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ON Γ-CONVERGENCE FOR PROBLEMS OF JUMPING TYPE

ALESSANDRO GROLI

Abstract. The convergence of critical values for a sequence of functionals (f_h) Γ-converging to a functional f_∞ is studied. These functionals are related to a classical "jumping problem", in which the position of two real parameters α, β plays a fundamental role. We prove the existence of at least three critical values for f_h, when α and β satisfy the usual assumption with respect to f_∞, but not with respect to f_h.

1. Introduction

Let (f_h) be a sequence of functionals from $H_0^1(\Omega)$ to $\mathbb{R}$ and f_∞ a functional from $H_0^1(\Omega)$ to $\mathbb{R}$. It is well known that the convergence of (possible) minima of f_h to those of f_∞ can be studied in an efficient way by the notion of Γ-convergence [7, 13] (epiconvergence, in the language of [2]).

The problem of the convergence of critical points, on the contrary, is much less clarified. A certain number of results is available in the literature, dealing with the case in which f_h is Γ-convergent to f_∞ and satisfies suitable uniform assumptions (see e.g. [9, 10, 11] and references therein).

In particular, let us remark that the applications to PDE's, so far considered, concern only functionals of the calculus of variations whose principal part is convex.

We are interested in a further case, which is not covered in the literature and is particularly interesting for critical point theory: that of "jumping problems". It can be considered as a perturbation of the functional $f_\infty : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$f_\infty(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x) D_i u D_j u \, dx - \frac{\alpha}{2}\int_\Omega (u^+)^2 \, dx - \frac{\beta}{2}\int_\Omega (u^-)^2 \, dx + \int_\Omega \phi_1 u \, dx,$$

where $\beta < \alpha$ and ϕ_1 is a positive eigenfunction of $-\sum D_j(A_{ij}^{(\infty)} D_i u)$ with homogeneous Dirichlet condition. The simplest type of perturbation, extensively considered in the literature, amounts to consider

$$\begin{aligned} f_h(u) =& \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x) D_i u D_j u \, dx - \frac{\alpha}{2}\int_\Omega (u^+)^2 \, dx \\ & - \frac{\beta}{2}\int_\Omega (u^-)^2 \, dx - \int_\Omega \frac{G_0(x, t_h u)}{t_h^2} \, dx + \int_\Omega \phi_1 u \, dx, \end{aligned}$$

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where $t_h \to +\infty$, and

$$\lim_{|s|\to+\infty} \frac{D_s G_0(x,s)}{s} = 0\,.$$

In such a case, very refined results have been obtained, starting from the pioneering paper [1], (see e.g. [16, 17, 18, 19] and references therein).
More recently, some results have been obtained when

$$\begin{aligned} f_h(u) =& \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,t_h u) D_i u D_j u\,dx - \frac{\alpha}{2}\int_\Omega (u^+)^2\,dx \\ & - \frac{\beta}{2}\int_\Omega (u^-)^2\,dx - \int_\Omega \frac{G_0(x,t_h u)}{t_h^2}\,dx + \int_\Omega \phi_1 u\,dx, \end{aligned}$$

where t_h and G_0 are as above (see [3, 4]). Observe that in this case the principal part is no longer convex.

Here we are interested in a more general perturbation of the form

$$\begin{aligned} f_h(u) =& \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u) D_i u D_j u\,dx - \frac{\alpha}{2}\int_\Omega (u^+)^2\,dx \\ & - \frac{\beta}{2}\int_\Omega (u^-)^2\,dx - \int_\Omega \frac{G_0(x,t_h u)}{t_h^2}\,dx + \int_\Omega \phi_1 u\,dx. \end{aligned}$$

Actually, for the sake of simplicity, we will consider only the case $G_0 = 0$, being the perturbation of the principal part the most interesting feature.

Let us mention that the result we are interested in, namely the existence of at least three critical points for f_h, is well known if $\beta < \mu_1^{(h)} < \mu_2^{(h)} < \alpha$, where $\mu_1^{(h)}, \mu_2^{(h)}$ are the first two eigenvalues of $-\sum D_j(A_{ij}^{(h)} D_i u)$, then

$$\lim_{s\to+\infty} a_{ij}^{(h)}(x,s) = \lim_{s\to-\infty} a_{ij}^{(h)}(x,s) = A_{ij}^{(h)}(x)$$

(see [4]). The point is that, under our assumptions, we have $\beta < \mu_1^{(h)}$. But it may happen that $\alpha < \mu_2^{(h)}$ for any $h \in \mathbb{N}$ (see Example 3.2). Nevertheless, the hypothesis that $\alpha > \mu_2$, where μ_2 is the second eigenvalue of $-\sum D_j(A_{ij}^{(\infty)} D_i u)$ combined with the Γ-convergence of f_h to f_∞, is sufficient to ensure, for h large, the existence of at least three critical points of f_h. In some sense, we find a genuine effect of Γ-convergence, which cannot be deduced by the usual study of the position of β and α with respect to the spectrum of $-\sum D_j(A_{ij}^{(h)} D_i u)$. Let us also mention that a relevant question, in jumping problem, is the position of α and β with respect to the Fučik spectrum (see e.g. [8]). However this seems to be important mainly for the verification of the Palais-Smale condition, while the persistence of the geometrical conditions on the functional under Γ-convergence is the key point in our problem.

This paper is organized as follows. In section 2 we recall some notions of nonsmooth analysis and prove a nonsmooth version of the classical "local saddle theorem". In section 3 we present the problem and the main result. Section 4 is devoted to show some minmax estimates which allow us to prove the main theorem in section 5.

2. Tools of nonsmooth analysis

In this section, we recall some by-products of the nonsmooth critical point theory developed in [6, 12]. Let X be a metric space endowed with the metric d and $r > 0$. Let us set $B_r(u) = \{v \in X : d(u, v) < r\}$ and $S_r(u) = \{v \in X : d(u, v) = r\}$.

Definition 2.1. Let $f : X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ satisfying

$$d(\mathcal{H}(v, t), v) \le t, \quad f(\mathcal{H}(v, t)) \le f(v) - \sigma t,$$

whenever $v \in B_\delta(u)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the weak slope of f at u.

The following two definitions are related to the notion above.

Definition 2.2. Let $f : X \to \mathbb{R}$ be a continuous function. An element $u \in X$ is said to be critical point of f, if $|df|(u) = 0$. A real number c is said to be a critical value for f, if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be a regular value of f.

Definition 2.3. Let $f : X \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$. The function f is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ for short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence converging in X.

The next result is an adaptation to a continuous functional of the classical local saddle theorem (see e.g. [17]).

