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SHIGESADA-KAWASAKI-TERAMOTO MODEL ON HIGHER DIMENSIONAL DOMAINS

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ABSTRACT. We investigate the existence of a global attractor for a class of triangular cross diffusion systems in domains of any dimension. These systems includes the Shigesada-Kawasaki-Teramoto (SKT) model, which arises in population dynamics and has been studied in two dimensional domains. Our results apply to the (SKT) system when the dimension of the domain is at most 5.

1. INTRODUCTION

There has been a great interest in using cross diffusion to model physical and biological phenomena. For example in population dynamics, the strongly coupled parabolic system

$$\begin{gathered}
\frac{\partial u}{\partial t} = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v), \\
\frac{\partial v}{\partial t} = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2u - c_2v), \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u^0(x), \quad v(x,0) = v^0(x), \quad x \in \Omega
\end{gathered} \tag{1.1}$$

was proposed by Shigesada, Kawasaki and Teramoto (see [15]) for studying spatial segregation of interacting species. Here, Ω is a bounded domain in $\mathbb{R}^n$ and the initial data u^0, v^0 are nonnegative functions. Considerable progress has been made on (1.1) for the triangular cross diffusion case $\alpha_{21} = 0$. For instance, existence of global solutions was studied in [14, 16, 17] and long time dynamics was recently investigated in [10, 13]. However, due to technical difficulties, Ω has been always assumed to be *two dimensional.* In [8], the results in [10] were extended to arbitrary dimensional domains if $\alpha_{21} = \alpha_{22} = 0$.

Obviously, it is of biological interest and importance to study (1.1) on 3-dimensional domains, and perhaps higher dimensional situations should be also considered for purely mathematical interests. In this paper we will consider a class of triangular

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cross diffusion systems, which includes (1.1) when $\alpha_{21} = 0$ and $\alpha_{22} > 0$, given on an open bounded domain Ω in $\mathbb{R}^n$ with $n \geq 3$.

Let us consider quasilinear differential operators

$$\mathcal{A}_u(u,v) = \nabla(P(x,t,u,v)\nabla u + R(x,t,u,v)\nabla v),$$
$$\mathcal{A}_v(v) = \nabla(Q(x,t,v)\nabla v) + c(x,t)v,$$

and the parabolic system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}_u(u,v) + g(u,v), \quad x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} &= \mathcal{A}_v(v) + f(u,v), \quad x \in \Omega, t > 0, \end{aligned} \tag{1.2}$$

with mixed boundary conditions for $x \in \partial\Omega$ and $t > 0$

$$\begin{aligned} \chi(x)\frac{\partial v}{\partial n}(x,t) + (1-\chi(x))v(x,t) = 0, \\ \bar{\chi}(x)\frac{\partial u}{\partial n}(x,t) + (1-\bar{\chi}(x))u(x,t) = 0, \end{aligned} \tag{1.3}$$

where $\chi, \bar{\chi}$ are given functions on $\partial\Omega$ with values in $\{0,1\}$. The initial conditions are

$$v(x,0) = v^0(x), \quad u(x,0) = u^0(x), \quad x \in \Omega \tag{1.4}$$

for nonnegative functions v^0, u^0 in $X = W^{1,p}(\Omega)$ for some $p > n$ (see [2]). In (1.2), P and Q represent the *self-diffusion* pressures, and R is the *cross-diffusion* pressure acting on the population u by v.

We are interested not only in the question of global existence of solutions to (1.2) but also in long time dynamics of the solutions. Roughly speaking, we establish the following.

> A solution (u,v) of (1.2) exists globally in time if $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_1$ do not blow up in finite time. Moreover, if these norms of the solutions are ultimately uniformly bounded then an absorbing set exists, and therefore there is a compact global attractor, with finite Hausdörff dimension, attracting all solutions.

The assumptions on the parameters defining (1.2) will be specified later in Section 2, where we consider arbitrary dimensional domains. The settings are general enough to cover many other interesting models investigated in literature. Furthermore, our conclusion is far more stronger, in some cases, than what have been known about those systems (see also [8]). Nevertheless, as an application of our general results, we will confine ourselves in this paper to (1.1) (when $\alpha_{21} = 0$ and $n \leq 5$) and state our findings in Section 3. When this work was completed, we learned that Choi, Lui and Yamada ([3]) were also able to prove global existence results for the SKT model when $n \leq 5$. Their method was pure PDE and did not provide time independent estimates so that they could only assert that the solutions exist globally. Not only that our method, using PDE and semigroup techniques, applies to more general systems and gives stronger conclusions; but it also requires a much weaker assumption in some cases to obtain the existence of global attractors. In particular, we only need L^1 estimates of u if the second equation is not quasilinear.

