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DOUBLY NONLINEAR PARABOLIC EQUATIONS RELATED TO THE p-LAPLACIAN OPERATOR: SEMI-DISCRETIZATION

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Abstract. We study the doubly nonlinear parabolic equation

$$\frac{\partial \beta(u)}{\partial t} - \triangle_p u + f(x,t,u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

with Dirichlet boundary conditions and initial data. We investigate a time-discretization of the continuous problem by the Euler forward scheme. In addition to proving existence, uniqueness and stability questions, we study the long time behavior of the solution to the discrete problem. We prove the existence of a global attractor, and obtain its regularity under additional conditions.

1. Introduction

In this paper we study a doubly nonlinear parabolic partial differential equation related to the p-Laplacian operator estudied in [7]. We examine the validity of numerical solutions as approximations to solutions for long times. This work is inspired, on one hand by the results of El Hachimi and El Ouardi [7], and, on the other hand, by the work of Eden, Michaux and Rakotoson [4]. It is a generalization in different directions of several results.

The problem under consideration has the form

$$\begin{gathered} \frac{\partial \beta(u)}{\partial t} - \triangle_p u + f(x,t,u) = 0 \quad \text{in } \Omega \times]0,\infty[, \\ u = 0 \quad \text{on } \partial\Omega \times]0,\infty[, \\ \beta\big(u(.,0)\big) = \beta(u_0) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $1 < p < +\infty$, β is a nonlinearity of porous medium type, and f is a nonlinearity of reaction-diffusion type. The continuous problem (1.1) has been extensively treated in [7] for $p > 1$, and for the case $p = 2$ in [3]. Here, we shall discretize (1.1) and replace it by

$$\begin{gathered} \beta(U^n) - \tau \triangle_p U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) \quad \text{in } \Omega, \\ U^n = 0 \quad \text{on } \partial\Omega, \\ \beta(U^0) = \beta(u_0) \quad \text{in } \Omega. \end{gathered}$$

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The case $p = 2$ of this equation is studied in [4]. Here, we study the case $p > 1$ to obtain existence, uniqueness and stability results. Furthermore, we obtain existence of absorbing sets and of a global attractor. Under some conditions on f and p, additional regularity result for the global attractor and, as a consequence, a stabilization result are obtained when $\beta(u) = u$.

This paper is organized as follows: In section 2, we give some preliminaries. In section 3, we show the existence and uniqueness of solutions of problem (2.1). The question of stability is studied in section 4, while the semi-discrete dynamical system study is done in section 5. finally, section 6 is dedicated to obtaining some regularity for the attractor.

2. Preliminaries

2.1. **Notations and useful lemmas.** Let β be a continuous function with $\beta(0) = 0$. For $t \in \mathbb{R}$, define

$$\psi(t) = \int_0^t \beta(s)ds.$$

The Legendre transform is defined as $\psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \psi(s)\}$. Let Ω stand for a regular open bounded set of $\mathbb{R}^d$, $d \geq 1$ and $\partial\Omega$ be it's boundary.

The norm in a space X will be denoted by

- $\|.\|_r$ if $X = L^r(\Omega)$, $1 \leq r \leq +\infty$;
- $\|.\|_{1,q}$ if $X = W^{1,q}(\Omega)$, $1 \leq q \leq +\infty$;
- $\|.\|_X$ otherwise

and $\langle .,. \rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

For $p \geq 1$ we define it's conjugate p' by $\frac{1}{p} + \frac{1}{p'} = 1$. In this paper, C_i and C will denote various positive constants. We shall use the following results.

Lemma 2.1 ([11]). *If $u \in W_0^{1,p}(\Omega)$ is a solution to the equation*

$$-\tau\Delta_p u + F(x,u) = T,$$

where $T \in W^{-1,r}(\Omega)$ and F satisfies $\xi F(x,\xi) \geq 0$ in $\Omega \times \mathbb{R}$, then we have the following estimates

(a) *If $r > \frac{d}{p-1}$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p}$.*

(b) *If $p' \leq r < \frac{d}{p-1}$, then $u \in L^{r^*}(\Omega)$ and $\|u\|_{r^*} \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p}$, where $\frac{1}{r^*} = \frac{1}{(p-1)r} - \frac{1}{d}$.*

(c) *If $r = \frac{d}{p-1}$ and $r \geq p'$ then $u \in L^q(\Omega)$ for any q, $1 \leq q < \infty$ and $\|u\|_q \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p}$.*