Theorem 2.4. *Let X be a Banach space and $f : X \to \mathbb{R}$ be a continuous function. Assume that there exist two closed subspaces X_1, X_2 of X with $\dim X_1 < +\infty$ and $X = X_1 \oplus X_2$. Let $u_0 \in X$ and U_1, U_2 be two bounded neighborhoods of 0 in respectively X_1 and X_2 with U_2 convex. Suppose that*

$$\sup f(u_0 + \partial U_1) < a = \inf f(u_0 + \overline{U_2}), \quad b = \sup f(u_0 + \overline{U_1}) < \inf f(u_0 + \partial U_2),$$

and f satisfies $(PS)_c$ for any $c \in [a, b]$. Then there exists at least a critical point for f in $f^{-1}([a, b])$.

Proof. Without loss of generality, we can suppose $u_0 = 0$. We argue by contradiction and assume that there are no critical values for f in $[a, b]$. Since f satisfies $(PS)_c$ for every $c \in [a, b]$, it is readily seen that, for some $\varepsilon > 0$, there are no critical values for f in $[a - \varepsilon, b]$ and that f satisfies $(PS)_c$ for any $c \in [a - \varepsilon, b]$. By [6, Theorem 2.15] or [5, Theorem 1.1.14] there exists a continuous map $\eta : X \times [0, 1] \to X$ such that

$$\eta(u, 0) = u \quad \forall\, u \in X,$$
$$\eta(u, t) = u \quad \forall\, t \in [0, 1], \forall u \in f^{a-\varepsilon},$$
$$\eta(u, 1) \in f^{a-\varepsilon} \quad \forall\, u \in f^b,$$
$$f(\eta(u, t)) \le f(u) \quad \forall t \in [0, 1], \quad \forall u \in X.$$

Since $\overline{U_1} \subset f^b$, $\eta(\overline{U_1} \times \{1\}) \subset f^{a-\varepsilon}$. On the other hand, since $f^{a-\varepsilon} \cap \overline{U_2} = \emptyset$, it follows that

$$\eta(\overline{U_1} \times \{1\}) \cap \overline{U_2} = \emptyset. \tag{2.1}$$

Now consider the continuous map

$$\begin{array}{rcl}\Phi:[-1,1]\times\overline{U_1} & \to & \mathbb{R}\times X_1\\ (s,u) & \mapsto & (\rho_{U_2}(P_2\eta(u,1))+s,P_1\eta(u,1))\end{array}$$

where $P_i : X \to X_i$ $(i = 1, 2)$ are the projections of X onto X_i and ρ_{U_2} : $X_2 \to [0, +\infty[$ is the Minkowski functional associated with U_2. Since $(0, 0) \notin \Phi\,(\partial(]-1, 1[\times U_1))$, the Brouwer degree (see e.g. [14])

$$\deg\,(\Phi,]-1, 1[\times U_1, (0, 0))$$

is well defined. Moreover the continuous function defined by

$$\mathcal{H}((s, u), t) = (\rho_{U_2}(P_2\eta(u, t)) + s, P_1\eta(u, t))$$

is a homotopy between the identity map and Φ.

Since $(0, 0) \notin \mathcal{H}\,(\partial(]-1, 1[\times U_1) \times [0, 1])$, it follows that

$$\deg\,(\Phi,]-1, 1[\times U_1, (0, 0)) = 1.$$

Therefore, there exists $(s, u) \in]-1, 1[\times U_1$ such that $\Phi(s, u) = (0, 0)$. Hence we have $\eta(u, 1) \in X_2$ and $\rho_{U_2}(\eta(u, 1)) = -s$, namely $\rho_{U_2}(\eta(u, 1)) \le 1$. Therefore, $\eta(u, 1) \in \overline{U_2}$ and we have

$$\eta(U_1 \times \{1\}) \cap \overline{U_2} \ne \emptyset$$

which contradicts (2.1). □

Let us recall the notion of Γ-convergence (epiconvergence in the language of [2]) from [13].

Definition 2.5. Consider a topological space X. For any $h \in \mathbb{N} \cup \{+\infty\}$, let $g_h : X \to \mathbb{R} \cup \{+\infty\}$ be a function. According to [2, 13], we write that

$$g_\infty = \Gamma(X^-)\lim_h g_h$$

if the following facts hold:

(i) if (u_h) is a sequence in X convergent to u, we have $g_\infty(u) \le \liminf_h g_h(u_h)$;

(ii) for every $u \in X$, there exists a sequence (u_h) in X convergent to u such that $g_\infty(u) = \lim_h g_h(u_h)$.

3. Position of the problem and main result

Let Ω be a connected bounded open subset of $\mathbb{R}^n$ (for the sake of simplicity we suppose $n \ge 3$). We assume that, for every $h \in \mathbb{N}$, the functions $a_{ij}^{(h)} : \Omega \times \mathbb{R} \to \mathbb{R}$ and the function $A_{ij}^{(\infty)} : \Omega \to \mathbb{R}$ $(1 \le i, j \le n)$ satisfy the following conditions:

(A1) For all $s \in \mathbb{R}$, $a_{ij}^{(h)}(\cdot, s)$ and $A_{ij}^{(\infty)}(\cdot)$ are measurable; for a.e. $x \in \Omega$, $a_{ij}^{(h)}(x, \cdot)$ is of class C^1; for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $a_{ij}^{(h)}(x, s) = a_{ji}^{(h)}(x, s)$, $A_{ij}^{(\infty)}(x) = A_{ji}^{(\infty)}(x)$.

(A2) There exists $C > 0$ such that for each $h \in \mathbb{N}$, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$, $1 \le i, j \le n$,

$$|a_{ij}^{(h)}(x, s)| \le C, \quad |A_{ij}^{(\infty)}(x)| \le C, \quad \Big|\sum_{i,j=1}^{n} sD_s a_{ij}^{(h)}(x, s)\xi_i\xi_j\Big| \le C|\xi|^2.$$

(A3) There exists $\nu > 0$ such that for each $h \in \mathbb{N}$, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$,
$$\sum_{i,j=1}^n a_{ij}^{(h)}(x,s)\xi_i\xi_j \geq \nu|\xi|^2, \quad \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)\xi_i\xi_j \geq \nu|\xi|^2.$$

(A4) For each $h \in \mathbb{N}$, there exists $R_h > 0$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$,
$$|s| > R_h \Rightarrow \sum_{i,j=1}^n sD_s a_{ij}^{(h)}(x,s)\xi_i\xi_j \geq 0.$$

(A5) For a.e. $x \in \Omega$, assume that
$$\lim_{s\to+\infty} a_{ij}^{(h)}(x,s) = \lim_{s\to-\infty} a_{ij}^{(h)}(x,s) = A_{ij}^{(h)}(x)$$
(observe that by (A4) such limits exist).

