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LARGE ENERGY SIMPLE MODES FOR A CLASS OF KIRCHHOFF EQUATIONS

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Abstract. It is well known that the Kirchhoff equation admits infinitely many simple modes, i.e., time periodic solutions with only one Fourier component in the space variable(s). We prove that for some form of the nonlinear term these simple modes are stable provided that their energy is large enough. Here stable means orbitally stable as solutions of the two-modes system obtained considering initial data with two Fourier components.

1. Introduction

Let H be a real Hilbert space, with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. Let A be a self-adjoint linear positive operator on H with dense domain $D(A)$ (i.e., $\langle Au,u\rangle>0$ for all $u\in D(A)$). We consider the evolution problem

$$u''(t)+m(|A^{1/2}u(t)|^2)Au(t)=0, \tag{1.1}$$

where $m:[0,+\infty)\to(0,+\infty)$ is a C^1 function. Equation (1.1) is an abstract setting of the hyperbolic PDE with a non-local non-linearity of Kirchhoff type

$$u_{tt}-m\Big(\int_\Omega|\nabla u|^2\,dx\Big)\Delta u=0,\quad \text{in } \Omega\times\mathbb{R}, \tag{1.2}$$

where $\Omega\subseteq\mathbb{R}^n$ is an open set, ∇u is the gradient of u with respect to space variables, and Δ is the Laplace operator. When Ω is an interval of the real line, this equation is a model for the small transversal vibrations of an elastic string.

When H admits a complete orthogonal system made of eigenvectors of A (this is the case e.g. in (1.2) if Ω is bounded), (1.1) may be thought as a system of ODEs with infinitely many unknowns, namely the components of u.

Many papers have been written about equations (1.1) and (1.2) after Kirchhoff's monograph [7]. The interested reader can find appropriate references in the surveys [1] and [8]. We just state that, at the present, the existence of global solutions for all initial data in C^∞ or in Sobolev spaces is still an open problem.

In this paper, we consider a particular class of global solutions of (1.1). Let us assume that λ is an eigenvalue of A, and e_λ is a corresponding eigenvector, which we assume normalized so that $|e_\lambda|=1$. If the initial data are multiples of e_λ, say

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$u(0) = w_0 e_\lambda$, $u'(0) = w_1 e_\lambda$, then the solution of (1.1) remains a multiple of e_λ for every $t \in \mathbb{R}$; i.e., we have that $u(t) = w(t)e_\lambda$, where $w(t)$ is the solution of the ODE

$$w''(t) + \lambda m(\lambda w^2(t))w(t) = 0, \quad w(0) = w_0, \quad w'(0) = w_1.$$

Such solutions are called *simple modes* of equation (1.1), and are known to be time periodic under very general assumptions on m.

In this paper, we are interested to stability of high energy simple modes for particular choices of m. This program is too optimist since stability problems are often hard already for systems with 3 unknowns, and we have seen that (1.1) has infinitely many degrees of freedom. For this reason we limit ourselves to consider the two-mode system

$$\begin{aligned} w''(t) + \lambda\, m(\lambda w^2(t) + \mu z^2(t))w(t) = 0, \\ z''(t) + \mu\, m(\lambda w^2(t) + \mu z^2(t))z(t) = 0, \end{aligned} \tag{1.3}$$

where $\mu \neq \lambda$ is another eigenvalue of A, corresponding to an eigenvector e_μ such that $|e_\mu| = 1$, and $u(t) = w(t)e_\lambda + z(t)e_\mu$. It is clear that simple modes are particular solutions of this system, corresponding to initial data with $z(0) = z'(0) = 0$. What we actually study is the stability of simple modes as solutions of (1.3).

To simplify the notation, let us set

$$\nu := \frac{\mu}{\lambda}, \quad u(t) := \sqrt{\lambda}\, w\big(\frac{t}{\sqrt{\lambda}}\big), \quad v(t) := \sqrt{\mu}\, z\big(\frac{t}{\sqrt{\lambda}}\big),$$

so that (1.3) is equivalent to

$$\begin{aligned} u''(t) + m(u^2(t) + v^2(t))u(t) = 0, \\ v''(t) + \nu\, m(u^2(t) + v^2(t))v(t) = 0. \end{aligned} \tag{1.4}$$

This system (as well as (1.3) and (1.1)) are Hamiltonian, with conserved energy

$$H(u, u', v, v') := \frac{1}{2}\Big\{[u']^2 + \frac{[v']^2}{\nu} + M(u^2 + v^2)\Big\}, \tag{1.5}$$

where $M(r) = \int_0^r m(s)\, ds$. As far as we know, stability of simple modes was studied in at least four papers (see section 2.2.1 for precise definitions).

- Dickey [3] proved that simple modes are *linearly stable* provided that their energy is *small* enough. Roughly speaking, linearly stable means that $v(t) \equiv 0$ is a stable solution for the linearization of the second equation in (1.4).
- In [4] it was proved that simple modes as solutions of (1.3) are *orbitally stable* provided that their energy is *small* enough. Roughly speaking, orbitally stable means that every solution $(u(t), v(t))$ of system (1.4) with initial data near $(u_0, u_1, 0, 0)$ remains close to the periodic orbit of the simple mode for every $t \in \mathbb{R}$.
- Cazenave and Weissler [2] assumed that there exists $\alpha > 0$ such that

  $$\lim_{\sigma \to +\infty} \frac{m(\sigma r)}{m(\sigma)} = r^\alpha,$$

  uniformly on bounded intervals (*e. g.* $m(r) = 1 + r^\alpha$). They showed that if

  $$\nu \in \bigcup_{m \in \mathbb{N}} \left((m+1)\left((\alpha+1)m + 1\right), (m+1)\left((\alpha+1)m + 1 + 2\alpha\right)\right),$$

then every simple mode of (1.4) with *large* enough energy is *unstable*. If $\alpha = 1$, and Ω is an interval of the real line, this result implies the instability of every simple mode of (1.2) with large enough energy.

- In [5] it was proved that if m is nondecreasing and for every $r \in [0,1)$ one has
$$\lim_{\sigma\to+\infty} \frac{m(\sigma r)}{m(\sigma)} = 0,$$
then every simple mode of (1.1) with large enough energy is unstable.

Remark 1.1. Let $m > 0$ be a continuous function such that for all $r \in (0,1)$ there exists
$$\lim_{\sigma\to+\infty} \frac{m(\sigma r)}{m(\sigma)}.$$
Since this limit is a multiplicative function, then there are only three possibilities:

- The limit is r^α for some $\alpha > 0$.
- The limit is 0 for every $r \in (0,1)$.
- The limit is 1 for every $r \in (0,1)$.

In [2] and in [5], the first two cases were treated (and proved instability). Here we treat the third case and prove orbital stability. Our main result is the following.

Theorem 1.2. *Let $\nu \neq 1$ be a positive real number. Let $m : [0,+\infty) \to (0,+\infty)$ be a smooth function with $m'(x) > 0$ for all $x > 0$ such that*

(H1) There exists a constant c such that, for every real $k > 0$, $\sup_{y\in(0,1)} \frac{ym'(ky)}{m'(k)} \leq c$

(H2) For every $y \in (0,1)$ $\lim_{k\to+\infty} \frac{m'(ky)}{m'(k)} = \frac{1}{y}$.

Then there exists $k_0 > 0$ such that, if $H(u_0,u_1,0,0) > k_0$, then the simple mode of (1.4) with $u(0) = u_0$, $u'(0) = u_1$ is orbitally stable.

Let us remark that any function with $m' > 0$ such that $xm'(x) \to l > 0$ as $x \to +\infty$ (*e. g.* $\log(2 + x^2)$) satisfies (H1) - (H2).

We conclude with a few comments on Theorem 1.2.

- Since there exists $\lim_{r\to+\infty} m(r)$, by (H2) we get $\lim_{k\to+\infty} m(ky)/m(k) = 1$ for every $y > 0$.
- Assumption $m'(x) > 0$ for all $x > 0$ is not essential but this would only complicate proofs without introducing new ideas.
- To prove orbital stability we use KAM theory to Poincaré map (see section 2.2). Smoothness of m is used only to give the smoothness required by KAM theory. To this end, $m \in C^5$ is enough.
- Since (1.4) is reversible (if $(u(t), v(t))$ is any solution, then $(u(-t), v(-t))$ is another solution), then a consequence of Theorem 1.2 is the following: "if the energy of a two-mode solution of (1.4) is large enough, then it is *not* possible that asymptotically all this energy is absorbed by one of the two components".

This paper is organized as follows: in section 2 we rescal the problem and give preliminaries on stability and Poincaré map; in section 3 we state our results; in section 4 we give the proofs. Section 4 is divided in four parts: in the first two parts we prove all we need about function m and simple mode; in the third part we get the proof of Theorem 3.2 and in the last one we prove Theorem 3.3.

2. Preliminaries

2.1. **Rescaling.** In this section we rescale the solutions of (1.4) and find an equivalent system, in which is simpler to work. Given $k > 0$, let us consider the simple mode $\overline{u}_k$ of system (1.4) which solves

$$\overline{u}_k''(t) + m(\overline{u}_k^2(t))\overline{u}_k(t) = 0, \quad \overline{u}_k(0) = k, \quad \overline{u}_k'(0) = 0. \tag{2.1}$$

We recall that $\overline{u}_k$ is a periodic function, and so we can assume $\overline{u}_k(0) > 0$ and $\overline{u}_k'(0) = 0$ without loss of generality. Moreover assuming that k is large is equivalent to assuming that the energy of $\overline{u}_k$ is large.

Let τ_k be the period of $\overline{u}_k$. Applying conservation low (1.5) it is easy to see that

$$\tau_k = 4\int_0^k \frac{1}{\sqrt{M(k^2) - M(y^2)}}\,dy = 4k\int_0^1 \frac{1}{\sqrt{M(k^2) - M(k^2y^2)}}\,dy. \tag{2.2}$$

Now let (u_k, v_k) be the solution of (1.4) with initial data $u_k(0) = a_1k$, $u_k'(0) = b_1k$, $v_k(0) = x_1$, $v_k'(0) = y_1$. Setting

$$w_k(t) = \frac{u_k(\tau_k t)}{k}, \quad z_k(t) = v_k(\tau_k t),$$

it turns out that (w_k, z_k) is the solution of

$$\begin{gathered} w_k''(t) + \tau_k^2 m(k^2w_k^2(t) + z_k^2(t))w_k(t) = 0, \quad w_k(0) = a, \quad w_k'(0) = b \\ z_k''(t) + \nu\tau_k^2 m(k^2w_k^2(t) + z_k^2(t))z_k(t) = 0, \quad z_k(0) = x, \quad z_k'(0) = y \end{gathered} \tag{2.3}$$

where $a = a_1$, $b = \tau_k b_1$, $x = x_1$ and $y = \tau_k y_1$. In the sequel we study the stability of simple modes of (2.3). Indeed following result holds, whose simple proof is omitted (see also Definition 2.2 below).