2. Main results

In this section, we will specify our assumptions on the general system (1.2) and state our main results. Let (u^0, v^0) be given functions in $X = W^{1,p_0}(\Omega)$, $p_0 > n$. Let (u, v) be the solution of system (1.2), and $I := I(u^0, v^0)$ be its maximal interval of existence (see [2]).

We will consider the following conditions on the parameters of the system.

(H1) There are differentiable functions $P(u, v), R(u, v)$ such that

$$\mathcal{A}_u(u, v) = \nabla(P(u, v)\nabla u + R(u, v)\nabla v).$$

There exist a continuous function Φ and positive constants C, d such that

$$P(u, v) \geq d(1 + u) > 0, \quad \forall u \geq 0, \tag{2.1}$$

$$|R(u, v)| \leq \Phi(v)u. \tag{2.2}$$

Moreover, the partial derivatives of P, R with respect to u, v can be majorized by some powers of u, v.

The operator $\mathcal{A}_v$ is regular linear elliptic in divergence form. That is, for some Hölder continuous functions $Q(x, t)$ and $c(x, t)$ with uniformly bounded norms

$$\mathcal{A}_v(v) = \nabla(Q(x, t)\nabla v) + c(x, t)v, \quad Q(x, t) \geq d > 0, \quad c(x, t) \leq 0. \tag{2.3}$$

We will impose the following assumption on the reaction terms.

(H2) There exists a nonnegative continuous function $C(v)$ such that

$$|f(u, v)| \leq C(v)(1 + u), \quad g(u, v)u^p \leq C(v)(1 + u^{p+1}), \tag{2.4}$$

for all $u, v \geq 0$ and $p > 0$.

We will be interested only in nonnegative solutions, which are relevant in many applications. Therefore, we will assume that the solution u, v stay nonnegative if the initial data u^0, v^0 are nonnegative functions. Conditions on f, g that guarantee such positive invariance can be found in [7].

Essentially, we will establish certain a priori estimates for various spatial norms of the solutions. In order to simplify the statements of our theorems and proof, we will make use of the following terminology taken from [10].

Definition 2.1. Consider the initial-boundary problem (1.2),(1.3) and (1.4). Assume that there exists a solution (u, v) defined on a subinterval I of $\mathbb{R}_+$. Let $\mathcal{O}$ be the set of functions ω on I such that there exists a positive constant C_0, which may generally depend on the parameters of the system and the W^{1,p_0} norm of the initial value (u^0, v^0), such that

$$\omega(t) \leq C_0, \quad \forall t \in I. \tag{2.5}$$

Furthermore, if $I = (0, \infty)$, we say that ω is in $\mathcal{P}$ if $\omega \in \mathcal{O}$ and there exists a positive constant C_∞ that depends only on the parameters of the system but does not depend on the initial value of (u^0, v^0) such that

$$\limsup_{t\to\infty} \omega(t) \leq C_\infty. \tag{2.6}$$

If $\omega \in \mathcal{P}$ and $I = (0, \infty)$, we will say that ω is *ultimately uniformly bounded.*

If $\|u(\cdot,t)\|_\infty$, $\|v(\cdot,t)\|_\infty$, as functions in t, satisfy (2.5) the supremum norms of the solutions to (1.2) do not blow up in any finite time interval and are bounded by some constant that may depend on the initial conditions. This implies that the solution exists globally (see [2]). Moreover, if these norms verify (2.6), then they can be majorized eventually by a universal constant independent of the initial data. This property implies that there is an absorbing ball for the solution and therefore shows the existence of the global attractor if certain compactness is proven (see [6]).

Our first result is the following global existence result.

Theorem 2.2. *Assume (H1) and (H2). Let (u,v) is a nonnegative solution to (1.2) with its maximal existence interval I. If $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_1$ are in $\mathcal{O}$ then there exists $\nu>1$ such that*

$$\|v(\cdot,t)\|_{C^\nu(\Omega)}, \quad \|u(\cdot,t)\|_{C^\nu(\Omega)} \in \mathcal{O}. \tag{2.7}$$

If we have better bounds on the norms of the solutions then a stronger conclusion follows.