(A6) For all $h \in \mathbb{N}$ there exists uniformly Lipschitz continuous bounded functions $\psi_h : \mathbb{R} \to [0,+\infty[$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^n$
$$\sum_{i,j=1}^n sD_s a_{ij}^{(h)}(x,s)\xi_i\xi_j \leq 2s\psi_h'(s)\sum_{i,j=1}^n a_{ij}^{(h)}(x,s)\xi_i\xi_j.$$
Also assume that
$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx\\ &= \Gamma(w-H_0^1(\Omega)^-)\lim_h \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u)D_iuD_ju\,dx\end{aligned} \tag{3.1}$$
where $w - H_0^1(\Omega)$ denotes the space $H_0^1(\Omega)$ endowed with the weak topology. Let μ_k, $\mu_k^{(h)}$ denote the eigenvalues of respectively the operators $-\sum D_j(A_{ij}^{(\infty)}D_iu)$ and $-\sum D_j(A_{ij}^{(h)}D_iu)$ with homogeneous Dirichlet condition and ϕ_k, $\phi_k^{(h)}$ the corresponding eigenfunctions. It is well known (see [15]) that $\phi_1 \in H_0^1(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)$ and that we can take $\phi_1(x) > 0$ for every $x \in \Omega$ and $\int_\Omega \phi_1^2\,dx = 1$.

(A7) Assume that $\lim_h \mu_1^{(h)} = \mu_1$.

Our purpose in this article is to study the existence of weak solutions of the family of problems:
$$\begin{gathered}-\sum_{i,j=1}^n D_j(a_{ij}^{(h)}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}^{(h)}(x,u)D_iuD_ju = \alpha u^+ - \beta u^- - \phi_1,\\ u \in H_0^1(\Omega),\end{gathered} \tag{3.2}$$
where α, β are two real numbers, $u^+ = \max\{u,0\}$, $u^- = \max\{-u,0\}$.

Under the assumptions above, we shall prove is the following result.

Theorem 3.1. *Assume that $\beta < \mu_1$ and $\alpha > \mu_2$. Then there exists $\overline{h}$ in $\mathbb{N}$ such that for all $h \geq \overline{h}$, the problem (3.2) has at least three weak solutions in $H_0^1(\Omega)$.*

For $\alpha > \mu_2^{(h)}$, this result corresponds to [4, Theorem 1.1]; however our assumptions do not imply that $\alpha > \mu_2^{(h)}$ for large h. As the following example shows, it may happen that $\mu_2 < \mu_2^{(h)}$ (and hence $\alpha \in]\mu_2, \mu_2^{(h)}[$).

Example 3.2. Let $\Omega =]0, \pi[$ and define the functions $a_{ij}^{(h)}(x, s)$ such that: for $x \in]0, \frac{\pi}{2}[$,

$$a_{ij}^{(h)}(x,s) = \begin{cases} \gamma\delta_{ij}(x) & s \in]-h, h[, \\ \delta_{ij}(x) & s \in \mathbb{R} \setminus [-2h, 2h]; \end{cases}$$

for $x \in]\frac{\pi}{2}, \pi[$

$$a_{ij}^{(h)}(x,s) = \begin{cases} \eta\delta_{ij}(x) & s \in]-h, h[, \\ \delta_{ij}(x) & s \in \mathbb{R} \setminus [-2h, 2h], \end{cases}$$

where $\delta_{ij}(x) = 1$ if $i = j$, $\delta_{ij}(x) = 0$ if $i \neq j$ and $\gamma, \eta \in \mathbb{R}$. Then, $A_{ij}^{(h)}(x) = \delta_{ij}(x)$. The eigenvalues $\mu_k^{(h)}$ of the Dirichlet problem

$$\begin{gathered} -u'' = \mu u, \\ u(0) = u(\pi) = 0, \end{gathered}$$

are $\mu_k^{(h)} = k^2$, for all $k \geq 1$. On the other hand, all the assumptions of Theorem 3.1 are satisfied with

$$A_{ij}^{(\infty)}(x) = \begin{cases} \gamma\delta_{ij}(x) & 0 < x < \frac{\pi}{2}, \\ \eta\delta_{ij}(x) & \frac{\pi}{2} < x < \pi. \end{cases}$$

Hence, the eigenvalues μ_k of the Dirichlet problem

$$\begin{gathered} -\left(A_{ij}^{(\infty)}(x)u'\right)' = \mu u, \\ u(0) = u(\pi) = 0, \end{gathered}$$

for η such that

$$\sqrt{\frac{1}{\eta}}\frac{\pi}{4} = \arctan\sqrt{5},$$

and $\gamma = 4\eta$, are $\mu_1 = \mu_1^{(h)} = 1$ and since $\arctan\sqrt{5} > \arctan\sqrt{3} = \frac{\pi}{3}$, it follows that

$$\mu_2 = \left(\frac{\pi - \arctan\sqrt{5}}{\arctan\sqrt{5}}\right)^2 < 4 = \mu_2^{(h)}.$$

4. Minmax estimates

We introduce the functionals $f_h, f_\infty, \widehat{f}_\infty : H_0^1(\Omega) \to \mathbb{R}$,

$$f_h(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u)D_iuD_ju\,dx - \frac{\alpha}{2}\int_\Omega (u^+)^2\,dx - \frac{\beta}{2}\int_\Omega (u^-)^2\,dx + \int_\Omega \phi_1 u\,dx,$$

$$f_\infty(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx - \frac{\alpha}{2}\int_\Omega (u^+)^2\,dx - \frac{\beta}{2}\int_\Omega (u^-)^2\,dx + \int_\Omega \phi_1 u\,dx,$$

$$\widehat{f}_\infty(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx + \int_\Omega \phi_1 u\,dx.$$

For later use, we also introduce $g_h, g_\infty : H_0^1(\Omega) \to \mathbb{R}$ as the "principal parts" of f_h and f_∞:

$$g_h(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u) D_i u D_j u \, dx,$$

$$g_\infty(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x) D_i u D_j u \, dx.$$

The following theorem provides a fundamental connection between the above abstract notion of weak slope and the concrete notion related to our problem.

Theorem 4.1. *Let $u \in H_0^1(\Omega)$ be a critical point of f_h. Then, u is a weak solution of* (3.2).

The proof of this theorem can be found in [4, Corollary 2.8].

To apply the local saddle theorem, we shall need two ingredients: the Palais Smale condition and some minmax estimates.