Theorem 2.1. *Let $k > 0$ be fixed and let $\overline{u_k}$ be defined in (2.1). If $U_k(t) = \overline{u_k}(\tau_k t)/k$ is orbitally stable as solution of (2.3) then $\overline{u_k}$ is orbitally stable as solution of (1.4).*

We remark that for (2.3), we can write the conserved energy as

$$H_k(w_k, w_k', z_k, z_k') = \frac{1}{2}\left\{[w_k']^2 + \frac{[z_k']^2}{k^2\nu} + \frac{\tau_k^2}{k^2}M(k^2w_k^2 + z_k^2)\right\}.$$

2.2. **Kam Theory and stability.** We recall the notion of stability, and then we describe the Poincaré map P_k associated with a simple mode U_k of system (2.3).

We refer to [6] for general facts about dynamical and Hamiltonian systems, and to [2, 4] for specific results related to the particular system (2.3). Before we enter into the details, we fix some notation.

We assume that $m : [0, +\infty) \to (0, +\infty)$ is a nondecreasing function of class C^5. We denote by $M_{2\times2}$ the set of 2×2 matrices. For each $A \in M_{2\times2}$, a_{ij} is the element in the i-th row and $j - th$ column, unless otherwise stated, and $\mathrm{Tr}A = a_{11} + a_{22}$ is the trace of A. For every $\omega \in \mathbb{R}$, R_ω denotes the rotation matrix

$$R_\omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

2.2.1. *Stability.* In this section we recall some definitions of stability from the classical theory of Hamiltonian systems. For the sake of simplicity, we adapt definitions to the case of simple modes for system (2.3). In the phase space $\mathbb{R}^4$ we consider the energy level

$$\mathcal{H}_k := \left\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 :\ H_k(x_1,x_2,x_3,x_4) = H_k(1,0,0,0)\right\},$$

and the orbit

$$\Gamma_k := \{(U_k(t), U_k'(t), 0, 0) :\ t \in \mathbb{R}\}.$$

Definition 2.2. The simple mode U_k is called *orbitally stable* if, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every solution $(w(t), z(t))$ of system (2.3), the following property holds: if the initial datum $(w(0), w'(0), z(0), z'(0))$ belongs to a δ neighborhood of $(1,0,0,0)$, then for every $t \in \mathbb{R}$ the point $(w(t), w'(t), z(t), z'(t))$ lies in an ϵ neighborhood of Γ_k.

Definition 2.3. The simple mode U_k is called *isoenergetically orbitally stable* if the condition of Definition 2.2 is satisfied with the restriction $(w(0), w'(0), z(0), z'(0)) \in \mathcal{H}_k$.

Definition 2.4. The simple mode U_k is said to be *linearly stable* if $z(t) \equiv 0$ is a stable solution of the linear equation $z''(t) + \nu\ \tau_k^2 m(k^2 U_k^2(t)) z(t) = 0$, (that is the linearization of the second equation in (2.3)), i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|(z(0), z'(0))\| < \delta \implies \|(z(t), z'(t))\| < \epsilon, \quad \forall t \in \mathbb{R}.$$

It is obvious that orbital stability implies isoenergetical orbital stability. In non-degenerate situations, isoenergetical orbital stability implies linear stability. Here "non-degenerate situation" means that $(0,0)$ is not a parabolic point for the associated Poincaré map (see section 2.2.2 and 2.2.3 below). It is not essential to explain now such condition; we just remark that it is satisfied by our large energy simple modes.

2.2.2. *The Poincaré map.* Let us consider the open set $\mathcal{U}_k \subseteq \mathbb{R}^2$ defined by

$$\mathcal{U}_k := \left\{(x,y) \in \mathbb{R}^2 :\ H_k(0,0,x,2\pi\sqrt{\nu}y) < H_k(1,0,0,0)\right\}.$$

For every $(x,y) \in \mathcal{U}_k$, let $\alpha(x,y) > 0$ be the unique positive number such that

$$H_k(\alpha(x,y), 0, x, 2\pi\sqrt{\nu}y) = H_k(1,0,0,0).$$

Let $(w(t), z(t))$ be the solution of system (2.3) with initial data

$$w(0) = \alpha(x,y), \quad w'(0) = 0, \quad z(0) = x, \quad z'(0) = 2\pi\sqrt{\nu}y.$$

Finally, let $T := T(x,y)$ be the smallest $t > 0$ such that $w'(t) = 0$ and $w(t) > 0$. The interested reader can verify that such a T exists for every $(x,y) \in \mathcal{U}_k$. On the other hand the existence of T is classical up to restricting $\mathcal{U}_k$.

The Poincaré map $P_k : \mathcal{U}_k \to \mathbb{R}^2$, relative to the simple mode U_k of (2.3), is defined by

$$P_k(x,y) := \Big(z(T), (4\pi^2\nu)^{-1/2} z'(T)\Big).$$

We point out that both z and T depend on (x,y) and k. When $(x,y) = (0,0)$, then $w(t) = U_k(t)$ and $z(t) = 0$ for every $t \in \mathbb{R}$. It follows that $P_k(0,0) = (0,0)$, i.e., $(0,0)$ is a fixed point of the Poincaré map.

The interested reader is referred to the quoted literature, and in particular to [4], for a heuristic description of the Poincaré map.

Now we recall the classical definition of stability of fixed points for planar maps.

Definition 2.5. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set containing $(0,0)$, and let $P : \mathcal{U} \to \mathbb{R}^2$ be a map such that $P(0,0) = (0,0)$. The fixed point $(0,0)$ is said to be *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$(x,y) \in \mathcal{U},\ \|(x,y)\| < \delta \implies \|P^n(x,y)\| < \epsilon \quad \forall n \in \mathbb{N},$$

where P^n denotes the n-th iteration of P.

The stability of U_k as a periodic solution is clearly related to the stability of $(0,0)$ as a fixed point of P_k. This relation is stated in Theorem 2.7 below.

2.2.3. *KAM theory for planar maps.* Stability of planar maps has long been studied. In this subsection we sum the basic results we need in the sequel. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set containing $(0,0)$, and let $P : \mathcal{U} \to \mathcal{U}$. The theory of planar maps has been developed for very general maps P; however we state the results under suitable assumptions which allow to simplify some notations, and are trivially satisfied in our case. Therefore let us assume that:

(P1) $P \in C^5(\mathcal{U}, \mathcal{U})$ and $P(0,0) = (0,0)$;
(P2) P is area-preserving;
(P3) if $P(x,y) = (a,b)$, then $P(a,-b) = (x,-y)$;
(P4) $P(-x,-y) = -P(x,y)$.

The first object to look at in order to study the stability of the fixed point $(0,0)$ is the differential of P at $(0,0)$, which we denote by L. It is well known that the canonical form of L is one of the following three.

- $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some $\lambda \in \mathbb{R}$, $|\lambda| > 1$. In this case $(0,0)$ is said to be *hyperbolic* and it is *unstable.*
- $\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$ for some $a \neq 0$. In this case $(0,0)$ is said to be *parabolic.* The map L is unstable, but nothing can be said about P. However, we will not find this degenerate case in this paper.
- R_ω for some $\omega \in \mathbb{R}$. In this case $(0,0)$ is said to be *elliptic.* The map L is stable, but this is in general not enough to guarantee the stability of P.

Therefore, L gives only necessary conditions for stability (i.e., non hyperbolicity). KAM theory provides sufficient conditions in the case of elliptic fixed points. To describe such conditions, it is better to write P in polar coordinates up to terms of order three. If we choose coordinates where L is written in the canonical form of a rotation, then, in the corresponding polar coordinates, P becomes

$$P \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \rho + a(\theta)\rho^3 \\ \theta - \omega + b(\theta)\rho^2 \end{pmatrix} + o(\rho^3),$$

where ω is the same as in the linear term L, and $a(\theta)$ and $b(\theta)$ are trigonometric polynomials of degree 4. The absence of even powers of ρ in the first component, and of odd powers of ρ in the second component, is due to (P4). Finally we set

$$\gamma(P) := \frac{1}{2\pi} \int_0^{2\pi} b(\theta)\, d\theta. \tag{2.4}$$

Then we have the following KAM result.

Theorem 2.6. *Let $P : \mathcal{U} \to \mathcal{U}$ be a planar map satisfying (P1)–(P4). Let $(0,0)$ be an elliptic fixed point, and let ω and γ be defined as above. Let us assume that*

(KAM 1) **:** $e^{hi\omega} \neq 1$ *for every* $h \in \{1,2,3,4\}$*;*

(KAM 2) **:** $\gamma(P) \neq 0$.

Then $(0,0)$ is stable for P according to Definition 2.5.

The following result relates stability, Poincaré maps, and KAM theory. It is the fundamental tool in our analysis.

Theorem 2.7. *Let U_k be a simple mode of system (2.3), and let P_k be the associated Poincaré map. Then*

- *U_k is* linearly *stable if and only if $(0,0)$ is an elliptic fixed point of P_k;*
- *U_k is* isoenergetically orbitally *stable if and only if $(0,0)$ is a stable fixed point of the Poincaré map P_k;*
- *if $(0,0)$ is an elliptic fixed point of P_k, and P_k satisfies (KAM 1) and (KAM 2), then U_k is* orbitally *stable.*

Thanks to Theorem 2.6 and Theorem 2.7, the orbital stability of a periodic solution in the four dimensional space can be proved by verifying that a planar map satisfies two algebraic conditions.

3. Statement of results

Let us denote by $P_k : \mathcal{U}_k \to \mathcal{U}_k$ the Poincaré map associated with U_k as in section 2.2.2, and by L_k its differential in the fixed point $(0,0)$. In the next result we sum up the main properties of P_k and L_k.

Theorem 3.1. *For every $k > 0$, let P_k and L_k be as above. Then*

(1) *P_k satisfies (P1)–(P4);*

(2) $\det L_k = 1$*;*

(3) *if L_k^{ij} are the entries of L_k, then $L_k^{11} = L_k^{22}$.*

We do not prove such properties, since they are well known in the literature (see [4]).

Thanks to Theorem 2.1 the main result of this paper (Theorem 1.2) is reduced to prove that U_k is orbitally stable if k is large. Thanks to Theorem 2.7 and Theorem 2.6, the main result will be proved if we show that P_k satisfies assumptions (KAM 1) and (KAM 2) of Theorem 2.6. Assumption (KAM 1) follows from statements (1)–(3) of the following result, where the behaviour of L_k for large k is considered.