Theorem 2.3. *Assume (H1) and (H2). Let (u,v) be a nonnegative solution to (1.2) with its maximal existence interval I. If $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_1$ are in $\mathcal{P}$ then there exists $\nu>1$ such that*

$$\|v(\cdot,t)\|_{C^\nu(\Omega)}, \quad \|u(\cdot,t)\|_{C^\nu(\Omega)} \in \mathcal{P}. \tag{2.8}$$

Therefore, if $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_1$ are in $\mathcal{P}$ for every solution (u,v) of (1.2), then there exists an absorbing ball where all solutions will enter eventually. Thus, if the system (1.2) is autonomous then there is a compact global attractor with finite Hausdorff dimension which attracts all solutions.

To include (1.1) in our study, we also allow $\mathcal{A}_v$ to be a quasilinear operator given by

$$\mathcal{A}_v(v) = \nabla(Q(v)\nabla v) + c(x,t)v, \ Q(v) \ge d > 0, \tag{2.9}$$

for some differentiable function Q. Additional a priori estimates will give the following statement.

Theorem 2.4. *Assume as in Theorem 2.2 (respectively, Theorem 2.3) but with $\mathcal{A}_v$ described as in (2.9). The conclusions of Theorem 2.2 (respectively, Theorem 2.3) continue to hold if $\|u\|_{q,r,[t,t+1]\times\Omega} = \big(\int_t^{t+1}\|u(\cdot,s)\|_{q,\Omega}^r ds\big)^{1/r}$ (as a function in t) is in $\mathcal{O}$ (respectively $\mathcal{P}$) for some q,r satisfying*

$$\frac{1}{r} + \frac{n}{2q} = 1-\chi, \quad q \in \big[\frac{n}{2(1-\chi)}, \infty\big], \quad r \in \big[\frac{1}{1-\chi}, \infty\big] \tag{2.10}$$

for some $\chi \in (0,1)$.

Remark 2.1. This theorem improves our previous result [10] where we had to assume that $\|u(\cdot,t)\|_p$ are in $\mathcal{P}$ for some $p \ge n$. Moreover, the theorem is our main tool in the study of (1.1) on higher dimensional domains in Section 3.

We first consider Theorem 2.2 and Theorem 2.3. Their proofs will be based on several lemmas. Hereafter, we will use $\omega(t), \omega_1(t), \dots$ to denote various continuous functions in $\mathcal{O}$ or $\mathcal{P}$. We first have the following fact on the component v and its spatial derivative.

Lemma 2.2. *There exist nonnegative functions ω_0, ω defined on the maximal interval of existence of v such that $\omega_0 \in \mathcal{P}$ and the followings hold for v. For some $\delta > 0$, $r > 1$, $\beta \in (0,1)$ such that $2\beta > 1 - n/q + n/r$, we have*

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\cdot,s)\|_r ds. \tag{2.11}$$

Moreover, ω belongs to $\mathcal{O}$, respectively $\mathcal{P}$, if $\|v(\cdot,t)\|_\infty$ does.

The proof of this lemma is identical to that of [10, Lemma 2.5 (ii)] except that we use the imbedding [10, (2.12)] for fractional power operators.

Our starting point is the following integro-differential inequality for the L^p norm of u.

Lemma 2.3. *Given the conditions of Theorem 2.2 (respectively Theorem 2.3). For any $p > \max\{n/2, 1\}$, we set $y(t) = \int_\Omega u^p\, dx$. We can find $\beta \in (0,1)$ and positive constants A, B, C, and functions $\omega_i \in \mathcal{O}$ (respectively, $\mathcal{P}$) such that the following inequality holds*

$$\begin{aligned} \frac{d}{dt} y \le &- A y^\eta + (\omega_0(t) + \|u(\cdot,t)\|_1) y + B\omega(t) \\ &+ C y^\theta \Big\{ \omega_1(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega_2(s) \|u(\cdot,s)\|_1^\zeta y^\vartheta(s) ds \Big\}^2. \end{aligned} \tag{2.12}$$

Here, $\eta = \frac{p+1}{p}$, $\theta = \frac{p-1}{p}$ and $\vartheta = \frac{(r-1)}{r(p-1)}$, $\zeta = \frac{(p-r)}{r(p-1)}$ for some $r \in (1,p)$. Moreover, $\eta > \theta + 2\vartheta$.