Theorem 4.2. *Let $\beta < \mu_1 < \alpha$. Then, for all $a, b \in \mathbb{R}$ there exists $\bar{h} \in \mathbb{N}$ such that f_h satisfies $(PS)_c$ for all $h \geq \bar{h}$ and every $c \in [a,b]$.*

Proof. In view of assumption (A7), $\beta < \mu_1^{(h)} < \alpha$ eventually, so we can apply [4, Theorem 3.1] and deduce the assertion. □

For the rest of this article, we shall consider $\beta < \mu_1$ and $\mu_k < \alpha \leq \mu_{k+1}$ with $k \geq 2$. Define

$$\overline{\phi}_1 = \frac{\phi_1}{\alpha - \mu_1},$$

$$H_k = \operatorname{span}\{\phi_1, \cdots, \phi_k\}, \quad H_k^\perp = \operatorname{span}\{\phi_{k+1}, \cdots\}.$$

Let $\psi_2, \dots, \psi_k \in C_c^\infty(\Omega)$. Consider the space

$$\widehat{H}_k = \operatorname{span}\{\phi_1, \psi_2 \cdots, \psi_k\}.$$

If $\psi_2, \dots, \psi_k$ are sufficiently close in the H_0^1-norm to $\phi_2, \dots, \phi_k$, then $H_0^1(\Omega) = \widehat{H}_k \oplus H_k^\perp$. Moreover, since $\overline{\phi}_1$ is a critical point for $\widehat{f}_\infty$, it is readily seen that

$$\forall \rho > 0 : \sup_{\widehat{H}_k \cap S_\rho(\overline{\phi}_1)} \widehat{f}_\infty < \widehat{f}_\infty(\overline{\phi}_1). \tag{4.1}$$

Lemma 4.3. *There exist $\varepsilon, \rho > 0$ such that for all $u \in \widehat{H}_k \cap B_\rho(\overline{\phi}_1)$ the condition $u(x) \geq \varepsilon \phi_1(x)$ holds a.e. in Ω.*

Proof. It is sufficient to recall that $\inf_K \phi_1 > 0$ for every compact subset K of Ω. □

Lemma 4.4. *There exist $u_0, \dots, u_m \in \widehat{H}_k$ such that if $S = \operatorname{conv}\{u_0, \dots, u_m\}$, then S is a neighborhood of $\overline{\phi}_1$ and*

$$\sup \{f_\infty(u) : u \in S\} \leq f_\infty(\overline{\phi}_1),$$

$$\sup \Big\{ f_\infty(u) : u \in \partial_{\widehat{H}_k} S \Big\} < f_\infty(\overline{\phi}_1).$$

Proof. If ρ is as in Lemma 4.3, recalling (4.1), we have

$$\sup\Big\{f_\infty(u) : u \in \overline{B_\rho(\overline{\phi}_1)} \cap \widehat{H}_k\Big\} \le f_\infty(\overline{\phi}_1),$$

$$\sup\Big\{f_\infty(u) : u \in \Big(\overline{B_\rho(\overline{\phi}_1)} \setminus B_{\frac{\rho}{2}}(\overline{\phi}_1)\Big) \cap \widehat{H}_k\Big\} < f_\infty(\overline{\phi}_1).$$

The assertions follow easily. □

Lemma 4.5. *Let S be as in Lemma 4.4. Then, there exists $R > 0$ such that, if $u \in \widehat{H}_k \cap S$ and*

$$u_h \to u \text{ weakly in } H^1_0(\Omega), \quad f_h(u_h) \to f_\infty(u),$$

then $\limsup_h \|u_h\|_{H^1_0(\Omega)} < R$.

Proof. Fix $u \in \widehat{H}_k \cap S$. In view of (3.1), there exists a sequence (u_h) such that $u_h \to u$ weakly in $H^1_0(\Omega)$ and $f_h(u_h) \to f_\infty(u)$. Eventually we have

$$f_h(u_h) < \sup\{f_\infty(u) : u \in \widehat{H}_k \cap S\} + 1.$$

Moreover we have

$$\begin{aligned}&\lim_h \Big\{-\frac{\alpha}{2}\int_\Omega (u_h^+)^2\,dx - \frac{\beta}{2}\int_\Omega (u_h^-)^2\,dx + \int_\Omega \phi_1 u_h\,dx\Big\}\\ &= -\frac{\alpha}{2}\int_\Omega (u^+)^2\,dx - \frac{\beta}{2}\int_\Omega (u^-)^2\,dx + \int_\Omega \phi_1 u\,dx.\end{aligned}$$

Therefore, $g_h(u_h)$, the principal part of $f_h(u_h)$, is (eventually) bounded. Hence, using (A3), we deduce the assertion. □

Let now X_1 be the eigenspace associated to μ_{k+1} and $X_2 = \operatorname{span}\{\phi_{k+2}, \dots\}$ so that

$$H_k^\perp = X_1 \oplus X_2.$$

Proposition 4.6. *Let R be as in Lemma 4.5. Then there exist a finite dimensional space $\widehat{X}_1 \subseteq C^\infty_c(\Omega)$, $\rho_1 > 0$ and $\rho_2 > R$ such that*

$$H^1_0(\Omega) = \widehat{H}_k \oplus \widehat{X}_1 \oplus X_2, \tag{4.2}$$

$$\liminf_h \Big[\inf\Big\{f_h(\overline{\phi}_1 + u) : u \in \partial_{\widehat{X}_1 \oplus X_2} Q\Big\}\Big] > f_\infty(\overline{\phi}_1), \tag{4.3}$$

$$\liminf_h \big[\inf\{f_h(\overline{\phi}_1 + u) : u \in Q\}\big] \ge f_\infty(\overline{\phi}_1), \tag{4.4}$$

where $Q = \Big(\widehat{X}_1 \cap \overline{B_{\rho_1}(0)}\Big) + \Big(X_2 \cap \overline{B_{\rho_2}(0)}\Big)$.

Proof. Since $k + 1 \ge 2$, there exists $\rho_1 > 0$ such that

$$\forall v \in X_1 : \overline{\phi}_1 + v \ge 0 \Rightarrow \|v\|_{H^1_0(\Omega)} < \rho_1.$$

Moreover, there exists $\rho_2 > R$ such that

$$f_\infty(\overline{\phi}_1) < \frac{\nu}{4}(\rho_2)^2 - \frac{C}{2}\int_\Omega |D(\overline{\phi}_1 + v)|^2\,dx - \frac{\alpha}{2}\int_\Omega (\overline{\phi}_1 + v)^2\,dx + \int_\Omega \phi_1(\overline{\phi}_1 + v)\,dx, \tag{4.5}$$

for every $v \in X_1 \cap B_{\rho_1}(0)$. We prove (4.2). Let $\{\varphi_1, \dots, \varphi_l\}$ be a L^2−orthonormal basis of X_1 and consider a sequence $\{\varphi_m^{(s)}\}$ $(m = 1, \dots, l)$ in $C^\infty_c(\Omega)$ such that $\varphi_m^{(s)} \to \varphi_m$ in $H^1_0(\Omega)$. Let

$$\widehat{X}_1^{(s)} = \operatorname{span}\{\varphi_1^{(s)}, \dots, \varphi_l^{(s)}\}$$