Theorem 3.2. *Let $\nu \neq 1$ be a positive real number. Then there exist $k_1 > 0$, $\omega : (k_1, +\infty) \to \mathbb{R}$, and $\delta : (k_1, +\infty) \to (0, +\infty)$ such that*

(1) *for every $k \geq k_1$ the eigenvalues of L_k are $\{e^{\pm i\omega(k)}\}$;*

(2) *$\omega(k) \to 2\pi\sqrt{\nu}$ as $k \to +\infty$;*

(3) *$\omega(k) \neq 2\pi\sqrt{\nu}$ for k large enough if $2\pi\sqrt{\nu} = h\pi$ for some $h \in \mathbb{Z}$;*

(4) *setting $D(k) = \begin{pmatrix} 1 & 0 \\ 0 & \delta(k) \end{pmatrix}$ we have that $[D(k)]^{-1} L_k D(k) = R_{\omega(k)}$;*

(5) *$\delta(k) \to \delta > 0$ as $k \to +\infty$.*

Statements (2) and (3) prevent $e^{i\omega(k)}$ from being a h-th root of 1 for $h \in \{1,2,3,4\}$ and k large. Indeed, if $e^{2\pi\sqrt{\nu}hi} \neq 1$ for $h \in \{1,2,3,4\}$, then by (2) the same holds true for $e^{\omega(k)hi}$, provided that k is large enough; if on the contrary

$e^{2\pi\sqrt{\nu}i}$ is a h-th root of 1 for some $h \in \{1,2,3,4\}$, then for k large $e^{\omega(k)i}$ is not because of (3). This shows in particular that $(0,0)$ is an elliptic fixed point of P_k for k large. Statement (4) says that L_k can be written in the canonical form by a diagonal matrix $D(k)$.

The following result implies that P_k satisfies assumption (KAM 2) for k large.

Theorem 3.3. *Let $\gamma_k := \gamma(P_k)$ be as in formula (2.4). Then $\gamma_k \neq 0$ for k large enough.*

We have therefore reduced the proof of Theorem 1.2 to the proof of Theorem 3.2 and Theorem 3.3.

4. Proofs

Throughout this section we denote by c various constants depending only on function m.

4.1. Properties of the function m.

In the following lemmata we state all properties of function m we need in the proofs.

Lemma 4.1. *As $k \to +\infty$,*

$$\lambda_k := \frac{k^2 m'(k^2)}{m(k^2)} \to 0. \tag{4.1}$$

Lemma 4.2. *For all $y \in (0,1)$,*

$$\frac{m(k^2 y)}{m(k^2)} = 1 + \lambda_k \log y + \phi_k(y)$$

where:

- (m1) $|\phi_k(y)| \le c\lambda_k |\log y|$,
- (m2) *for all* $0 < a < 1$, $\lim_{k\to+\infty} \sup_{a\le y\le 1} \lambda_k^{-1}|\phi_k(y)| = 0$.

Lemma 4.3. *For all $x > 0$:*

- (M1) $m(x) \le c(1 + \sqrt[8]{x})$;
- (M2) $M(x) = xm(x) - \int_0^x sm'(s)\,ds$;
- (M3) $k^2 m(k^2)(M(k^2))^{-1} = 1 + \lambda_k + o(\lambda_k)$;
- (M4) $\lim_{k\to+\infty} M(kx)(M(k))^{-1} = x$;
- (M5) $\sup\{(1-y^2)M(k^2)(M(k^2) - M(k^2y^2))^{-1} : \ y \in (0,1)\} \le c$.

In the next lemma we state some properties of τ_k, as defined in (2.2).

Lemma 4.4. *The following equalities hold*

$$\frac{\tau_k\sqrt{M(k^2)}}{4k} = \frac{\pi}{2} + \frac{\lambda_k}{2}\int_0^1 \frac{y^2\log y^2}{(1-y^2)^{3/2}}dy + o(\lambda_k) =: \frac{\pi}{2} + \lambda_k h_0 + o(\lambda_k). \tag{4.2}$$

Moreover

$$\tau_k^2 m(k^2) = 4\pi^2 + (4\pi^2 + 16h_0\pi)\lambda_k + o(\lambda_k).$$

Proof of Lemma 4.1. Since

$$\lambda_k^{-1} = \frac{\int_0^{k^2} m'(s)\,ds + m(0)}{k^2 m'(k^2)} \ge \int_0^1 \frac{m'(k^2y)}{m'(k^2)}\,dy,$$

for all $\varepsilon > 0$, using hypotheses (H1)–(H2) we get

$$\liminf_{k\to+\infty} \lambda_k^{-1} \geq \liminf_{k\to+\infty} \int_\varepsilon^1 \frac{m'(k^2y)}{m'(k^2)}\,dy = \int_\varepsilon^1 \frac{1}{y}\,dy.$$

Since ε is arbitrary we have obtained that $\lim_{k\to+\infty} \lambda_k^{-1} = +\infty$. □

Proof of Lemma 4.2. Firstly let us observe that

$$\frac{m(k^2y) - m(k^2)}{m(k^2)} - \lambda_k \log y = -\int_y^1 \frac{k^2 m'(k^2 s)}{m(k^2)} ds + \int_y^1 \frac{\lambda_k}{s} ds = \phi_k(y).$$

To prove (m1) it suffices to remark that by hypothesis (H1), we obtain

$$|\phi_k(y)| \leq \lambda_k \int_y^1 \frac{1}{s}\left|\frac{m'(k^2s)s}{m'(k^2)} - 1\right| ds \leq -\lambda_k c \log y.$$

Moreover for $y \geq a > 0$ holds true

$$|\phi_k(y)| \leq \lambda_k \int_a^1 \frac{1}{s}\left|\frac{m'(k^2s)s}{m'(k^2)} - 1\right| ds.$$

Passing now to the limit using Lebesgue's Theorem for the dominate convergence and (H2) we have (m2). □

Proof of Lemma 4.3. To prove (M1) it suffices to remark that by (4.1) we have $m'(x)/m(x) \leq 1/(8x)$ for large x. To show property (M2) it is enough integrate by parts the function m. Using property (M2) and hypothesis (H1) we then get:

$$\begin{aligned}
\frac{k^2 m(k^2)}{M(k^2)} - 1 - \lambda_k &= \int_0^1 \frac{k^4 y m'(k^2 y)}{M(k^2)} dy - \lambda_k \\
&= \lambda_k \Big(\frac{\int_0^1 y m'(k^2y)(m'(k^2))^{-1} dy}{M(k^2)(k^2 m(k^2))^{-1}} - 1 \Big) \\
&= \lambda_k \Big(\frac{\int_0^1 y m'(k^2y)(m'(k^2))^{-1} dy}{1 - \lambda_k \int_0^1 y m'(k^2y)(m'(k^2))^{-1} dy} - 1 \Big) \\
&= \lambda_k \Big(\frac{\int_0^1 y m'(k^2y)(m'(k^2))^{-1} dy}{1 + o(1)} - 1 \Big) =: \lambda_k R_k.
\end{aligned}$$

Since by Lebesgue's Theorem $\lim_{k\to+\infty} R_k = 0$, we have obtained (M3).

Property (M4) follows from L' Hopital's Theorem. Finally, to show property (M5) it is enough to remark that, using Cauchy's Theorem and the monotonicity of m we find

$$\frac{M(k^2y^2)}{M(k^2)} = \frac{m(\xi y^2)}{m(\xi)} y^2 \leq y^2.$$

□

Proof of Lemma 4.4. Let

$$\psi_k(y) = \frac{1}{\sqrt{1-y^2}\sqrt{1 - M(k^2y^2)/M(k^2)}(\sqrt{1-y^2} + \sqrt{1 - M(k^2y^2)/M(k^2)})}.$$

Applying Lemma 4.3 and Lemma 4.2 it turns out that

$$
\begin{aligned}
&\frac{\tau_k\sqrt{M(k^2)}}{4k}-\frac{\pi}{2}\\
&=\int_0^1 \frac{1}{\sqrt{1-M(k^2y^2)/M(k^2)}}-\frac{1}{\sqrt{1-y^2}}\,dy\\
&=\int_0^1 \psi_k(y)\Big(-y^2+\frac{M(k^2y^2)}{M(k^2)}\Big)\,dy\\
&=\int_0^1 \Big(y^2k^2(m(k^2y^2)-m(k^2))+y^2\int_0^{k^2} sm'(s)\,ds-\int_0^{y^2k^2} sm'(s)\,ds\Big)\frac{\psi_k(y)}{M(k^2)}\,dy\\
&=-\int_0^1 y^2\Big[1-\frac{m(k^2y^2)}{m(k^2)}\Big](1+o(1))\psi_k(y)\,dy+\\
&\quad+\int_0^1 (1+o(1))\frac{\lambda_k}{k^4}\Big[y^2\int_0^{k^2}\frac{sm'(s)}{m'(k^2)}\,ds-\int_0^{k^2y^2}\frac{sm'(s)}{m'(k^2)}\,ds\Big]\psi_k(y)\,dy\\
&=\int_0^1 -y^2\left[-\lambda_k\log y^2-\phi_k(y^2)\right](1+o(1))\psi_k(y)\,dy+\\
&\quad+\lambda_k\int_0^1(1+o(1))\Big[(y^2-1)\int_0^1\frac{sm'(k^2s)}{m'(k^2)}\,ds+\int_{y^2}^1\frac{sm'(k^2s)}{m'(k^2)}\,ds\Big]\psi_k(y)\,dy.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big(\frac{\tau_k\sqrt{M(k^2)}}{4k}-\frac{\pi}{2}-h_0\lambda_k\Big)\lambda_k^{-1}\\
&=\int_0^1 y^2\log y^2\Big(\psi_k(y)-\frac{1}{2(1-y^2)^{3/2}}\Big)\,dy+\\
&\quad+\int_0^1 o(1)y^2\log y^2\psi_k(y)+(1+o(1))y^2\psi_k(y)\frac{\phi_k(y^2)}{\lambda_k}\,dy+\\
&\quad+\int_0^1(1+o(1))\psi_k(y)(1-y^2)\Big[-\int_0^1\frac{sm'(k^2s)}{m'(k^2)}\,ds+\frac{1}{1-y^2}\int_{y^2}^1\frac{sm'(k^2s)}{m'(k^2)}\,ds\Big].
\end{aligned}
$$

Let us remark that by Lemma 4.3, properties (M4)–(M5), $(1-y^2)^{3/2}\psi_k(y)$ is a bounded function and converges for $y\in(0,1)$ to the function identically $=1/2$, hence, using once more hypotheses (H1)–(H2) and Lemma 4.2, by Lebesgue's Theorem we obtain

$$\lim_{k\to+\infty}\Big(\frac{\tau_k\sqrt{M(k)}}{4k}-\frac{\pi}{2}-h_0\lambda_k\Big)\lambda_k^{-1}=0.$$

In order to prove the second part of the lemma it suffices to observe that

$$\tau_k^2m(k^2)=\Big(\frac{\tau_k\sqrt{M(k^2)}}{4k}\Big)^2 16k^2\frac{m(k^2)}{M(k^2)}=16(\frac{\pi}{2}+h_0\lambda_k+o(\lambda_k))^2(1+\lambda_k+o(\lambda_k)).$$

□

4.2. **Properties of the simple mode U_k.** Let us recall that U_k is the solution of the problem:

$$U_k''+\tau_k^2m(k^2U_k^2)U_k=0\quad U_k(0)=1,\;U_k'(0)=0. \tag{4.3}$$

In the sequel we need the following simple properties of U_k:

(U1) U_k is a 1-periodic function, and for every $t \in [0, 1/4]$,

$$U_k(t) = U_k(1-t) = -U_k(1/2 - t) = -U_k(1/2 + t);$$

(U2) U_k is decreasing in $[0, 1/2]$ and increasing in $[1/2, 1]$;

(U3) for every $t \in [0, 1]$ we have that

$$|U_k'|^2 + \frac{\tau_k^2}{k^2} \int_0^{k^2 U_k^2} m(s)\, ds = \frac{\tau_k^2}{k^2} \int_0^{k^2} m(s)\, ds;$$

(U4) $|U_k(t)| \le 1$ for every $t \in [0, 1]$;

(U5) $|U_k'(t)| \le \frac{\tau_k}{k} \sqrt{M(k^2)}$ for every $t \in [0, 1]$.