Proof. We assume the conditions of Theorem 2.3 as the proof for the other case is identical. We multiply the equation for u by u^{p-1} and integrate over Ω. Using integration by parts and noting that the boundary integrals are all zero thanks to the boundary condition on u, we see that

$$\begin{aligned} &\int_\Omega u^{p-1} \frac{d}{dt} u\, dx + \int_\Omega P(u,v) \nabla u \nabla (u^{p-1})\, dx \\ &\le \int_\Omega (-R(u,v) \nabla (u^{p-1}) \nabla v + g(u,v) u^{p-1})\, dx. \end{aligned}$$

Using the conditions (2.1) and (2.2) , we derive (for some positive constants $C(d,p)$, $\epsilon, C(\epsilon, d, p)$)

$$\int_\Omega P(u,v) \nabla u \nabla (u^{p-1})\, dx \ge C(d,p) \int_\Omega u^{p-1} |\nabla u|^2\, dx,$$

$$\begin{aligned} -\int_\Omega R(u,v) \nabla (u^{p-1}) \nabla v\, dx &\le C(d,p) \int_\Omega u^{p-1} \Phi(v) \nabla u \nabla v\, dx \\ &\le \epsilon \int_\Omega u^{p-1} |\nabla u|^2\, dx + C(\epsilon, d, p) \int_\Omega u^{p-1} \Phi^2(v) |\nabla v|^2\, dx. \end{aligned}$$

From this inequality and (2.4), we obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u^p\, dx + C(d,p) \int_\Omega u^{p-1} |\nabla u|^2\, dx \\ &\le C(\epsilon, d, p) \int_\omega (u^{p-1} \Phi^2(v) |\nabla v|^2 + C(v)(u^p + 1)\, dx. \end{aligned} \tag{2.13}$$

Furthermore, the second term on the left-hand side can be estimated as

$$\begin{aligned}\int_\Omega u^{p-1}|\nabla u|^2\,dx &= C(p)\int_\Omega |\nabla(u^{(p+1)/2})|^2\,dx\\ &\ge C\int_\Omega u^{p+1}\,dx - C\Big(\int_\Omega u^{(p+1)/2}\,dx\Big)^2\\ &\ge C\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}} - C\|u\|_1\int_\Omega u^p\,dx.\end{aligned}$$

Here, we have used the Hölder's inequality $\big(\int_\Omega u^{(p+1)/2}\,dx\big)^2 \le \|u\|_1\int_\Omega u^p\,dx$.

Next, we consider the first integral on the right of (2.13). By our assumption on L^∞ norm of v, $\Phi(v)\le\omega_1(t)$ for some $\omega_1\in\mathcal{P}$. Using the Hölder inequality, we have

$$\begin{aligned}\int_\Omega u^{p-1}\Phi^2(v)|\nabla v|^2\,dx &\le \omega_1(t)\Big(\int_\Omega u^p\,dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |\nabla v|^{2p}\,dx\Big)^{1/p}\\ &= \omega_1(t)y^{\frac{p-1}{p}}\|\nabla v\|_{2p}^2.\end{aligned}$$

Since $p>\max\{n/2,1\}$, there exists $r\in(1,p)$ such that

$$\frac{1}{n}+\frac{1}{2p}>\frac{1}{r}>\frac{1}{p}. \tag{2.14}$$

This implies $2>1-n/2p+n/r$. Hence, we can find $\beta\in(0,1)$ such that $2\beta>1-n/2p+n/r$. From (2.11), with $q=2p>r$, we have

$$\|\nabla v\|_{2p}\le\omega_0(t)+\int_0^t(t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r ds.$$

Applying the above estimates in (2.13), we derive the following inequality for $y(t)$

$$\begin{aligned}\frac{d}{dt}y+C(d,p)y^{\frac{p+1}{p}} &\le Cy^{\frac{p-1}{p}}\omega_1(t)\Big\{\omega_0(t)+\int_0^t(t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r ds\Big\}^2\\ &\quad + C(\omega_2(t)+\|u\|_1)y+B\omega_2(t).\end{aligned} \tag{2.15}$$

Since $1<r<p$, we can use Hölder's inequality

$$\|u\|_r\le\|u\|_1^{1-\lambda}\|u\|_p^\lambda=\|u\|_1^{1-\lambda}y^{\frac{\lambda}{p}}$$

with $\lambda=\frac{1-1/r}{1-1/p}=\frac{p(r-1)}{r(p-1)}$. Applying this in (2.15) and re-indexing the functions ω_i, we prove (2.12). The last assertion of the lemma follows from the following equivalent inequalities

$$\eta>\theta+2\vartheta\Leftrightarrow\frac{p+1}{p}>\frac{p-1}{p}+\frac{2(r-1)}{r(p-1)}\Leftrightarrow\frac{1}{p}>\frac{(r-1)}{r(p-1)}\Leftrightarrow rp-r>pr-p\Leftrightarrow p>r.$$

This completes the proof. $\square$

Next, we will show that the L^p norm of u is in the class $\mathcal{O}$ or $\mathcal{P}$ for any $p\ge1$.