Eventually as $s \to +\infty$ we have

$$H_0^1(\Omega) = \widehat{H}_k \oplus \widehat{X}_1^{(s)} \oplus X_2.$$

For proving (4.3) we argue by contradiction. Suppose that, up to a subsequence,

$$\lim_s f_{h_s}\left(\overline{\phi}_1 + v_s + w_s\right) \le f_\infty(\overline{\phi}_1),$$

with $u_s = v_s + w_s \in \partial_{\widehat{X}_1^{(s)} \oplus X_2} Q$. Up to a further subsequence, u_s weakly converges to some u. Then $v_s \to v \in X_1$, while $w_s \to w$ weakly in X_2, where $u = v + w$. Using (3.1) we deduce that $\widehat{f}_\infty(\overline{\phi}_1 + v + w) \le f_\infty(\overline{\phi}_1 + v + w) \le f_\infty(\overline{\phi}_1)$. By definition of X_1 and X_2 we have $w = 0$ and $\widehat{f}_\infty(\overline{\phi}_1 + v) = f_\infty(\overline{\phi}_1 + v)$, namely that $\overline{\phi}_1 + v \ge 0$. By the choice of ρ_1, we have $\|v\|_{H_0^1(\Omega)} < \rho_1$. Therefore $\|v_s\|_{H_0^1(\Omega)} < \rho_1$ and $\|w_s\|_{H_0^1(\Omega)} = \rho_2$ eventually. Using $(A2)$ and $(A3)$, we get

$$\begin{aligned}
& f_{h_s}\left(\overline{\phi}_1 + u_s\right) \\
&= f_{h_s}\left(\overline{\phi}_1 + v_s + w_s\right) \\
&= \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h_s)}(x, \overline{\phi}_1 + u_s) D_i(\overline{\phi}_1 + v_s) D_j(\overline{\phi}_1 + v_s)\, dx \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h_s)}(x, \overline{\phi}_1 + u_s) D_i(\overline{\phi}_1 + v_s) D_j w_s\, dx \\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h_s)}(x, \overline{\phi}_1 + u_s) D_i w_s D_j w_s\, dx \\
&\quad - \frac{\alpha}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^+)^2\, dx - \frac{\beta}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^-)^2\, dx + \int_\Omega \phi_1(\overline{\phi}_1 + u_s)\, dx \\
&\ge \frac{1}{4}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h_s)}(x, \overline{\phi}_1 + u_s) D_i w_s D_j w_s\, dx \\
&\quad - \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h_s)}(x, \overline{\phi}_1 + u_s) D_i(\overline{\phi}_1 + v_s) D_j(\overline{\phi}_1 + v_s)\, dx \\
&\quad - \frac{\alpha}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^+)^2\, dx - \frac{\beta}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^-)^2\, dx + \int_\Omega \phi_1(\overline{\phi}_1 + u_s)\, dx \\
&\ge \frac{\nu}{4}\int_\Omega |Dw_s|^2\, dx - \frac{C}{2}\int_\Omega |D(\overline{\phi}_1 + v_s)|^2\, dx - \frac{\alpha}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^+)^2\, dx \\
&\quad - \frac{\beta}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^-)^2\, dx + \int_\Omega \phi_1(\overline{\phi}_1 + u_s)\, dx \\
&= \frac{\nu}{4}(\rho_2)^2 - \frac{C}{2}\int_\Omega |D(\overline{\phi}_1 + v_s)|^2\, dx - \frac{\alpha}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^+)^2\, dx \\
&\quad - \frac{\beta}{2}\int_\Omega ((\overline{\phi}_1 + u_s)^-)^2\, dx + \int_\Omega \phi_1(\overline{\phi}_1 + u_s)\, dx.
\end{aligned}$$

Hence, as $s \to +\infty$ we have

$$f_\infty(\overline{\phi}_1) \ge \frac{\nu}{4}(\rho_2)^2 - \frac{C}{2}\int_\Omega |D(\overline{\phi}_1 + v)|^2\, dx - \frac{\alpha}{2}\int_\Omega (\overline{\phi}_1 + v)^2\, dx + \int_\Omega \phi_1(\overline{\phi}_1 + v)\, dx,$$

which contradicts (4.5). Finally let us prove (4.4). Since

$$f_\infty(\overline{\phi}_1) = \inf_{\overline{\phi}_1 \oplus (\widehat{X}_1 \oplus X_2)} f_\infty, \tag{4.6}$$

the assertion follows. □

Lemma 4.7. *For any $u \in \widehat{H}_k \setminus \{0\}$ there exists a sequence $(u_h) \subset H_0^1(\Omega)$ such that*

$$(u_h - u) \in \widehat{H}_k \oplus X_2, \tag{4.7}$$

$$u_h \to u \text{ weakly in } H_0^1(\Omega), \quad f_h(u_h) \to f_\infty(u), \tag{4.8}$$

$$\forall h \in \mathbb{N} : \ \frac{u_h - u}{\overline{\phi}_1} \in L^\infty(\Omega), \quad \frac{u_h - u}{\overline{\phi}_1} \to 0 \text{ in } L^\infty(\Omega). \tag{4.9}$$

Proof. Fix $u \in \widehat{H}_k \setminus \{0\}$. In view of (3.1), there exists $(\tilde{u}_h)$ such that

$$\tilde{u}_h \to u \text{ weakly in } H_0^1(\Omega), \quad \lim_h f_h(\tilde{u}_h) = f_\infty(u). \tag{4.10}$$

Consider a strictly increasing sequence $(h_k) \subset \mathbb{N}$ such that

$$\forall h \geq h_k : \ \mathcal{L}^n\big(\big\{x \in \Omega : |\tilde{u}_h - u| > \frac{1}{k}\overline{\phi}_1\big\}\big) < \frac{1}{k},$$

where $\mathcal{L}^n$ denotes the Lebesgue measure. Set

$$\varepsilon_h = \begin{cases} 2 & \text{if } h < h_1, \\ \frac{1}{k} & \text{if } h_k \leq h < h_{k+1}. \end{cases}$$

Then $\varepsilon_h > 0$, $\varepsilon_h \to 0$ and $\mathcal{L}^n\left(\left\{x \in \Omega : |\tilde{u}_h - u| > \varepsilon_h\overline{\phi}_1\right\}\right) < \frac{1}{k}$ if $h_k \leq h < h_{k+1}$. In particular

$$\lim_h \mathcal{L}^n\left(\left\{x \in \Omega : |\tilde{u}_h - u| > \varepsilon_h\overline{\phi}_1\right\}\right) = 0. \tag{4.11}$$