Properties (U1) and (U2) follow from the symmetries of U_k; (U3) follows from the conservation of the Hamiltonian for U_k, and (U4) and (U5) are consequences of (U3).

The simple mode verifies also the following properties.

Lemma 4.5. *One has:*

(B1) $U_k(t) = \cos(2\pi t) + o(1)$ *where* $o(1)$ *is uniform in* t *in bounded intervals;*

(B2) *there exist* $\lambda > 0$, $0 < t_0 < 1/4$ *and* $k_0 \in \mathbb{R}$ *such that for all* $k \ge k_0$, $|U_k'(t)| \ge \lambda$ *for all* $t \in \big[t_0, \frac{1}{4}\big]$;

(B3) *there exists* $c(t) \in L^1([0, 1])$ *such that for* k *large:* $|\log(U_k^2(t))| \le c(t)$ *for all* $t \in [0, 1]$.

Throughout the paper we need also some properties of integrals of U_k.

Lemma 4.6. *The following inequalities hold true*

$$\int_0^1 \Big| 1 - \frac{\tau_k^2}{4\pi^2} m(k^2 U_k(t)) \Big|\, dt \le c\lambda_k; \tag{4.4}$$

$$\int_0^1 \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt \le c\, \frac{\log k}{k^2}; \tag{4.5}$$

$$\int_0^1 m'(k^2 U_k(s)^2)\, ds \le \frac{c}{k}. \tag{4.6}$$

Proof of Lemma 4.5. By Lemma 4.2 and Lemma 4.4 we obtain

$$U_k'' + (4\pi^2 + o(1))(1 + \lambda_k \log U_k^2 + \phi_k(U_k^2))U_k = 0,$$

that we can rewrite as

$$U_k'' + 4\pi^2 U_k = (4\pi^2 + o(1))(\lambda_k \log U_k^2 + \phi_k(U_k^2))U_k + o(1)U_k.$$

Since, thanks to (m1) and (U4)

$$\big[\log U_k^2 + \frac{\phi_k(U_k^2)}{\lambda_k})\big] U_k$$

is a bounded function, we get $U_k'' + 4\pi^2 U_k = o(1)$. Moreover, setting $v_k(t) = U_k(t) - \cos(2\pi t)$, we find $v_k'' + 4\pi^2 v_k = o(1)$, hence (B1) follows from

$$\sqrt{|v_k'(t)|^2 + 4\pi^2 |v_k(t)|^2} \le o(1)t.$$

By (B1) there exists $t_0 \in (0,1/4)$ such that $|U_k(t)| \le 1/2$ for all $t \in [t_0, 1/4]$ and k large. To prove (B2) it is enough to remark that, by Lemma 4.2, Lemma 4.4 we have

$$\inf_{t_0 \le t \le 1/4} |U_k'(t)| = \inf_{t_0 \le t \le 1/4} \int_{U_k^2(t)}^1 \tau_k^2 m(k^2 s)\, ds \ \ge \int_{1/4}^1 \tau_k^2 m(k^2 s)\, ds \to 3\pi^2.$$

Let now t_0 be as in (B2). By (B1) and (U4) for large k we get $1 \ge |U_k(t)| \ge c > 0$ for all $t \in [0, t_0]$. Using (B2) and Cauchy's Theorem, for large k and $t \in [t_0, 1/4]$ we get

$$\frac{U_k(t)}{1/4 - t} = -U_k'(\xi_t) \ge \lambda,$$

hence

$$\Big| \frac{\sqrt{U_k(t)} \log(U_k^2(t))}{\sqrt{U_k(t)}} \Big| \le \frac{c}{\sqrt{1/4 - t}}.$$

Then (B3) follows from the symmetries of the function U_k. □

Proof of Lemma 4.6. To show (4.4) it suffices to observe that, by Lemma 4.2, Lemma 4.4 and Lemma 4.5, we have that

$$\begin{aligned}\int_0^1 \big|1 - \frac{\tau_k^2}{4\pi^2} m(k^2 U_k(s))\big|\, ds &= \int_0^1 \big|1 - \big(1 + (1 + 4\frac{h_0}{\pi})\lambda_k + o(\lambda_k)\big) \frac{m(k^2 U_k(s))}{m(k^2)}\big|\, ds \\ &\le c\lambda_k \int_0^1 1 + \big|\log(U_k(s)^2) + \frac{\phi_k(U_k(s)^2)}{\lambda_k}\big|\, ds \le c\lambda_k.\end{aligned}$$

Thanks to (U1), in order to prove (4.5) it is enough to show that

$$\int_0^{1/4} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt \le c \frac{\log k}{k^2}.$$

Let us take t_0 as in Lemma 4.5, (B2) and let us divide the integral as follows

$$\int_0^{1/4} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt = \int_0^{t_0} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt + \int_{t_0}^{1/4} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt.$$

Now we can estimate the two terms separately. By (U4) and Lemma 4.5, (B1), for large k we have that: $1 \ge |U_k(t)| \ge c$ for all $t \in [0, t_0]$, hence

$$\int_0^{t_0} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt \le \frac{c}{k^2}.$$

Let λ be as in Lemma 4.5, (B2). Since $U_k(1/4) = 0$, we get

$$\int_{t_0}^{1/4} \frac{|U_k(t)|}{1 + k^2 U_k^2(t)} dt = -\int_{t_0}^{1/4} \frac{U_k(t) U_k'(t)}{(1 + k^2 U_k^2(t)) |U_k'(t)|} dt \le \frac{1}{\lambda k^2} \log(1 + k^2 U_k^2(t_0)).$$

To prove (4.6) we can proceed as to prove (4.5). Only we remark that, since by (4.1) and (M1) one has

$$m'(z^2) \le \frac{c\, m(z^2)}{(1 + z^2)} \le \frac{c}{1 + z^{7/4}},$$

then in $[0,t_0]$ we get $m'(k^2U_k^2(t)) \leq c/k^{7/4}$ and in $[t_0,1/4]$:

$$\int_{t_0}^{1/4} m'(k^2U_k^2(s))\,ds = -\int_{t_0}^{1/4} \frac{m'(k^2U_k^2(s))U_k'(s)}{|U_k'(s)|}\,ds \leq -c\int_{t_0}^{1/4} m'(k^2U_k^2(s))U_k'(s)$$
$$= -\frac{c}{k}\int_{kU_k(t_0)}^{0} m'(z^2)\,dz \leq \frac{c}{k}\int_0^{+\infty} m'(z^2)\,dz < +\infty.$$

□

4.3. **Linear stability.**

4.3.1. *Preliminary results for k fixed.* Let P_k be the Poincaré map associated with U_k, and let L_k be its differential in $(0,0)$. Then the linear operator $L_k : \mathbb{R}^2 \to \mathbb{R}^2$ can be characterized in the following way.

Given $(x,y) \in \mathbb{R}^2$, let $z_k(t)$ be the solution of the linear problem

$$z_k''(t) + \nu\,\tau_k^2 m(k^2U_k^2(t))z_k(t) = 0, \quad z_k(0) = x, \quad z_k'(0) = 2\pi\sqrt{\nu}\,y. \tag{4.7}$$

This problem is the linearization of the second equation of system (2.3). Then we have that

$$L_k(x,y) := \Big(z_k(1), (4\pi^2\nu)^{-1/2}z_k'(1)\Big)\,.$$

We do not give the proof of this characterization, since it is completely analogous to the proof of [2, Proposition 2.1]. However, L_k is the first term in the Taylor expansion of P_k and in section 4.4 we find the first three terms of the expansion of P_k.

The fundamental tool in the analysis of the behaviour of L_k for k large is the following linear algebra result, whose proof is omitted since it is analogous of the corresponding one in [4] (Proposition 4.1).

Proposition 4.7. *Let $k_0 > 0$, and let $A : (k_0,+\infty) \to M_{2\times 2}$. Let us assume that*

(i) $\det A(k) = 1$ *for every* $k \geq k_0$*;*
(ii) $a_{11}(k) = a_{22}(k)$ *for every* $k \geq k_0$*;*
(iii) *there exist* $\omega_0 \in \mathbb{R}$ *and* $B \in M_{2\times 2}$ *such that, for* $k \to +\infty$, $A(k) = R_{\omega_0} + \lambda_k B + o(\lambda_k)$*;*
(iv) ω_0 *and* B *satisfy one of the following conditions:*
  (iv-1) $\omega_0 \neq h\pi$ *for every* $h \in \mathbb{Z}$*;*
  (iv-2) $\omega_0 = h\pi$ *for some* $h \in \mathbb{Z}$, $\mathrm{Tr}B = 0$, *and* $b_{12}\cdot b_{21} < 0$.

Then there exist $k_1 \geq k_0$, $\omega : (k_1,+\infty) \to \mathbb{R}$, *and* $\delta : (k_1,+\infty) \to (0,+\infty)$ *such that*

(1) *for every* $k \geq k_1$ *the eigenvalues of* $A(k)$ *are* $\{e^{\pm i\omega(k)}\}$*;*
(2) $\omega(k) \to \omega_0$ *as* $k \to +\infty$*;*
(3) $\omega(k) \neq \omega_0$ *for* k *large enough if* $\omega_0 = h\pi$ *for some* $h \in \mathbb{Z}$*;*
(4) *setting* $D(k) = \begin{pmatrix} 1 & 0 \\ 0 & \delta(k) \end{pmatrix}$ *we have that* $[D(k)]^{-1}A(k)D(k) = R_{\omega(k)}$*;*
(5) $\delta(k) \to \delta$ *as* $k \to +\infty$ *when* $\delta = 1$ *if* $\omega_0 \neq h\pi$ *for all* $h \in \mathbb{Z}$ *and* $\delta = \sqrt{-b_{21}/b_{12}}$ *otherwise.*

We use also the following lemma.

