosen uniformly for $t \in [0, T_1]$. We put $A(t) = L(t) - \omega I$ as in Corollary 7.7.

Lemma 7.8. *Let $(L(t), B_j(t))$ be a regular elliptic pair for each $t \in [0, T_1]$ and let $A(t) = L(t) - \omega I$. Then for every $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ the mapping $t \mapsto (\lambda I - A(t))^{-1}$ belongs to $C^1([0, T_1], L(H^0))$. There exist $M_2, M_3 > 0$ depending only on E_0, Ω and on $\|\partial_t A\|_0 + \sum_j \|\partial_t B_j\|_{m-m_j}$ such that for all $\lambda \in \Sigma_{\theta_0}$ and $t, \tau \in [0, T_1]$ we have*

$$\|\partial_t (A(t) - \lambda I)^{-1}\|_{L(H^0)} \le M_2/|\lambda| \tag{7.19}$$

$$\|\partial_t (A(t)^{-1}) - \partial_t (A(\tau)^{-1})\|_{L(H^0)} \le M_3 |t - \tau|. \tag{7.20}$$

The proof of this lemma can be found in [50], Lemma 5.3.6. It is based on Theorem 7.6.

8. The nonlinear parabolic problem

We consider the nonlinear initial boundary value problem (4.9). We assume that Ω is a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^∞ and that $\mathcal{F}$ and $\mathcal{B} = (\mathcal{B}_j)_{j=1}^{m/2}$ are smooth differential operators defined by (4.2), (4.6) in $[0, T] \times U$ as in section 4. We fix an initial value $\phi \in U \subset H^\infty(\Omega)$ and a boundary value $h \in C^\infty([0, T], H^\infty(\partial\Omega)^{m/2})$. We suppose the necessary compatibility conditions (4.11) coupling ϕ and h. By Theorem 4.4 we have to solve the linear problem (4.28).

We assume that the pair $(\mathcal{F}_u(0, \phi), \mathcal{B}_u(0, \phi))$ is a regular elliptic pair in the sense of Definition 7.5. We can choose $0 < T_1 \le T$ and an open neighbourhood V of ϕ in $H^\infty(\Omega)$ such that $(\mathcal{F}_u(t, u(t)), \mathcal{B}_u(t, u(t)))$ is a regular elliptic pair with a uniform characteristic constant E_0 for all $t \in [0, T_1], u \in W$ where

$$W = \{u \in C^\infty([0, T_1], H^\infty(\Omega)) : u(t) \in V, t \in [0, T_1]\}. \tag{8.1}$$

Theorem 8.1. *Let $\mathcal{F}, \mathcal{B}$ be smooth differential operators. Let the initial value $\phi \in H^\infty(\Omega)$ and the boundary value $h \in C^\infty([0, T], H^\infty(\partial\Omega)^{m/2})$ satisfy the compatibility conditions* (4.11). *Assume that $(\mathcal{F}_u(0, \phi), \mathcal{B}_u(0, \phi))$ is a regular elliptic pair. Then there exist $T_0 > 0$ and a unique solution $u \in C^\infty([0, T_0], H^\infty(\Omega))$ of the nonlinear initial value problem* (4.9).

Proof. We have to verify the assumptions of Theorem 4.4. The existence of the required mappings R_u is proved in Corollary 5.6. We choose T_1, V, W as above such that $(\mathcal{F}_u(t, u(t)), \mathcal{B}_u(t, u(t)))$ is a regular elliptic pair for every $t \in [0, T_1], u \in W$. We fix $0 < T \le T_1$ and consider the linear problem (4.28) where $f_1 \in C_0^\infty([0, T], H^\infty(\Omega))$. Since $f_1^{(j)}(0) = 0$ for all j this is a problem with trivial (vanishing) compatibility relations. By classical results on linear parabolic boundary value problems (cf. [24], Ch. IV, 6.4) there is a unique solution $w \in C_0^\infty([0, T], H^\infty(\Omega))$ of problem (4.28).

We have to show estimates (4.29). We write $L(t) = \mathcal{F}_u(t, u(t)), B(t) = \mathcal{B}_u(t, u(t))$ observing that the following holds uniformly for $u \in W$. Using 7.7, 7.8 we choose $\omega, \theta_0, M_1, M_2, M_3$ such that $A(t) = L(t) - \omega I$ satisfies conditions (P_1), (P_2), (P_3). By Theorem 7.4 conditions (P_0), (P_4), (P_6) hold for $A(t)$ with uniform constants M_0, M_4, M_k. Choosing W sufficiently small we obtain condition (P_5) for $A(t), B(t)$ with a uniform constant M_5. Hence Theorem 6.4 applies to the pair $(A(t), B(t))$. We put $f(t) = e^{-\omega t} f_1(t)$ and $z(t) = e^{-\omega t} w(t)$. Then z is a solution of problem

(6.13). By Theorem 6.4 thus z satisfies the estimate (6.38) with a uniform constant C depending on k. Replacing (z, f) in (6.38) by (w, f_1) we get the estimate

$$\|w\|_{km} = \sum_{i=0}^{k} |w|_{i,k(m-i)} \le C \sum_{i=0}^{k} [A, B]_{3m,(k-i)m} \|f_1\|_{mi} \le C[A, B; f_1]_{3m,k}$$

and thus $\|w\|_k \le C[A, B; f_1]_{4m,k}$ for any k, shrinking V, W if necessary. Since $\|A\|_i + \|B\|_i \le C[u]_{m+b,i}$ for $b = [n/2] + 1$ (cf. (4.4), (4.8)) this implies $\|w\|_k \le C[u; f_1]_{5m+b,k}$ for any k. We proved (4.29) and thus the result. □

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