Lemma 2.4. *Given the conditions of Theorem 2.2 (respectively Theorem 2.3), for any finite $p\ge1$, there exists a function $\omega_p\in\mathcal{O}$ (respectively $\mathcal{P}$) such that*

$$\|u(\cdot,t)\|_p\le\omega_p(t). \tag{2.16}$$

To prove this, we apply the following facts from [10] to the differential inequality (2.12).

Lemma 2.5 ([10, Lemma 2.17]). *Let $y : \mathbb{R}^+ \to \mathbb{R}$ satisfy*

$$y'(t) \le \mathcal{F}(t, y), \quad y(0) = y_0, \quad t \in (0, \infty), \tag{2.17}$$

where $\mathcal{F}$ is a functional from $\mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$. Assume that

- F1 *There is a function $F(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathcal{F}(t, y) \le F(y(t), Y)$ if $y(s) \le Y$ for all $s \in [0, t]$.*
- F2 *There exists a real M such that $F(Y, Y) < 0$ if $Y \ge M$.*

Then there exists finite M_0 such that $y(t) \le M_0$ for all $t \ge 0$.

Proposition 2.5 ([10, Prop 2.18]). *Assume* (2.17) *and assume that*

- G1 *There exists a continuous function $G(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ such that for τ sufficiently large, if $t > \tau$ and $y(s) \le Y$ for every $s \in [\tau, t]$ then there exists $\tau' \ge \tau$ such that*
$$\mathcal{F}(t, y) \le G(y(t), Y) \quad \text{if } t \ge \tau' \ge \tau. \tag{2.18}$$
- G2 *The set $\{z : G(z, z) = 0\}$ is not empty and $z_* = \sup\{z : G(z, z) = 0\} < \infty$. Moreover, $G(M, M) < 0$ for all $M > z_*$.*
- G3 *For $y, Y \ge z_*$, $G(y, Y)$ is increasing in Y and decreasing in y.*

If $\limsup_{t\to\infty} y(t) < \infty$ then

$$\limsup_{t\to\infty} y(t) \le z_*. \tag{2.19}$$

Remark 2.6. Examples of functions F, G satisfying the conditions of the above two lemmas includes

$$F(y(t), Y),\ G(y(t), Y) = -Ay^\eta(t) + D(y^\gamma + 1) + y^\theta(B + CY^\vartheta)^k, \tag{2.20}$$

with positive constants $A, B, C, D, \eta, \theta, \vartheta, k$ satisfies $\eta > \theta + k\vartheta$ and $\eta > \gamma$.

Proof of Lemma 2.4. Assume first the conditions of Theorem 2.2. From (2.12), we deduce the following integro-differential inequality

$$\frac{d}{dt} y \le -Ay^\eta + \omega_1(t)y + B\omega_2(t) + Cy^\theta\{\omega_0(t) + K(t)\}^2, \tag{2.21}$$

where

$$K(t) := \int_0^t (t - s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s) ds$$

for some $\omega_0, \omega_1, \omega \in \mathcal{O}$ (because $\|u(\cdot, t)\|_1 \in \mathcal{O}$). We will show that Lemma 2.5 can be used here to assert that $y(t)$ is bounded in any finite interval. This means $\|u\|_p \in \mathcal{O}$. We define the functional

$$\mathcal{F}(t, y) = -Ay^\eta + \omega_1(t)y + B + Cy^\theta\{\omega_0(t) + K(t)\}^2. \tag{2.22}$$

Since $\omega_i \in \mathcal{O}$, we can find a positive constant C_ω, which may still depend on the initial data, such that $\omega_i(t) \le C_\omega$ for all $t > 0$. Let

$$C_1 := \sup_{t>0} \int_0^t (t - s)^{-\beta} e^{-\delta(t-s)} ds \le \int_0^\infty s^{-\beta} e^{-\delta s} ds < \infty,$$

because $\beta \in (0, 1)$ and $\delta > 0$. We then set

$$F(y, Y) = -Ay^\eta + C_\omega(y + B) + Cy^\theta(C_\omega + C_\omega C_1 Y^\vartheta)^2.$$

Because $\eta > \theta + 2\vartheta$, by Lemma 2.3, and Remark 2.6, the functionals $\mathcal{F}, F$ satisfy the conditions (F.1),(F.2). Hence, Lemma 2.5 applies and gives

$$y(t) \le C_0(v^0, u^0), \quad \forall t > 0. \tag{2.23}$$

For some constant $C_0(v^0, u^0)$ which may still depend on the initial data since F does. We have shown that $y(t) \in \mathcal{O}$.