Lemma 4.8. *The following equalities hold true.*

(S1) *For all $h \geq 1$ we have that*

$$\alpha_h := \int_0^{\pi/2} \sin(2hx)\frac{\sin x}{\cos x}\,dx = (-1)^{h+1}\frac{\pi}{2}.$$

(S2) *If h_0 is as in (4.2) then*

$$h_0 = -\frac{\pi}{2} - \frac{1}{2}\int_0^{\pi/2} \log(\cos^2 x)\,dx.$$

(S3) *For all $h \geq 1$, $h \in \mathbb{N}$ one has*

$$\int_0^{\pi/2} 2\sin^2(hx)\log(\cos^2 x) = (-1)^h\frac{\pi}{2h} + \int_0^{\pi/2} \log(\cos^2 x)\,dx.$$

Proof of Lemma 4.8. We prove (S1). Obviously we have $\alpha_1 = \pi/2$; moreover for $h > 1$ thesis follows from

$$\begin{aligned}
&\int_0^{\pi/2} \sin(2(h-1)x + 2x)\frac{\sin x}{\cos x}\,dx \\
&= \int_0^{\pi/2} 2\cos(2(h-1)x)\sin^2 x\,dx + \int_0^{\pi/2} \sin(2(h-1)x)(2\cos^2 x - 1)\frac{\sin x}{\cos x}\,dx \\
&= \int_0^{\pi/2} \cos(2(h-1)x)(1-\cos 2x)\,dx + \int_0^{\pi/2} \sin(2(h-1)x)\sin 2x\,dx - \alpha_{h-1} \\
&= -\alpha_{h-1} - \int_0^{\pi/2} \cos((2(h-1)+2)x)\,dx + \frac{\sin(2(h-1)x)}{2(h-1)}\Big|_0^{\pi/2} = -\alpha_{h-1}
\end{aligned}$$

To prove (S2) it suffices a change of variables and an integration by parts. To prove (S3) we need only to remark that

$$\begin{aligned}
&\int_0^{\pi/2} 2\sin^2(hx)\log(\cos^2 x)\,dx \\
&= \int_0^{\pi/2} \log(\cos^2 x)\,dx - \int_0^{\pi/2} \cos(2hx)\log(\cos^2 x)\,dx \\
&= \int_0^{\pi/2} \log(\cos^2 x)\,dx - \frac{\sin(2hx)}{2h}\log(\cos^2 x)\Big|_0^{\pi/2} - \frac{\alpha_h}{h}.
\end{aligned}$$

□

4.3.2. *Polar coordinates for $z_k(t)$.* We write (4.7) as a first order system. To this end we set $x_k(t) = z_k(t)$, $y_k(t) = (\nu 4\pi^2)^{-1/2} z_k'(t)$, so that (4.7) becomes

$$x_k'(t) = \sqrt{\nu 4\pi^2}\, y_k(t), \quad y_k'(t) = -(\nu 4\pi^2)^{-1/2}\,\nu\,\tau_k^2 m(k^2 U_k^2(t))\, x_k(t),$$

with initial data $x_k(0) = x$, $y_k(0) = y$.

If $(x, y) \neq (0, 0)$, then $(x_k(t), y_k(t)) \neq (0, 0)$ for every $t \in \mathbb{R}$. We can therefore study this system introducing polar coordinates $\rho_k(t)$, $\theta_k(t)$ such that

$$x_k(t) = \rho_k(t)\cos\theta_k(t), \quad y_k(t) = \rho_k(t)\sin\theta_k(t). \tag{4.8}$$

In a standard way it turns out that ρ_k and θ_k solve the system

$$\rho_k' = 2\pi\sqrt{\nu}\,\rho_k \sin\theta_k \cos\theta_k \big\{1 - \tau_k^2 \frac{m(k^2U_k^2)}{4\pi^2}\big\}, \tag{4.9}$$

$$\theta_k' = -2\pi\sqrt{\nu}\big\{\sin^2\theta_k + \tau_k^2 \frac{m(k^2U_k^2)}{4\pi^2}\cos^2\theta_k\big\}, \tag{4.10}$$

with initial data $\rho_k(0) = \rho$, $\theta_k(0) = \theta$, such that $x = \rho\cos\theta$, $y = \rho\sin\theta$.

4.3.3. *Behaviour of ρ_k and θ_k for large k.* We look for functions $\rho_{0,k}$, $\rho_{2,k}$, $\theta_{0,k}$, $\theta_{2,k}$ such that, as $k \to +\infty$:

$$\rho_k(t) = \rho_{0,k}(t) + \rho_{2,k}(t)\lambda_k + o(\lambda_k), \tag{4.11}$$

$$\theta_k(t) = \theta_{0,k}(t) + \theta_{2,k}(t)\lambda_k + o(\lambda_k), \tag{4.12}$$

where $o(\lambda_k)$ is uniform in $t \in [0,1]$. We prove that $\rho_{0,k}(t) \equiv \rho$ and $\rho_{2,k}(t)$ solves

$$\rho_{2,k}'(t) = -\rho\pi\sqrt{\nu}\Big(1 + \frac{4h_0}{\pi} + \log(U_k^2(t))\Big)\sin(2\theta - 4\pi\sqrt{\nu}t) \quad \rho_{2,k}(0) = 0,$$

while $\theta_{0,k}(t) = \theta - 2\pi\sqrt{\nu}t$, and $\theta_{2,k}$ solves

$$\theta_{2,k}'(t) = -2\pi\sqrt{\nu}\Big(1 + \frac{4h_0}{\pi} + \log(U_k^2(t))\Big)\cos^2(\theta - 2\pi\sqrt{\nu}t) \quad \theta_{2,k}(0) = 0.$$

Thanks to (4.9) we find:

$$\rho_k(t) \le c\int_0^t \rho_k(s)\Big|1 - \tau_k^2\frac{m(k^2U_k^2(s))}{4\pi^2}\Big|\,ds + \rho,$$

hence using (4.4) we obtain that, for all $t \in [0,1]$, $|\rho_k(t) - \rho| \le c\lambda_k$. In the same way we also get $|\theta_k(t) + 2\pi\sqrt{\nu}t - \theta| \le c\lambda_k$. Moreover using Lemma 4.2 and Lemma 4.4 for $t \in [0,1]$,

$$\begin{aligned}
&\frac{|\rho_k(t) - \rho - \lambda_k\rho_{2,k}(t)|}{\lambda_k}\\
&\le \frac{c}{\lambda_k}\int_0^t \Big|1 - \tau_k^2\frac{m(k^2U_k^2(s))}{4\pi^2} + \lambda_k(1 + \frac{4h_0}{\pi} + \log(U_k^2(s)))\Big|\,ds\\
&\quad + \frac{c}{\lambda_k}\int_0^t \Big|1 - \tau_k^2\frac{m(k^2U_k^2(s))}{4\pi^2}\Big|(|\rho_k(s) - \rho| + |\sin(2\theta_k(s)) - \sin(2\theta - 4\pi\sqrt{\nu}s)|)\,ds\\
&\le \frac{c}{\lambda_k}\int_0^t |\lambda_k o(1)\log(U_k^2(s))| + o(\lambda_k) + |\phi_k(U_k^2(s))|\,ds\\
&\quad + \frac{c}{\lambda_k}\int_0^t (\lambda_k + |\theta_k(s) - \theta + 2\pi\sqrt{\nu}s|)\Big|1 - \tau_k^2\frac{m(k^2U_k^2(s))}{4\pi^2}\Big|\,ds\\
&\le c\int_0^1 \Big|1 - \tau_k^2\frac{m(k^2U_k^2(s))}{4\pi^2}\Big| + o(1)(|\log(U_k^2(s))| + 1)\,ds + c\int_0^1 \frac{\phi_k(U_k^2(s))}{\lambda_k}ds.
\end{aligned}$$

By Lemma 4.2, (m1)–(m2) and Lemma 4.5, (B3), in a standard way we get

$$\lim_{k\to+\infty}\int_0^1 \frac{\phi_k(U_k^2(s))}{\lambda_k}ds = 0,$$

hence using once more (B3) and (4.4) it turns out that:

$$\lim_{k\to+\infty}\sup_{t\in[0,1]}\frac{|\rho_k(t) - \rho - \lambda_k\rho_{2,k}(t)|}{\lambda_k} = 0,$$

that is (4.11). In a similar way one can prove also (4.12). By Lemma 4.5 we can now pass to limit using Lebesgue's Theorem, then

$$\lim_{k\to+\infty}\rho_{2,k}(1)=-\pi\sqrt{\nu}\rho\int_0^1\sin(2\theta-4\pi\sqrt{\nu}t)(1+\frac{4h_0}{\pi}+\log(\cos^2(2\pi t))\,dt:=\rho_{1,\rho,\theta},$$

$$\lim_{k\to+\infty}\theta_{2,k}(1)=-2\pi\sqrt{\nu}\int_0^1\cos^2(\theta-2\pi\sqrt{\nu}t)(1+\frac{4h_0}{\pi}+\log(\cos^2(2\pi t))\,dt:=\theta_{1,\rho,\theta},$$

hence

$$\rho_k(1)=\rho+\lambda_k\rho_{1,\rho,\theta}+o(\lambda_k),\quad\text{and}\quad\theta_k(1)=\theta-2\pi\sqrt{\nu}+\lambda_k\theta_{1,\rho,\theta}+o(\lambda_k).\tag{4.13}$$

4.3.4. *Behaviour of L_k for large k.* Now let us denote by L_k^{ij} the entries of the matrix L_k. Then it holds true that $(L_k^{11},L_k^{21})=\big(z_k(1),(4\pi^2\nu)^{-1/2}z_k'(1)\big)$, where z_k has initial data $x=1$, $y=0$, corresponding to $\rho=1$, $\theta=0$. By (4.8) and (4.13) we obtain that

$$\begin{aligned}L_k^{11}&=\cos(2\pi\sqrt{\nu})+\lambda_k(\rho_{1,1,0}\cos(2\pi\sqrt{\nu})+\theta_{1,1,0}\sin(2\pi\sqrt{\nu}))+o(\lambda_k),\\ L_k^{21}&=-\sin(2\pi\sqrt{\nu})+\lambda_k(-\rho_{1,1,0}\sin(2\pi\sqrt{\nu})+\theta_{1,1,0}\cos(2\pi\sqrt{\nu}))+o(\lambda_k).\end{aligned}$$

Making the same computations with initial data $x=0$, $y=1$, corresponding to $\rho=1$, $\theta=\pi/2$, we find that $L_k^{22}=L_k^{11}$, and

$$L_k^{12}=\sin(2\pi\sqrt{\nu})+\lambda_k(\rho_{1,1,\pi/2}\sin(2\pi\sqrt{\nu})-\theta_{1,1,\pi/2}\cos(2\pi\sqrt{\nu}))+o(\lambda_k).$$

We have thus proved that

$$L_k=R_{\omega_0}+\lambda_kB+o(\lambda_k),$$

where $\omega_0=2\pi\sqrt{\nu}$, and B is a matrix whose entries are

$$\begin{aligned}b_{11}=b_{22}&=(\rho_{1,1,0}\cos(2\pi\sqrt{\nu})+\theta_{1,1,0}\sin(2\pi\sqrt{\nu})),\\ b_{12}&=(\rho_{1,1,\pi/2}\sin(2\pi\sqrt{\nu})-\theta_{1,1,\pi/2}\cos(2\pi\sqrt{\nu})),\\ b_{21}&=(-\rho_{1,1,0}\sin(2\pi\sqrt{\nu})+\theta_{1,1,0}\cos(2\pi\sqrt{\nu})).\end{aligned}$$