Lemma 2.2. *Let $g(x,s,\xi)$ be a Caratheodory function such that $\operatorname{sign}\xi\, g(x,s,\xi) \geq -C_1$ and $|g(x,s,\xi)| \leq b(|s|)(|\xi|^p + c(x))$, where b is a continuous and increasing function with (finite) values on $\mathbb{R}^+$, $c \in L^1(\Omega)$, $c \geq 0$ and C_1 is a nonnegative real. Also let $h \in W^{-1,p'}(\Omega)$. Then the problem*

$$-\Delta_p u + g(x,u,\nabla u) = h \quad \text{in } \mathcal{D}'(\Omega),$$
$$u \in W_0^{1,p}(\Omega),$$

has at least one solution.

Remark 2.3. Since in [1], $C_1 = 0$, a slight modification has to be introduced in the proof therein. Indeed, we consider $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$-\Delta_p u_\varepsilon + g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = h,$$

where $g_\varepsilon = g/(1 + \varepsilon|g|)$. Thanks to the *sign condition*, it is easy to obtain a $W_0^{1,p}(\Omega)$-estimate on u_ε. By extracting a subsequence, u_ε tends to u in $W_0^{1,p}(\Omega)$ weak. The problem will be solved whenever the convergence is proved to be strong in $W_0^{1,p}(\Omega)$, and this follows the same lines as in [1] provided we replace h by $h+C_1$.

2.2. **Assumptions and definition of solution.** For (1.1), we consider the Euler forward scheme

$$\begin{gathered}\beta(U^n) - \tau\triangle_p U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) \quad \text{in } \Omega, \\ U^n = 0 \quad \text{on } \partial\Omega, \\ \beta(U^0) = \beta(u_0) \quad \text{in } \Omega,\end{gathered} \tag{2.1}$$

where $N\tau = T$, T a fixed positive real, and $1 \le n \le N$. We shall be concerned with one of the following two cases:

case 1 $u_0 \in L^\infty(\Omega)$, and we assume the following hypotheses:

- (H1) The function β is an increasing and continuous from $\mathbb{R}$ to $\mathbb{R}$, and $\beta(0) = 0$.
- (H2) For $\xi \in \mathbb{R}$, the map $(x,t) \mapsto f(x,t,\xi)$ is measurable and, a.e. in $\Omega \times \mathbb{R}^+$, $\xi \mapsto f(x,t,\xi)$ is continuous. Furthermore we assume that there exists $C_1 > 0$, such that for a.e. $(x,t) \in \Omega \times \mathbb{R}^+$ $\operatorname{sign}\xi f(x,t,\xi) \ge -C_1$.
- (H3) There is $C_2 > 0$, such that for almost $(x,t) \in \Omega\times\mathbb{R}^+$, $\xi \mapsto f(x,t,\xi)+C_2\beta(\xi)$ is increasing.

Case 2 $u_0 \in L^2(\Omega)$, and we assume the following hypotheses:

- (H1') The function β is increasing and continuous from $\mathbb{R}$ to $\mathbb{R}$, $\beta(0) = 0$, and for some $C_3 > 0$, $C_4 > 0$, $\beta(\xi) \le C_3|\xi| + C_4$ for all $\xi \in \mathbb{R}$.
- (H2') For any ξ in $\mathbb{R}$, the map $(x,t) \mapsto f(x,t,\xi)$ is measurable and, a.e, in $\Omega\times\mathbb{R}^+$, $\xi \mapsto f(x,t,\xi)$ is continuous. Furthermore we assume that there exist $q > sup(2,p)$ and positives constants C_5, C_6 and C_7 such that

  $$\operatorname{sign}\xi f(x,t,\xi) \ge C_5|\xi|^{q-1} - C_6.$$

  Also assume that $|f(x,t,\xi)| \le a(|\xi|)$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and

  $$\limsup_{t\to 0^+} |f(x,t,\xi)| \le C_7(|\xi|^{q-1} + 1).$$
- (H3') There is $C_2 > 0$ such that for almost all $(x,t) \in \Omega \times \mathbb{R}^+$, $\xi \mapsto f(x,t,\xi) + C_2\beta(\xi)$ is increasing.

Remark 2.4. In the hypothesis (H2'), the monotonicity condition on a is not restrictive since we can replace a by the increasing function $\widetilde{a}(s) = \sup_{0\le t\le s} a(t)$.