We now seek for uniform estimates and assume the conditions of Theorem 2.3. From Lemma 2.3 we again obtain (2.21) with ω_i are now in $\mathcal{P}$. If a function ω belong to $\mathcal{P}$, by Definition 2.1, we can find $\tau_1 > 0$ such that $\omega(s) \le \bar{C}_\infty = C_\infty + 1$ if $s > \tau_1$. We emphasize the fact that $\bar{C}_\infty$ is independent of the initial data. Let $t > \tau \ge \tau_1$ and assume that $y(s) \le Y$ for all $s \in [\tau, t]$. Let us write

$$K(t) = \int_0^\tau (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s) ds + \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s) ds = J_1 + J_2.$$

By (2.23), there exists some constant $C(v^0, u^0)$ such that $\omega(s) y^\vartheta(s) \le C(v^0, u^0)$ for every s. Hence, we can find $\tau' > \tau$ such that $J_1 \le 1$ if $t > \tau'$. Thus,

$$K(t) \le 1 + \bar{C}_\infty C_* Y^\vartheta, \quad \text{where} \quad C_* = \sup_{t>\tau, \tau>0} \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} ds < \infty.$$

Therefore, for $t > \tau'$ we have $f(t, y) \le G(y(t), Y)$ with

$$G(y(t), Y) = -Ay^\eta(t) + \bar{C}_\infty(y + B) + y^\theta(\bar{C}_\infty + 1 + \bar{C}_\infty C_* Y^\vartheta)^2. \tag{2.24}$$

We see that G is independent of the initial data and satisfies (G1)-(G3) as $\eta > \theta + 2\vartheta$ (see Remark 2.6). Therefore, Proposition 2.5 applies here to complete the proof. $\square$

We conclude this section by giving the following proofs.

Proof of Theorems 2.2 and 2.3. Having established the fact that $\|u(\cdot, t)\|_p \in \mathcal{O}$ (respectively, $\|u(\cdot, t)\|_p \in \mathcal{P}$) for any $p > 1$, we can follow the proof of [10, Theorem 2] to assert (2.7) (respectively, (2.8)). $\square$

Proof of Theorem 2.4. The proof is exactly the same as that of Theorem 2.3 if we can regard $\mathcal{A}_v$ as a *linear* regular elliptic operator with Hölder continuous coefficients (whose norms are also ultimately uniformly bounded) so that Lemma 2.2 is applicable. To this end, we need only to show that $Q(v(x, t))$, as a function in (x, t), is Hölder continuous. Since we assume that $\|v(\cdot, t)\|_\infty \in \mathcal{P}$ and (2.4) holds, the assumption of the theorem implies that $\|f(u, v)\|_{q,r,[t,t+1]\times\Omega} \in \mathcal{P}$. The range of q, r in (2.10) and well known regularity theory for quasilinear parabolic equations (see [9, Chap.5, Theorem 1.1] or [12]) assert that there is $\alpha > 0$ such that $v \in C^{\alpha,\alpha/2}(\Omega \times (0, \infty))$ with uniformly bounded norm. So is $Q(v(x, t))$. In fact, by [5], we also have that $\nabla v \in C^{\alpha,\alpha/2}(\Omega \times (0, \infty))$. $\square$

3. Shigesada-Kawasaki-Teramoto model on higher dimensional domains

In this section we show that the assumption of Theorem 2.4 is verified for (1.2) if the dimension $n \le 5$ and the reaction terms are of Lotka-Volterra type used in (1.1).

$$f(u, v) = v(c_1 - c_{11}v - c_{12}u), \quad g(u, v) = u(c_2 - c_{21}v - c_{22}u), \tag{3.1}$$

where c_{ij} are given constants. The main result of this section is the following.

Theorem 3.1. *Assume that $\mathcal{A}_v$ is of the form (2.9), $n \le 5$, and that $c_{11}, c_{12}, c_{22} > 0$. For any given $p_0 > n$, the system (1.2), (1.3) with (3.1) possesses a global attractor with finite Hausdorff dimension in*

$$X = \{(u, v) \in W^{1,p_0}(\Omega) \times W^{1,p_0}(\Omega) : u(x), v(x) \ge 0, \quad \forall x \in \Omega\}.$$

For given nonnegative initial data $u^0, v^0 \in X$, it is standard to show that the solution stays nonnegative (see [7]). We consider the dynamical system associated with (1.2),(1.3) on X (see [2]). Clearly, the functions f, g satisfy the condition (H2). We need only to verify the hypotheses of Theorem 2.4. We first have the following facts from [10, Lemmas 3.1-3.3] which hold for *any* dimension n.