4.3.5. *Properties of B.* If $\omega_0=h\pi$ for some $h\in\mathbb{Z}$, then the matrix B becomes

$$B=\pm\begin{pmatrix}\rho_{1,1,0}&-\theta_{1,1,\pi/2}\\ \theta_{1,1,0}&\rho_{1,1,0}\end{pmatrix}.$$

In this case we have $\rho_{1,1,0}=0$, indeed it is the integral of a periodic (of period 1) odd function over the interval $[0,1]$; therefore $\mathrm{Tr}B=0$. Moreover it holds

$$\begin{aligned}-\theta_{1,1,\pi/2}&=\pi h\int_0^1\sin^2(\pi ht)\Big(1+\frac{4h_0}{\pi}+\log(\cos^2(2\pi t)\Big)\,dt\\ \theta_{1,1,0}&=-\pi h\int_0^1\cos^2(\pi ht)\Big(1+\frac{4h_0}{\pi}+\log(\cos^2(2\pi t)\Big)\,dt.\end{aligned}$$

If h is odd, by Lemma 4.8, (S2), computing, one has

$$\begin{aligned}-\frac{\theta_{1,1,\pi/2}}{\pi h}&=\int_0^1\frac{1-\cos(2h\pi t)}{2}\Big(1+\frac{4h_0}{\pi}+\log(\cos^2 2\pi t)\Big)\,dx\\ &=\big(1+\frac{4h_0}{\pi}\big)\frac{1}{2}+\frac{1}{\pi}\int_0^{\pi/2}\log(\cos^2x)\,dx-\frac{1}{4\pi}\int_0^{2\pi}\cos(ht)\log(\cos^2t)\,dt\\ &=-1/2;\end{aligned}$$

indeed in this case

$$\int_0^\pi \cos(ht)\log(\cos^2 t)\,dt = -\int_\pi^{2\pi} \cos(ht)\log(\cos^2 t)\,dt.$$

In an analogous way we get also

$$\frac{\theta_{1,1,0}}{\pi h} = \frac{1}{2} = -\frac{\theta_{1,1,\pi/2}}{\pi h}.$$

If $h = 2j$ is even, then using once more Lemma 4.8 and computing, one has

$$\begin{aligned}
-\frac{\theta_{1,1,\pi/2}}{2\pi j} &= \int_0^1 (1+\frac{4h_0}{\pi})\sin^2(2\pi jt)dt + \frac{1}{2\pi}\int_0^{2\pi} \sin^2(jx)\log(\cos^2 x)\,dx \\
&= (1+\frac{4h_0}{\pi})\frac{1}{2} + \frac{1}{\pi}\Big(\frac{(-1)^j\pi}{2j} + \int_0^{\pi/2}\log(\cos^2 x)\,dx\Big) \\
&= -\frac{1}{2} + \frac{(-1)^j}{2j} < 0.
\end{aligned}$$

Since $\nu \neq 1$, hence we get $j \neq 1$; then work as before we find

$$\frac{\theta_{1,1,0}}{2\pi j} = \frac{1}{2} + \frac{(-1)^j}{2j} > 0.$$

We have hence proved that in all cases B satisfies hypothesis (iv) of Proposition 4.7.

4.3.6. *Proof of Theorem 3.2.* By Subsections 4.3.4 and 4.3.5 and Theorem 3.1 we get that L_k satisfies all the assumptions of Proposition 4.7. Therefore statements (1)–(5) of Theorem 3.2 follow from the corresponding statements of Proposition 4.7.

4.4. **Orbital stability.** In proofs, we need expansions of solutions of Cauchy problems depending on some small parameter. We will always work formally as follows. Assume that the Cauchy problem is

$$Z' = F(Z,\mu), \quad Z(0) = \Phi(\mu), \tag{4.14}$$

where μ is the small parameter, and $Z(t) \in \mathbb{R}^k$ is the unknown. Then we look for an expansion like

$$Z(t) = Z_0(t) + Z_1(t)\mu + Z_2(t)\mu^2 + \ldots + Z_h(t)\mu^h + o(\mu^h). \tag{4.15}$$

We replace Z in (4.14) with this expression, and using the Taylor formula, we write also $F(Z,\mu)$ and $\Phi(\mu)$ as polynomials of degree h in μ (in the first case the coefficients depend on $Z_0, Z_1, \ldots Z_h$) plus $o(\mu^h)$. Finally, considering the coefficients of $\mu^0, \mu^1, \ldots, \mu^h$, we find the Cauchy problems solved by $Z_0, Z_1, \ldots Z_h$.

It is well known that, if F and Φ are smooth enough, then this procedure can be rigorously justified, and that (4.15) turns out to be uniform on bounded time intervals. To avoid useless terms in writing expansion (4.15), we always omit from the beginning the terms which *a posteriori* would turn out to be zero.

4.4.1. *Taylor expansions in ρ for k fixed.* In this section k will be fixed. We compute the first three terms in the Taylor expansion in a neighborhood of $(0,0)$ of the Poincaré map P_k associated with the simple mode U_k given in (4.3). In order to fix notations, we write once more the definition of P_k, following section 2.2.2.

Given $(x,y) \in \mathcal{U}_k$ we consider the solution of the system

$$W'' + \tau_k^2 m(k^2W^2 + Z^2)W = 0, \quad W(0) = \alpha, \quad W'(0) = 0, \tag{4.16}$$

$$Z'' + \nu\,\tau_k^2 m(k^2W^2 + Z^2)Z = 0, \quad Z(0) = x, \quad Z'(0) = 2\pi\sqrt{\nu}\,y, \tag{4.17}$$

where α is the positive solution of

$$\frac{4\pi^2\,y^2}{k^2} + \frac{\tau_k^2}{k^2}M(k^2\alpha^2 + x^2) = \frac{\tau_k^2}{k^2}M(k^2).$$

Let T be the smallest $t > 0$ such that $W'(t) = 0$ and $W(t) > 0$. Then

$$P_k(x,y) := \Big(Z(T), (4\pi^2\nu)^{-1/2}Z'(T)\Big).$$

Since we plan to use polar coordinates we assume that $x = \rho\cos\theta$, $y = \rho\sin\theta$.

Formally this definition is very similar to the definition of L_k. However the situation is here much more complicated, because α depends on k, ρ, θ, hence also W, Z and T depend on k, ρ, θ. We use capital letters to avoid confusion with the corresponding functions used in the study of the linear term. We also write $W(k,\rho,\theta,t)$, $\alpha(k,\rho,\theta)$, and so on, to recall the dependence on all these variables. The symbol $'$ will always denote differentiation with respect to the time variable t.

In this first part of the proof we consider the asymptotic behaviour of these functions as $\rho \to 0^+$ (k fixed). All the terms $o(\rho^j)$ we introduce are uniform on $\theta \in [0, 2\pi]$, and on t belonging to any bounded time interval.

Asymptotic behaviour of α. We prove that as $\rho \to 0^+$, we have that

$$\alpha(k,\rho,\theta) = 1 - \frac{1}{2k^2}\Big[\frac{4\pi^2\sin^2\theta}{\tau_k^2 m(k^2)} - \cos^2\theta\Big]\rho^2 + o(\rho^3). \tag{4.18}$$

Since $\alpha(k,0,\theta) = 1$ we look for an expansion of α as

$$\alpha(k,\rho,\theta) = 1 + \alpha_2(k,\theta)\rho^2 + o(\rho^3).$$

Since

$$\frac{4\pi^2\rho^2\sin^2\theta}{k^2} + \int_0^{\alpha^2+(\rho^2\cos^2\theta)/k^2} \tau_k^2 m(k^2s)\,ds = \int_0^1 \tau_k^2 m(k^2s)\,ds,$$

then taking into account the Taylor expansions,

$$\frac{4\pi^2\rho^2\sin^2\theta}{k^2} + \tau_k^2 m(k^2)\left(1 - \alpha^2 - \rho^2\frac{\cos^2\theta}{k^2}\right) + o(\rho^3) = 0.$$

Hence thesis follows immediately by

$$\frac{4\pi^2\rho^2\sin^2\theta}{k^2} + \tau_k^2 m(k^2)\left(-2\alpha_2(k,\theta) - \frac{\cos^2\theta}{k^2}\right)\rho^2 + o(\rho^3) = 0.$$

Polar coordinates for Z. We argue as in section 4.3.2. Setting

$$X(k,\rho,\theta,t) = Z(k,\rho,\theta,t), \quad Y(k,\rho,\theta,t) = (\nu 4\pi^2)^{-1/2}Z'(k,\rho,\theta,t),$$

and using polar coordinates $R(k,\rho,\theta,t)$, $\Theta(k,\rho,\theta,t)$ such that $X = R\cos\Theta$, $Y = R\sin\Theta$, it turns out that R and Θ solve the system

$$R' = 2\pi\sqrt{\nu}\, R\,\sin\Theta\cos\Theta\big\{1 - \frac{\tau_k^2 m(k^2W^2 + R^2\cos^2\Theta)}{4\pi^2}\big\}, \quad R(k,\rho,\theta,0) = \rho, \tag{4.19}$$

$$\Theta' = -2\pi\sqrt{\nu}\big\{\sin^2\Theta + \frac{\tau_k^2 m(k^2W^2 + R^2\cos^2\Theta)}{4\pi^2}\cos^2\Theta\big\}, \quad \Theta(k,\rho,\theta,0) = \theta. \tag{4.20}$$

Asymptotic behaviour of W, R, Θ. We look for functions W_0, W_2, R_1, R_3, Θ_0, Θ_2 such that, as $\rho \to 0^+$,

$$W(k,\rho,\theta,t) = W_0(k,\theta,t) + W_2(k,\theta,t)\rho^2 + o(\rho^3), \tag{4.21}$$

$$R(k,\rho,\theta,t) = R_1(k,\theta,t)\rho + R_3(k,\theta,t)\rho^3 + o(\rho^3), \tag{4.22}$$

$$\Theta(k,\rho,\theta,t) = \Theta_0(k,\theta,t) + \Theta_2(k,\theta,t)\rho^2 + o(\rho^3). \tag{4.23}$$

Using these expansions, we have

$$\begin{aligned} m(k^2W^2 + Z^2) &= m\left(k^2W_0^2 + (2k^2W_0W_2 + R_1^2\cos^2\Theta_0)\rho^2 + o(\rho^3)\right) \\ &= m(k^2W_0^2) + m'(k^2W_0^2)(2W_0W_2k^2 + R_1^2\cos^2\Theta_0)\rho^2 + o(\rho^3). \end{aligned} \tag{4.24}$$