Consider now

$$\check{u}_h = u + \left[\big((\tilde{u}_h - u) \vee (-\varepsilon_h\overline{\phi}_1)\big) \wedge (\varepsilon_h\overline{\phi}_1)\right],$$

and denote by $\Pi_{\widehat{X}_1}$ the projection on $\widehat{X}_1$ associated to the decomposition (4.2). Let $v_h = -\Pi_{\widehat{X}_1}(\check{u}_h - u)$, then

$$u_h = \check{u}_h - \Pi_{\widehat{X}_1}(\check{u}_h - u) = \check{u}_h + v_h.$$

satisfies all the requirements (4.7)-(4.9).
Requirement (4.7) is straightforward. Furthermore, since $|\check{u}_h - u| \leq \varepsilon_h\overline{\phi}_1$ a.e. in Ω, (4.9) follows. Since $\check{u}_h \to u$ weakly in $H_0^1(\Omega)$, then $v_h \to 0$ strongly and $u_h \to u$ weakly in $H_0^1(\Omega)$. To show that $f_h(u_h) \to f_\infty(u)$, it suffices to prove that $g_h(u) \to g_\infty(u)$, namely that

$$\lim_h \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x, u_h)D_iu_hD_ju_h\,dx = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx. \tag{4.12}$$

We obtain (4.12) by combining the two following facts:

$$\lim_h \frac{1}{2}\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}^{(h)}(x, u_h)D_iu_hD_ju_h - \sum_{i,j=1}^n a_{ij}^{(h)}(x, \check{u}_h)D_i\check{u}_hD_j\check{u}_h\Big]\,dx = 0 \tag{4.13}$$

and

$$\lim_h \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x, \check{u}_h)D_i\check{u}_hD_j\check{u}_h\,dx = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx. \tag{4.14}$$

Now we prove (4.13). We have

$$\begin{aligned}
&a_{ij}^{(h)}(x,u_h)D_iu_hD_ju_h - a_{ij}^{(h)}(x,\breve{u}_h)D_i\breve{u}_hD_j\breve{u}_h\\
&= a_{ij}^{(h)}(x,u_h)D_i(\breve{u}_h+v_h)D_j(\breve{u}_h+v_h) - a_{ij}^{(h)}(x,\breve{u}_h)D_i\breve{u}_hD_j\breve{u}_h =\\
&= [a_{ij}^{(h)}(x,u_h) - a_{ij}^{(h)}(x,\breve{u}_h)]D_i\breve{u}_hD_j\breve{u}_h + 2a_{ij}^{(h)}(x,u_h)D_i\breve{u}_hD_jv_h\\
&\quad + a_{ij}^{(h)}(x,u_h)D_iv_hD_jv_h.
\end{aligned}$$

Clearly, by assumption (A2),

$$\lim_h \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u_h)D_i\breve{u}_hD_jv_h dx = 0,$$

$$\lim_h \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u_h)D_iv_hD_jv_h dx = 0.$$

On the other hand, there exists $\vartheta \in]0,1[$ such that

$$[a_{ij}^{(h)}(x,u_h) - a_{ij}^{(h)}(x,\breve{u}_h)] = D_sa_{ij}^{(h)}(x,u_h+\vartheta v_h)v_h = D_sa_{ij}^{(h)}(x,u+\eta\overline{\phi}_1+\vartheta v_h)v_h,$$

where $\eta \in \mathbb{R}$ and we have used (4.9) in the last identity. Since there exists $\delta_h > 0$ ($\delta_h \to 0^+$) such that

$$|v_h| \le \delta_h|u+\eta\overline{\phi}_1+\vartheta v_h|$$

using (A2), we deduce that

$$\lim_h \int_\Omega \sum_{i,j=1}^n [a_{ij}^{(h)}(x,u_h) - a_{ij}^{(h)}(x,\breve{u}_h)]D_i\breve{u}_hD_j\breve{u}_h\, dx = 0;$$

hence (4.13) holds. To prove (4.14) denote by χ_F the characteristic function of a set F. We have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,\breve{u}_h)D_i\breve{u}_hD_j\breve{u}_h\, dx\\
&= \frac{1}{2}\int_{\{x:|\tilde{u}_h-u|\le\varepsilon_h\overline{\phi}_1\}} \sum_{i,j=1}^n a_{ij}^{(h)}(x,\tilde{u}_h)D_i\tilde{u}_hD_j\tilde{u}_h\, dx\\
&\quad + \frac{1}{2}\int_{\{x:(\tilde{u}_h-u)>\varepsilon_h\overline{\phi}_1\}} \sum_{i,j=1}^n a_{ij}^{(h)}(x,u+\varepsilon_h\overline{\phi}_1)D_i(u+\varepsilon_h\overline{\phi}_1)D_j(u+\varepsilon_h\overline{\phi}_1)\, dx\\
&\quad + \frac{1}{2}\int_{\{x:(\tilde{u}_h-u)<-\varepsilon_h\overline{\phi}_1\}} \sum_{i,j=1}^n a_{ij}^{(h)}(x,u-\varepsilon_h\overline{\phi}_1)D_i(u-\varepsilon_h\overline{\phi}_1)D_j(u-\varepsilon_h\overline{\phi}_1)\, dx\\
&\le \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,\tilde{u}_h)D_i\tilde{u}_hD_j\tilde{u}_h\, dx\\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u+\varepsilon_h\overline{\phi}_1)D_i(u+\varepsilon_h\overline{\phi}_1)D_j(u+\varepsilon_h\overline{\phi}_1)\chi_{\{x:(\tilde{u}_h-u)>\varepsilon_h\overline{\phi}_1\}}\, dx\\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u-\varepsilon_h\overline{\phi}_1)D_i(u-\varepsilon_h\overline{\phi}_1)D_j(u-\varepsilon_h\overline{\phi}_1)\chi_{\{x:(\tilde{u}_h-u)<-\varepsilon_h\overline{\phi}_1\}}\, dx.
\end{aligned}$$

Using (4.10) and (4.11) we deduce

$$\limsup_h \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,\check{u}_h)D_i\check{u}_h D_j\check{u}_h\,dx \le \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}^{(\infty)}(x)D_iuD_ju\,dx.$$