Lemma 3.1. *For the component u, we have*

$$\|u(\cdot,t)\|_1 \in \mathcal{P}, \tag{3.2}$$

$$\int_t^{t+1}\int_\Omega u^2\,dx \in \mathcal{P}. \tag{3.3}$$

Furthermore, for the v component, we have $\|v(\cdot,t)\|_\infty \in \mathcal{P}$ and

$$\|\nabla v(\cdot,t)\|_2 \in \mathcal{P}, \tag{3.4}$$

$$\int_t^{t+1}\int_\Omega v_t^2(x,s)\,dx\,ds \in \mathcal{P}. \tag{3.5}$$

For $n = 3$, we note that the assumptions of Theorem 2.4 immediately follow from this lemma if we take $q = 2 > n/2$ and $r = \infty$ in (2.10). However, we will present a unified proof for all $n \le 5$ below.

We will also need the following variance of the Gronwall inequality whose proof is elementary.

Lemma 3.2 (The Uniform Gronwall Lemma)**.** *Let g,h,y be three nonnegative locally integrable functions on $(t_0,+\infty)$ such that y' is locally integrable on $(t_0,+\infty)$, and*

$$y'(t) \le g(t)y(t) + h(t), \quad \text{for } t \ge t_0, \tag{3.6}$$

and the following functions in t satisfy

$$\int_t^{t+1} y(s)ds, \quad \int_t^{t+1} g(s)ds, \quad \int_t^{t+1} h(s)ds \in \mathcal{P}. \tag{3.7}$$

Then $y(t) \in \mathcal{P}$.

Lemma 3.3. *For any $q \le 2^* = 2n/(n-2)$, we have*

$$\int_t^{t+1} \|\nabla v(\cdot,s)\|_q^2 ds \in \mathcal{P}. \tag{3.8}$$

Proof. By standard Sobolev embedding theorem [1, Theorem 5.4], we have

$$\|\nabla v\|_{2^*}^2 \le \frac{1}{d^2}\Big(\int_\Omega |Q\nabla v|^{2^*}\,dx\Big)^{2/2^*} \le C\int_\Omega (|Q\nabla v|^2 + |\nabla(Q\nabla v)|^2)\,dx \tag{3.9}$$

From the equation for v and the condition on f, we have

$$|\nabla(Q\nabla v)|^2 \le |f(u,v)|^2 + |v_t|^2 \le \omega(t)(u^2+1) + |v_t|^2.$$

This and (3.9) imply

$$\|\nabla v\|_{2^*}^2 \le C\omega_1(t)\int_\Omega (|\nabla v|^2 + |u|^2 + |v_t|^2)\,dx.$$

We then integrate the above inequality over $[t, t+1]$ and make use of Lemma 3.1 to get (3.8) for $q = 2^*$. Finally, if $q < 2^*$, we have $\|\nabla v\|_q \le C\|\nabla v\|_{2^*}$ (due to Hölder's inequality and the fact that Ω is bounded) for some constant C and complete the proof. □

Multiplying the equation for u by u^{2p-1} $(p > 1/2)$ and using the boundary condition, we derive

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^{2p}\,dx + \frac{2p-1}{p}\int_\Omega P|\nabla u^p|^2\,dx \\ &\le C(p)\int_\Omega |R\nabla u^{2p-1}\nabla v|\,dx + \omega(t)(\int_\Omega (u^{2p}+1)\,dx. \end{aligned} \tag{3.10}$$

Using the conditions on P, R and Young's inequality, we have

$$\int_\Omega P|\nabla u^p|^2\,dx \ge d(\int_\Omega u|\nabla u^p|\,dx + \int_\Omega |\nabla u^p|\,dx),$$

$$\begin{aligned} \int_\Omega |R\nabla u^{2p-1}\nabla v|\,dx &\le \omega(t)\int_\Omega |u^p\nabla u^p\nabla v|\,dx \\ &\le \epsilon\int_\Omega u|\nabla u^p|^2\,dx + C(\epsilon)\omega(t)\int_\Omega u^{2p-1}|\nabla v|^2\,dx. \end{aligned}$$

for any $\epsilon > 0$. Moreover,

$$\int_\Omega u^{2p-1}|\nabla v|^2\,dx \le \Big(\int_\Omega u^{2p}\,dx\Big)^{1-1/2p}\|\nabla v\|_{4p}^2 \le \Big(\int_\Omega u^{2p}\,dx + 1\Big)\|\nabla v\|_{4p}^2$$