Setting (4.21), (4.24), and (4.18) in equation (4.16), and looking at the terms without ρ, we find that W_0 solves

$$W_0'' + \tau_k^2 m(k^2W_0^2)W_0 = 0, \quad W_0(k,\theta,0) = 1, \quad W_0'(k,\theta,0) = 0, \tag{4.25}$$

while, looking at the terms in ρ^2, we find that W_2 solves

$$W_2'' + \tau_k^2 m(k^2W_0^2)W_2 + \tau_k^2 m'(k^2W_0^2)\left(2W_0W_2k^2 + R_1^2\cos^2\Theta_0\right)W_0 = 0, \tag{4.26}$$

with initial data

$$W_2(k,\theta,0) = -\frac{1}{2k^2}\Big[\frac{4\pi^2\sin^2\theta}{\tau_k^2 m(k^2)} - \cos^2\theta\Big], \quad W_2'(k,\theta,0) = 0. \tag{4.27}$$

From (4.25) we can see that W_0 is just the simple mode U_k. In particular, it is independent on θ, and so from now on we write $U_k(t)$, instead of $W_0(k,\theta,t)$. Setting (4.22), (4.23), and (4.24) in equation (4.19), and looking at the terms in ρ, we find that R_1 solves

$$R_1' = 2\pi\sqrt{\nu}\Big\{1 - \frac{\tau_k^2 m(k^2U_k^2)}{4\pi^2}\Big\}R_1\cos\Theta_0\sin\Theta_0, \quad R_1(k,\theta,0) = 1. \tag{4.28}$$

In an analogous way, looking at the terms without ρ in (4.20), we find that Θ_0 solves

$$\Theta_0' = -2\pi\sqrt{\nu}\Big\{\sin^2\Theta_0 + \frac{\tau_k^2 m(k^2U_k^2)}{4\pi^2}\cos^2\Theta_0\Big\}, \quad \Theta_0(k,\theta,0) = \theta. \tag{4.29}$$

Finally, using in equation (4.20) expansions (4.22), (4.23), and (4.24), and recalling that by Taylor formula

$$\begin{aligned} \sin^2\Theta &= \sin^2\Theta_0 + 2\rho^2\Theta_2\cos\Theta_0\sin\Theta_0 + o(\rho^2), \\ \cos^2\Theta &= \cos^2\Theta_0 - 2\rho^2\Theta_2\cos\Theta_0\sin\Theta_0 + o(\rho^2), \end{aligned}$$

looking at the terms in ρ^2, we find that Θ_2 solves

$$\begin{aligned}\Theta_2' = -2\pi\sqrt{\nu}\Big\{&2\Theta_2\cos\Theta_0\sin\Theta_0\big[1-\frac{\tau_k^2 m(k^2U_k^2)}{4\pi^2}\big] \\ &+\frac{\tau_k^2 m'(k^2U_k^2)}{4\pi^2}\cos^2\Theta_0\,\big[2k^2U_kW_2+R_1^2\cos^2\Theta_0\big]\Big\},\quad \Theta_2(k,\theta,0)=0.\end{aligned}\tag{4.30}$$

We do not write the equation for R_3 because we don't need it in the sequel.

Asymptotic behaviour of T. We prove that, as $\rho\to 0^+$,

$$T(k,\rho,\theta)=1+T_2(k,\theta)\rho^2+o(\rho^3),\tag{4.31}$$

where

$$T_2(k,\theta)=\frac{W_2'(k,\theta,1)}{\tau_k^2\,m(k^2)}.\tag{4.32}$$

It is natural to look for an expansion like (4.31) since for $\rho=0$, W is exactly the simple mode U_k. Replacing the expansions of W and T in the condition $W'(T)=0$ we obtain that

$$\begin{aligned}0&=W'(k,\rho,\theta,T(k,\rho,\theta))\\ &=U_k'(T(k,\rho,\theta))+\rho^2W_2'(k,\theta,T(k,\rho,\theta))+o(\rho^3)\\ &=U_k'(1)+\rho^2\left\{U_k''(1)T_2(k,\theta)+W_2'(k,\theta,1)\right\}+o(\rho^3).\end{aligned}\tag{4.33}$$

The first summand is zero. Moreover by equation (4.3)

$$U_k''(1)=U_k''(0)=-\tau_k^2 m\left(k^2U_k^2(0)\right)U_k(0)=-\tau_k^2 m(k^2).$$

Setting equal to zero the coefficient of ρ^2 in (4.33), and using the last equality, we get (4.32). It is easy to see that with this choice also condition $U(T)>0$ is satisfied for ρ small.

Asymptotic behaviour of the Poincaré map. Using the expansions in (4.22)-(4.23)-(4.31), we obtain that

$$\begin{aligned}\Theta(k,\rho,\theta,T(k,\rho,\theta))&=\Theta_0(k,\theta,T(k,\rho,\theta))+\Theta_2(k,\theta,T(k,\rho,\theta))\rho^2+o(\rho^3)\\ &=\Theta_0(k,\theta,1)+\left\{\Theta_0'(k,\theta,1)T_2(k,\theta)+\Theta_2(k,\theta,1)\right\}\rho^2+o(\rho^3),\end{aligned}$$

and similarly

$$R(k,\rho,\theta,T(k,\rho,\theta))=R_1(k,\theta,1)\rho+\left\{R_1'(k,\theta,1)T_2(k,\theta)+R_3(k,\theta,1)\right\}\rho^3+o(\rho^3).$$

Therefore, in polar coordinates the Poincaré map is

$$P_k\begin{pmatrix}\rho\\ \theta\end{pmatrix}=\begin{pmatrix}R(k,\rho,\theta,T(k,\rho,\theta))\\ \Theta(k,\rho,\theta,T(k,\rho,\theta))\end{pmatrix}=\begin{pmatrix}\alpha_1(k,\theta)\rho\\ \beta_0(k,\theta)\end{pmatrix}+\begin{pmatrix}\alpha_3(k,\theta)\rho^3\\ \beta_2(k,\theta)\rho^2\end{pmatrix}+o(\rho^3),$$

where

$$\begin{aligned}\alpha_1(k,\theta)&=R_1(k,\theta,1),\\ \alpha_3(k,\theta)&=R_1'(k,\theta,1)T_2(k,\theta)+R_3(k,\theta,1),\\ \beta_0(k,\theta)&=\Theta_0(k,\theta,1),\\ \beta_2(k,\theta)&=\Theta_0'(k,\theta,1)T_2(k,\theta)+\Theta_2(k,\theta,1).\end{aligned}\tag{4.34}$$

Let us now recall that if we choose coordinates where L_k is written in the canonical form of a rotation, then, in the corresponding polar coordinates, P_k becomes

$$P_k\begin{pmatrix}I\\ \sigma\end{pmatrix}=\begin{pmatrix}I+a(k,\sigma)I^3\\ \sigma-\omega_k+b(k,\sigma)I^2\end{pmatrix}+o(I^3).$$

The coordinate change from the cartesian coordinates (X,Y) to the original coordinates (x,y) is given by the diagonal matrix $D(k)$ introduced in Theorem 3.2. The expression of the corresponding change $D_*(k)$ from (I,σ) to (ρ,θ) is not so simple: it is given by

$$\rho = \alpha_*(k,\sigma)I, \quad \theta = \delta_*(k,\sigma), \tag{4.35}$$

where $\alpha_*(k,\sigma) = \left\{\cos^2\sigma + \delta^2(k)\sin^2\sigma\right\}^{1/2}$, and $\delta_*(k,\sigma) = \arctan(\delta(k)\tan\sigma)$ for $\sigma \in (-\pi/2, \pi/2)$, and similarly for all other values of σ. If we denote the inverse change by $D^*(k) := [D_*(k)]^{-1}$, then his components $\alpha^*(k,\theta)\rho$ and $\delta^*(k,\theta)$ are defined in analogy with α_*, δ_*, but with $\delta^{-1}(k)$ instead of $\delta(k)$.

Considering the second component of $P_k^* = D^*(k)P_kD_*(k)$ we have that, up to $o(I^3)$,

$$\sigma - \omega(k) + b(k,\sigma)I^2 = \delta^*\left[k, \beta_0(k,\delta_*(k,\sigma)) + \beta_2(k,\delta_*(k,\sigma))\cdot\alpha_*^2(k,\sigma)\,I^2\right],$$

so that, making the Taylor expansion of the right hand side and looking at the coefficients of I^2, it turns out that

$$b(k,\sigma) = \frac{\partial\delta^*}{\partial\theta}\left[k,\beta_0(k,\delta_*(k,\sigma))\right]\cdot\beta_2(k,\delta_*(k,\sigma))\cdot\alpha_*^2(k,\sigma).$$

4.4.2. *Behaviour for large k of coefficients in Taylor expansions.* **Behaviour of the coordinate change** By statement (5) of Theorem 3.2 we obtain:

$$\alpha_*(k,\sigma) \to (\cos^2\sigma + \delta^2\sin^2\sigma)^{1/2} = \mu_0(\sigma) \geq c > 0, \tag{4.36}$$

$$\delta_*(k,\sigma) \to \arctan(\delta\tan(\sigma)) = \mu_1(\sigma), \quad \sigma \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right), \tag{4.37}$$

$$\frac{\partial\delta_*}{\partial\sigma}(k,\sigma) \to \delta\frac{1+\tan^2\sigma}{1+\delta^2\tan^2\sigma} = \mu_2(\sigma) \geq c > 0, \tag{4.38}$$

as $k \to +\infty$, uniformly in σ. The same holds true for the inverse change $D^*(k)$.
Behaviour of W_2 We prove that

$$|W_2(k,\theta,t)| + |W_2(k,\theta,t)'| \leq c\,\frac{\log k}{k^2} \quad \forall\, t \in [0,1]. \tag{4.39}$$

Indeed let us set

$$E(k,t) = |W_2(k,\theta,t)|^2 + |W_2(k,\theta,t)'|^2.$$

Thanks to (4.26), since $m' \geq 0$, in a standard way we get

$$\begin{aligned} E'(k,t) \leq& E(k,t)[1 + \tau_k^2 m(k^2U_k(t)^2) + 2\tau_k^2 m'(k^2U_k(t)^2)k^2U_k(t)^2] \\ &+ \tau_k^2 m'(k^2U_k(t)^2)R_1^2(k,\theta,t)|U_k(t)|\sqrt{E(k,t)}. \end{aligned}$$

Now, using (U4), $m' \geq 0$, Lemma 4.4 and (4.1) we obtain:

$$\tau_k^2 m(k^2U_k(t)^2) \leq \tau_k^2 m(k^2) \leq c; \tag{4.40}$$

$$\tau_k^2 m'(k^2U_k(t)^2)k^2U_k(t)^2 \leq \tau_k^2 m(k^2)\frac{m'(k^2U_k(t)^2)k^2U_k(t)^2}{m(k^2U_k(t)^2)} \leq c; \tag{4.41}$$

$$\tau_k^2 m'(k^2U_k(t)^2)|U_k(t)| \leq c\frac{m'(k^2U_k(t)^2)}{m(k^2U_k(t)^2)}|U_k(t)| \leq c\,\frac{|U_k(t)|}{1+k^2U_k(t)^2}. \tag{4.42}$$

Moreover, by (4.28) we obtain that the function $R_1(k,\theta,t)$ is bounded in $[0,1]$. Therefore, integrating on $[0,t]$ the inequality

$$E'(k,s) \leq c\,E(k,s) + c\,\sqrt{E(k,s)}\frac{|U_k(s)|}{1+k^2U_k(s)^2}$$

and using Lemma 4.6 and (4.27) we get the desired estimate.
Behaviour of Θ_2 We prove that

$$\liminf_{k\to+\infty} -km(k^2)\int_0^{2\pi}\frac{\partial\delta^*}{\partial\theta}\,[k,\beta_0(k,\delta_*(k,\sigma))]\,\Theta_2(k,\delta_*(k,\sigma),1)\alpha_*^2(k,\sigma)\,d\sigma>0. \tag{4.43}$$

To this end we need the following lemmata.