Assumption (3.1) gives us the conclusion. □

Theorem 4.8. *Let $m \in \mathbb{Z}^+$. For all $r, \varepsilon > 0$ there exists $\delta > 0$ such that if $u_0, \dots, u_m \in \widehat{H}_k \cap B_r(\overline{\phi}_1)$ and*

$$\begin{gathered}\forall j = 0,\dots,m: \quad \operatorname{essinf}_\Omega \frac{u_j}{\overline{\phi}_1} \ge \varepsilon,\\ u_j^{(h)} \to u_j \quad (\textit{as in Lemma 4.7}), \\ \sup\big\{\|\frac{u-v}{\overline{\phi}_1}\|_\infty : u,v \in \operatorname{conv}\{u_0,\dots,u_m\}\big\} < \delta,\end{gathered} \tag{4.15}$$

then

$$\begin{aligned}&\limsup_h \Big\{\sup\Big\{f_h(v_h) : v_h \in \operatorname{conv}\{u_0^{(h)},\dots,u_m^{(h)}\}\Big\}\Big\}\\ &\le \sup\{f_\infty(u) : u \in \operatorname{conv}\{u_0,\dots,u_m\}\} + \varepsilon.\end{aligned} \tag{4.16}$$

Proof. Let $r, \varepsilon > 0$, $u_0, \dots, u_m$, $(u_j^{(h)})$ be as in (4.15). Since $u_j^{(h)} \to u_j$ strongly in $L^2(\Omega)$, then it is sufficient to prove that

$$\begin{aligned}&\limsup_h \Big\{\sup\Big\{g_h(v_h) : v_h \in \operatorname{conv}\{u_0^{(h)},\dots,u_m^{(h)}\}\Big\}\Big\}\\ &\le \sup\{g_\infty(u) : u \in \operatorname{conv}\{u_0,\dots,u_m\}\} + \varepsilon.\end{aligned} \tag{4.17}$$

where g_h and g_∞ are respectively the "principal parts" of f_h, f_∞.
Consider $\widetilde{f}_h : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{f}_h(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x,u_0)D_iuD_ju\,dx.$$

It is readily seen that $\widetilde{f}_h$ is convex. Therefore to prove (4.17) it suffices to verify that

$$\limsup_h \Big\{\sup\Big\{|g_h(v_h) - \widetilde{f}_h(v_h)| : v_h \in \operatorname{conv}\{u_0^{(h)},\dots,u_m^{(h)}\}\Big\}\Big\} < \frac{\varepsilon}{2}. \tag{4.18}$$

Of course, if $v_h \in \operatorname{conv}\{u_0^{(h)},\dots,u_m^{(h)}\}$, we have

$$g_h(v_h) - \widetilde{f}_h(v_h) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \Big[a_{ij}^{(h)}(x,v_h) - a_{ij}^{(h)}(x,u_0)\Big] D_iv_hD_jv_h\,dx. \tag{4.19}$$

It is not difficult to see that, if $v_h \in \operatorname{conv}\{u_0^{(h)},\dots,u_m^{(h)}\}$, then there exist $\delta > 0$, $c, d, e_h \in L^\infty(\Omega)$, with $\operatorname{essinf}_\Omega c \ge \varepsilon$, $\|d\|_\infty < \delta$ and $\|e_h\|_\infty \to 0$ such that

$$v_h = u_0 + (d + e_h)\overline{\phi}_1 = (c + d + e_h)\overline{\phi}_1.$$

By Lagrange Theorem, there exists $0 < \eta < 1$ such that

$$\begin{aligned}&a_{ij}^{(h)}(x,v_h) - a_{ij}^{(h)}(x,u_0)\\ &= \overline{\phi}_1(d + e_h)D_sa_{ij}^{(h)}\big(x,(c + \eta(d + e_h))\overline{\phi}_1\big)\end{aligned}$$

$$= \frac{(d+e_h)}{c+\eta(d+e_h)}\left((c+\eta(d+e_h))\overline{\phi}_1\right) D_s a_{ij}^{(h)}\left(x,(c+\eta(d+e_h))\,\overline{\phi}_1\right).$$

Therefore, if δ is small enough, by using (A2), we deduce that

$$\limsup_h \|a_{ij}^{(h)}(x,v_h) - a_{ij}^{(h)}(x,u_0)\|_\infty$$

is also small. Since f_∞ is bounded in $\widehat{H}_k \cap B_r(\overline{\phi}_1)$, we can assume without loss of generality that (eventually)

$$f_h(u_j^{(h)}) < \sup\{f_\infty(u) : u \in \widehat{H}_k \cap B_r(\overline{\phi}_1)\} + 1.$$

So, in view of $(A3)$ we may deduce that $\|u_j^{(h)}\|_{H_0^1}$ is bounded; hence also $\|v_h\|_{H_0^1}$ is bounded. By using all these facts in (4.19) we obtain that, for δ small enough, (4.18) holds. □

Remark 4.9. We point out that Theorem 4.8 is still valid if, in (4.15), we replace assumption $\operatorname{essinf}_\Omega \frac{u_j}{\overline{\phi}_1} \geq \varepsilon$ with $\operatorname{esssup}_\Omega \frac{u_j}{\overline{\phi}_1} \leq -\varepsilon$.

Now, let S be as in Lemma 4.4 and Q be as in Proposition 4.6. Let also $\varepsilon > 0$. We can suppose that

$$\sup\Big\{f_\infty(u) : u \in \partial_{\widehat{H}_k} S\Big\} < f_\infty(\overline{\phi}_1) - 2\varepsilon, \tag{4.20}$$

$$\liminf_h \Big[\inf\Big\{f_h(\overline{\phi}_1 + u) : u \in \partial_{\widehat{X}_1 \oplus X_2} Q\Big\}\Big] > f_\infty(\overline{\phi}_1) + 2\varepsilon. \tag{4.21}$$

For $r = \rho$ where ρ is introduced in Lemma 4.3 and ε given as above, take $\delta > 0$ as in Theorem 4.8. Let now

$$S = \bigcup_{j=1}^N S_j,$$

where S_j are the convex sets generated by the points $u_0^{(j)}, \dots, u_m^{(j)} \in \widehat{H}_k \cap B_r(\overline{\phi}_1)$, such that

$$\sup\big\{\|\frac{u-v}{\overline{\phi}_1}\|_\infty : u, v \in S_j\big\} < \delta.$$

For $k = 0, \dots, m$, we consider $(u_{k,h}^{(j)})_h$ the approximating sequence introduced in Theorem 4.8 and let

$$P_h = \bigcup_{j=1}^N \operatorname{conv}\{u_{0,h}^{(j)}, \dots, u_{m,h}^{(j)}\}.$$

Proposition 4.10. *Take ε as above, then there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$ we have*

$$\sup_{P_h} f_h < \inf_{\overline{\phi}_1 + \partial Q} f_h, \quad b_1 = \sup_{P_h} f_h < f_\infty(\overline{\phi}_1) + \varepsilon,$$
$$\sup_{\partial P_h} f_h < \inf_{\overline{\phi}_1 + Q} f_h, \quad a_1 = \inf_{\overline{\phi}_1 + Q} f_h > f_\infty(\overline{\phi}_1) - \varepsilon.$$