By choosing appropriately small ϵ, we derive from (3.10) and the above inequalities the following key inequality

$$\frac{d}{dt}y(t) + C_p\int_\omega (1+u)|\nabla u^p|^2\,dx \le g(t)y(t) + h(t), \tag{3.11}$$

where $y(t) = \int_\Omega u^{2p}\,dx$, $g(t) = \|\nabla v\|_{4p}^2 + \omega(t) + C(p)$, $h(t) = \omega(t) + C(p)$ for some $\omega \in \mathcal{P}$ and $C_p, C(p) > 0$.

We then have the following lemma.

Lemma 3.4. *For $\lambda = \min\{n/(n-2), 2\}$, we have $\|u(\cdot,t)\|_\lambda \in \mathcal{P}$.*

Proof. We choose p in (3.11) such that $2p = \lambda$. Firstly, $h(t)$ in (3.11) satisfies (3.7). On the other hand, as $4p = 2\lambda \le 2^*$ we see that $\|\nabla v(\cdot,t)\|_{4p}^2 \in \mathcal{P}$ by Lemma 3.3. Thus, $g(t)$ in (3.11) also verifies (3.7). Thanks to (3.3) and because $\lambda \le 2$, we see that $y(t) = \int_\Omega u^\lambda\,dx$ verifies the assumption of Lemma 3.2. This gives our lemma. $\square$

We conclude this article with the following proof.

Proof of Theorem 3.1. Thanks to Lemma 3.1, we need only to verify the last assumption on $\|u\|_{q,r}$ of the theorem. Let $p = \lambda/2$ and $l = \frac{\lambda+1}{2}$ in (3.11), and $U = u^l$. We integrate (3.11) over $[t, t+1]$ and use the above lemma to get

$$\int_t^{t+1}\int_\Omega \|\nabla U|^2\,dx\,ds = \big(1 + \frac{1}{2p}\big)^2\int_t^{t+1}\int_\Omega u|\nabla u^p|^2\,dx\,ds \in \mathcal{P}. \tag{3.12}$$

The function $W = U - \int_\Omega U\,dx$ has zero average and we can use the Gagliardo-Nirenberg inequality to get

$$\|W\|_{2^*,\Omega} \le C\|\nabla W\|_{2,\Omega} \Rightarrow \|U\|_{2^*,\Omega} \le C(\|\nabla U\|_{2,\Omega} + \|U\|_{1,\Omega}).$$

For $r = 2l$, $q = l2^*$, we derive

$$\int_t^{t+1}\|u\|_{q,\Omega}^r\,ds = \int_t^{t+1}\|U\|_{2^*,\Omega}^2\,ds \le C\Big(\int_t^{t+1}\|\nabla U\|_{2,\Omega}^2\,ds + \sup_{[t,t+1]}\|U\|_{1,\Omega}^2\Big).$$

As $l \leq \lambda$, $\|U(\cdot,t)\|_{1,\Omega} = \|u(\cdot,t)\|_{l,\Omega}^{l} \in \mathcal{P}$ (see Lemma 3.4). Thus, (3.12) and the above show that $\|u\|_{q,r,[t,t+1]\times\Omega} \in \mathcal{P}$, with r, q satisfying

$$1-\chi := \frac{1}{r} + \frac{n}{2q} = \frac{1}{l}(\frac{1}{2} + \frac{n}{22^*}) = \frac{n}{4l}$$

Set $A := q - \frac{n}{2(1-\chi)} = q - 2l$, $B := r - \frac{1}{1-\chi} = 2l - \frac{4l}{n}$. To see that q, r satisfy the condition (2.10) of Theorem 2.4, we show that $\chi \in (0,1)$ and $A, B \geq 0$. Computing the values of χ, A, B for $n = 3, 4, 5$ gives:

$$n = 3: \quad \chi = 1/2, \quad A = 6, \quad B = 1.$$
$$n = 4: \quad \chi = 1/3, \quad A = 3, \quad B = 3/2.$$
$$n = 5: \quad \chi = 1/16, \quad A = 16/9, \quad B = 8/5.$$

The assumptions of Theorem 2.4 are fulfilled and our proof is complete (we should also remark that $\chi = -1/5 < 0$ if $n = 6$). $\square$

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