Lemma 4.9. *The function* Θ_2 *satisfies* $km(k^2)|\Theta_2(k,\theta,t)|\le c$ *for all* $t\in[0,1]$, $k\ge\overline{k}$.

Proof. Integrating (4.30) we find

$$\begin{aligned}
\Theta_2(k,\theta,t) &= -2\pi\sqrt{\nu}\int_0^t \sin(2\Theta_0(k,\theta,s))\Big(1-\frac{\tau_k^2}{4\pi^2}m(k^2U_k^2(s))\Big)\Theta_2(k,\theta,s)\,ds\\
&\quad -2\pi\sqrt{\nu}\int_0^t 2k^2\frac{\tau_k^2}{4\pi^2}m'(k^2U_k^2(s))\cos^2(\Theta_0(k,\theta,s))U_k(s)W_2(k,\theta,s)\,ds\\
&\quad -2\pi\sqrt{\nu}\int_0^t \frac{\tau_k^2}{4\pi^2}m'(k^2U_k^2(s))\cos^4(\Theta_0(k,\theta,s))R_1(k,\theta,s)\,ds.
\end{aligned}$$

Hence by $m'\ge 0$, Lemma 4.4, (4.39), (4.42) and Lemma 4.6 we obtain

$$\begin{aligned}
|\Theta_2(k,\theta,t)| &\le c\int_0^t \Big|1+\frac{\tau_k^2}{4\pi^2}m(k^2)\Big||\Theta_2(k,\theta,s)|+\frac{m'(k^2U_k^2(s))}{m(k^2)}(|U_k(s)|\log k+1)\,ds\\
&\le c\int_0^t|\Theta_2(k,\theta,s)|\,ds+\frac{c}{km(k^2)}+c\log k\int_0^t\frac{|U_k(s)|}{1+k^2U_k^2(s)}\,ds\\
&\le c\int_0^t|\Theta_2(k,\theta,s)|\,ds+\frac{c}{km(k^2)}+\frac{c\log^2 k}{k^2}.
\end{aligned}$$

Hence thesis follows by Lemma 4.3, (M1), and Gronwall's Lemma. □

Lemma 4.10. *The next equality holds:*

$$\liminf_{k\to+\infty}-km(k^2)\Theta_2(k,\theta,1)=2\pi\sqrt{\nu}\liminf_{k\to+\infty}\int_0^1 km'(k^2U_k(s)^2)\cos^4(\theta-2\pi\sqrt{\nu}s)\,ds.$$

Proof. Integrating equation (4.30) we find:

$$\begin{aligned}
&-km(k^2)\Theta_2(k,\theta,1)\\
&=2\pi\sqrt{\nu}\int_0^1\sin(2\Theta_0(k,\theta,t))km(k^2)\Big(1-\frac{\tau_k^2}{4\pi^2}m(k^2U_k^2(t))\Big)\Theta_2(k,\theta,t)\,dt\\
&\quad+2\pi\sqrt{\nu}\int_0^1 2\frac{\tau_k^2}{4\pi^2}k^3m(k^2)\cos^2(\Theta_0(k,\theta,t))m'(k^2U_k(t)^2)U_k(t)W_2(k,\theta,t)\,dt\\
&\quad+2\pi\sqrt{\nu}\int_0^1\frac{\tau_k^2}{4\pi^2}m(k^2)kR_1^2(k,\theta,t)m'(k^2U_k(t)^2)\cos^4(\Theta_0(k,\theta,t))\,dt.
\end{aligned}$$

Now we estimate all terms in previous equality. Using Lemma 4.9 and Lemma 4.6 we obtain

$$\begin{aligned}
&\int_0^1\Big|\sin(2\Theta_0(k,\theta,t))km(k^2)\Big(1-\frac{\tau_k^2}{4\pi^2}m(k^2U_k^2(t))\Big)\Theta_2(k,\theta,t)\Big|\,dt\\
&\qquad\le\int_0^1\Big|1-\frac{\tau_k^2}{4\pi^2}m(k^2U_k^2(t))\Big|\,dt\le c\lambda_k.
\end{aligned} \tag{4.44}$$

Thanks to (4.39), (4.42), Lemma 4.6, Lemma 4.3 (property (M1)) we also obtain

$$\begin{aligned}\int_0^1 \Big|\frac{\tau_k^2}{4\pi^2}k^3 m(k^2)\cos^2(\Theta_0(k,\theta,t))m'(k^2U_k(t)^2)U_k(t)W_2(k,\theta,t)\Big|\,dt\\ \le c\int_0^1 k\log k|U_k(t)|m'(k^2U_k(t)^2)\,dt\\ \le c\int_0^1 k\log k\, m(k^2U_k^2(t))\frac{|U_k(t)|}{1+k^2U_k^2(t)}\,dt\\ \le c\frac{\log^2 k}{k}m(k^2)\le c\frac{1}{\sqrt{k}}.\end{aligned} \tag{4.45}$$

Let us remark that, thanks to Lemma 4.6 and (4.28)–(4.29), $R_1(k,\theta,t)=1+o(1)$, $\Theta_0(k,\theta,t)=\theta-2\pi\sqrt{\nu}t+o(1)$ (where $o(1)$ do not depends on $t\in[0,1]$), and, by Lemma 4.4, $\tau_k^2 m(k^2)=4\pi^2+o(1)$, hence

$$\begin{aligned}&\int_0^1 \frac{\tau_k^2}{4\pi^2}m(k^2)kR_1^2(k,\theta,t)m'(k^2U_k(t)^2)\cos^4(\Theta_0(k,\theta,t))\,dt\\ &=\int_0^1 (1+o(1))^3 km'(k^2U_k(t)^2)(\cos^4(\theta-2\pi\sqrt{\nu}t)+o(1))\,dt.\end{aligned}$$

Now we can use once more Lemma 4.6 and obtain

$$\lim_{k\to+\infty}\int_0^1 o(1)km'(k^2U_k(t)^2)[1+\cos^4(\theta-2\pi\sqrt{\nu}t)]\,dt=0. \tag{4.46}$$

□

We are now able to prove (4.43). Lemma 4.10 says that, for large k, $-\Theta_2(k,\theta,1)$ is a non negative function, hence, by (4.36) - (4.38), and a change of variable

$$-\int_0^{2\pi}\frac{\partial\delta^*}{\partial\theta}\,[k,\beta_0(k,\delta_*(k,\sigma))]\,\Theta_2(k,\delta_*(k,\sigma),1)\alpha_*^2(k,\sigma)\,d\sigma\ge c\int_0^{2\pi}\Theta_2(k,\theta,1)\,d\theta. \tag{4.47}$$

As in the proof of Lemma 4.10 (using (4.44), (4.45), (4.46)) we get

$$\begin{aligned}&\liminf_{k\to+\infty} -km(k^2)\int_0^{2\pi}\Theta_2(k,\theta,1)\,d\theta\\ &=\liminf_{k\to+\infty} 2\pi\sqrt{\nu}\int_0^1 km'(k^2U_k(s)^2)\int_0^{2\pi}\cos^4(\theta-2\pi\sqrt{\nu}s)\,d\theta\,ds\\ &=\frac{3}{2}\pi^2\sqrt{\nu}\liminf_{k\to+\infty}\int_0^1 km'(k^2U_k(s)^2)\,ds.\end{aligned}$$

Let t_0 be as in Lemma 4.5, (B2). Since thanks also to (U5) and Lemma 4.4 we find $0<c_1\le|U_k'(t)|<c_2$ for $t\in[t_0,1/4]$, it turns out that:

$$\int_0^1 km'(k^2U_k(s)^2)\,ds\ge\int_{t_0}^{1/4}km'(k^2U_k(s)^2)\frac{-U_k'(s)}{|U_k'(s)|}\,ds\ge c\int_0^{kU_k(t_0)}m'(y^2)\,dy,$$

and passing to the limit

$$\int_0^{kU_k(t_0)}m'(y^2)\,dy\to\int_0^{+\infty}m'(y^2)\,dy>0\quad\text{as}\quad k\to+\infty.$$

Hence, we have

$$\liminf_{k\to+\infty} -km(k^2)\int_0^{2\pi} \Theta_2(k,\theta,1)\,d\theta \geq c\liminf_{k\to+\infty}\int_{t_0}^{1/4} km'(k^2U_k(s)^2)\,d\theta > 0.$$

Thanks to (4.47) we then obtain (4.43).

Behaviour of $b(k,\sigma)$**.** We prove that, for k large,

$$\int_0^{2\pi} b(k,\sigma)\,d\sigma = \int_0^{2\pi} \frac{\partial\delta^*}{\partial\theta}\,[k,\beta_0(k,\delta_*(k,\sigma))]\cdot\beta_2(k,\delta_*(k,\sigma))\cdot\alpha_*^2(k,\sigma)\,d\sigma < 0. \tag{4.48}$$

Let us recall that in (4.34), $\beta_2(k,\theta) = \Theta_0'(k,\theta,1)T_2(k,\theta) + \Theta_2(k,\theta,1)$. Since α_*, $\frac{\partial\delta^*}{\partial\theta}$ and $\Theta_0'(k,\theta,t)$, are bounded functions, by (4.32) and (4.39), we obtain

$$\begin{aligned}&\int_0^{2\pi} \Big|\frac{\partial\delta^*}{\partial\theta}\,[k,\beta_0(k,\delta_*(k,\sigma))]\,\Theta_0'(k,\delta_*(k,\sigma),1)T_2(k,\delta_*(k,\sigma))\alpha_*^2(k,\sigma)\Big|\,d\sigma\\ &\leq c\int_0^{2\pi} |W_2'(k,\delta_*(k,\sigma),1)|\,d\sigma \leq c\frac{\log k}{k^2}\,.\end{aligned}$$

Since $\log k/k = o(1/m(k^2))$ to get the desired inequality it is now enough to remark that (4.43) says that for large k:

$$-\int_0^{2\pi} \frac{\partial\delta^*}{\partial\theta}\,[k,\beta_0(k,\delta_*(k,\sigma))]\,\Theta_2(k,\delta_*(k,\sigma),1)\alpha_*^2(k,\sigma)\,d\sigma \geq \frac{c}{km(k^2)}.$$

Proof of Theorem 3.3. To proof Theorem 3.3, we have to show that for large k, $\gamma_k < 0$, but this is exactly (4.48).

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