Proof. By (4.21) and (4.4) we deduce that there exists $\overline{h}_1 \in \mathbb{N}$ such that for every $h \geq \overline{h}_1$

$$\inf_{\overline{\phi}_1 + \partial Q} f_h > f_\infty(\overline{\phi}_1) + \varepsilon, \quad \inf_{\overline{\phi}_1 + Q} f_h > f_\infty(\overline{\phi}_1) - \varepsilon.$$

Using Lemma 4.4, Theorem 4.8 and (4.20) we see that there exists $\overline{h}_2 \in \mathbb{N}$ such that for every $h \geq \overline{h}_2$ we have

$$\sup_{P_h} f_h < f_\infty(\overline{\phi}_1) + \varepsilon, \quad \sup_{\partial P_h} f_h < f_\infty(\overline{\phi}_1) - \varepsilon.$$

The assertions follow, taking $\overline{h} = \max\{\overline{h}_1, \overline{h}_2\}$. □

Theorem 4.11. *For every $\varepsilon > 0$, there exists $\overline{h} \in \mathbb{N}$ such that for all $h \geq \overline{h}$, the functional f_h has a critical point $u_3^{(h)}$ with*

$$\big| f_h(u_3^{(h)}) - f_\infty(\overline{\phi}_1) \big| < \varepsilon. \tag{4.22}$$

Proof. Let $\Pi_1 : H_0^1(\Omega) \to \widehat{H}_k$ be projection induced by the decomposition $H_0^1(\Omega) = \widehat{H}_k \oplus (\widehat{X}_1 \oplus X_2)$. Then, for h large, the restriction of Π_1 to P_h is an injective map with inverse Lipschitz continuous and such that $x - \Pi_1(x) \in \widehat{X}_1 \oplus X_2$. Let $\varphi_h : \widehat{H}_k \to \widehat{X}_1 \oplus X_2$ be a Lipschitz continuous function such that

$$\Pi_1(x) + \varphi_h(\Pi_1(x)) = x \quad \forall x \in P_h.$$

If $\Phi_h : H_0^1(\Omega) \to H_0^1(\Omega)$ is defined by $\Phi_h(x) = \varphi_h(\Pi_1(x)) + x$, then Φ_h is a Lipschitz homeomorphism with inverse Lipschitz continuous. Moreover,

$$\Phi_h(\Pi_1(x)) = x \quad \forall x \in P_h.$$

Define $\widetilde{f}_h = f_h \circ \Phi_h$. Clearly, f_h satisfies $(PS)_c$ if and only if $\widetilde{f}_h$ satisfies $(PS)_c$; furthermore $u^{(h)}$ is a critical point of $\widetilde{f}_h$ if and only if $\Phi_h(u^{(h)})$ is a critical point of f_h. Using Proposition 4.10, it follows that

$$\sup_{\Pi_1(P_h)} \widetilde{f}_h < \inf_{\overline{\phi}_1 - \varphi_h(\overline{\phi}_1) + \partial Q} \widetilde{f}_h,$$
$$\sup_{\Pi_1(\partial P_h)} \widetilde{f}_h < \inf_{\overline{\phi}_1 - \varphi_h(\overline{\phi}_1) + Q} \widetilde{f}_h.$$

We have

$$a_1 = \inf_{\overline{\phi}_1 + Q} f_h = \inf_{\overline{\phi}_1 - \varphi_h(\overline{\phi}_1) + Q} \widetilde{f}_h, \quad b_1 = \sup_{P_h} f_h = \sup_{\Pi_1(P_h)} \widetilde{f}_h.$$

By Theorem 2.4, we deduce that there exists a critical point $\widetilde{u}_3^{(h)}$ for $\widetilde{f}_h$ with $\widetilde{f}_h(\widetilde{u}_3^{(h)}) \in [a_1, b_1]$. Therefore, there exists a critical point $u_3^{(h)}$ for f_h with $f_h(u_3^{(h)}) \in [a_1, b_1]$. Proposition 4.10 now gives (4.22). □

5. Proof of the main result

Theorem 5.1. *Let $\beta < \mu_1$ and $\alpha > \mu_2$. Then, there exist $\overline{h} \in \mathbb{N}$, $\varepsilon > 0$ such that for all $h \geq \overline{h}$, the functional f_h has at least two critical points $u_1^{(h)}$, $u_2^{(h)}$ with*

$$f_h(u_1^{(h)}) < f_h(u_2^{(h)}) < f_\infty(\overline{\phi}_1) - \varepsilon.$$

Proof. First of all, let us point out that from the definition of f_∞ and hypothesis on α and β, it can be easily seen that there exists $\rho > 0$ such that

$$\inf_{S_\rho\left(\frac{\phi_1}{\beta - \mu_1}\right)} f_\infty > f_\infty\Big(\frac{\phi_1}{\beta - \mu_1}\Big).$$

By [4, Lemma 4.1], there exist a continuous curve $\gamma : [0,1] \to H_0^1(\Omega)$, $\varepsilon > 0$ such that

$$\gamma(0) = \frac{\phi_1}{\beta - \mu_1}, \quad \gamma(1) \notin \overline{B_\rho\big(\frac{\phi_1}{\beta - \mu_1}\big)}, \quad \sup_{s\in[0,1]} f_\infty(\gamma(s)) < f_\infty(\overline{\phi}_1) - \varepsilon.$$

The same argument of [3, Theorem 4.2] shows that there exists $\overline{h} \in \mathbb{N}$ such that for all $h \geq \overline{h}$

$$\inf_{S_\rho\left(\frac{\phi_1}{\beta-\mu_1}\right)} f_h > f_\infty\left(\frac{\phi_1}{\beta - \mu_1}\right).$$

On the other hand, the argument used in the proof of Theorem 4.8 allows us to build a polygonal curve γ_h with

$$\gamma_h(0) \in B_\rho\left(\frac{\phi_1}{\beta - \mu_1}\right), \quad \gamma_h(1) \notin \overline{B_\rho\big(\frac{\phi_1}{\beta - \mu_1}\big)}, \quad \sup_{s\in[0,1]} f_h(\gamma_h(s)) < f_\infty(\overline{\phi}_1) - \varepsilon.$$

In view of $(A7)$ we can follow the same argument used in the proof of [4, Theorem 4.2] and deduce the assertion. □

Proof of Theorem 3.1. By Theorem 5.1 and Theorem 4.11 we deduce that for $h \geq \overline{h}$ the functional f_h has at least three critical points. Hence, by Theorem 4.1, when $h \geq \overline{h}$, problem (3.2) has at least three distinct weak solutions. □

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Alessandro Groli
Department of Mathematics, University of Brescia, Via Valotti 9, 25133 Brescia, Italy
E-mail address: alessandro.groli@ing.unibs.it