Definition 2.5. By a weak solution to the discretized problem, we mean a sequence $(U^n)_{0\le n\le N}$ such that $\beta(U^0) = \beta(u_0)$, and U^n is defined by induction as a weak solution of the problem

$$\begin{gathered}\beta(U) - \tau\triangle_p U + \tau f(x, n\tau, U) = \beta(U^{n-1}) \quad \text{in } \Omega, \\ U \in W_0^{1,p}(\Omega).\end{gathered}$$

3. Existence and uniqueness result

Case 1: $u_0 \in L^\infty(\Omega)$. Assume (H1)–(H3), we derive an a priori estimate.

Lemma 3.1. *The function U^n is in $L^\infty(\Omega)$ for $n = 0, \dots, N$.*

Proof. In this case $U^0 \in L^\infty(\Omega)$. To show that $U^1 \in L^\infty(\Omega)$, we can write (2.1) as

$$-\tau \triangle_p U^1 + F_1(x, U^1) = \beta(u_0) + C_1 \operatorname{sign}(U^1) = \varphi_1$$
$$U^1 \in W_0^{1,p}(\Omega),$$

where $F_1(x,\xi) = \tau f(x,\tau,\xi) + \beta(\xi) + C_1 \operatorname{sign}(\xi)$, and $\varphi_1 \in L^\infty(\Omega)$. According to (H1) and (H2), $\xi F_1(x,\xi) \geq 0$ for all $\xi \in \mathbb{R}$. By lemma 2.1 we can conclude that $U^1 \in L^\infty(\Omega)$. By a simple induction, we deduce that $U^n \in L^\infty(\Omega)$ for all $n = 0, \dots, N$. □

Theorem 3.2. *For $n = 1, \dots, N$, there exists a unique solution U^n of (2.1) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ provided that $0 < \tau < \frac{1}{C_2}$.*

Proof. We can write (2.1) as

$$-\tau \triangle_p U + F(x, U) = h,$$
$$U \in W_0^{1,p}(\Omega),$$

where $U = U^n$, $h = \beta(U^{n-1})$ and $F(x,\xi) = \tau f(x, n\tau, \xi) + \beta(\xi)$. According to (H1) and (H2),

$$\operatorname{sign} \xi \, F(x,\xi) \geq -\tau C_1 \quad \text{and} \quad h \in W^{-1,p'}(\Omega).$$

Hence the existence follows from lemma 2.2.

Next, we obtain uniqueness. For simplicity, we set

$$w = U^n, \quad \overline{f}(x,w) = f(x, n\tau, U^n), \quad \text{and} \quad g(x) = \beta(U^{n-1})$$

Then problem (2.1) reads

$$\begin{gathered} -\tau \triangle_p w + \tau \overline{f}(x,w) + \beta(w) = g(x), \\ w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{gathered} \tag{3.1}$$

If w_1 and w_2 are two solutions of (3.1), then

$$-\tau \Delta_p w_1 + \tau \Delta_p w_2 + \tau(\overline{f}(x,w_1) - \overline{f}(x,w_2)) + \beta(w_1) - \beta(w_2) = 0. \tag{3.2}$$

Multiplying (3.2) by $w_1 - w_2$ and integrating over Ω, gives

$$\begin{aligned} \langle -\tau \Delta_p w_1 + \tau \Delta_p w_2, w_1 - w_2 \rangle + \tau \int_\Omega \big(\overline{f}(x,w_1) - \overline{f}(x,w_2)\big)(w_1 - w_2) dx \\ + \int_\Omega \big(\beta(w_1) - \beta(w_2)\big)(w_1 - w_2) dx = 0. \end{aligned} \tag{3.3}$$

Applying (H3) yields

$$\int_\Omega \big(\overline{f}(x,w_1) - \overline{f}(x,w_2)\big)(w_1 - w_2) dx \geq -C_2 \int_\Omega \big(\beta(w_1) - \beta(w_2)\big)(w_1 - w_2) dx. \tag{3.4}$$

Using this equation and the monotonicity condition of the p-Laplacian operator, (3.3) reduces to

$$(1 - \tau C_2) \int_\Omega \big(\beta(w_1) - \beta(w_2)\big)(w_1 - w_2) \, dx \leq 0.$$

Then by (H1), if $\tau < 1/C_2$, we get $w_1 = w_2$. □

Uniqueness can be also obtained under the following assumption:

(H3") For all $M > 0$, there exists $C_M > 0$ such that, if $|\xi| + |\xi'| \le M$ then
$$|f(t,x,\xi) - f(t,x,\xi')|^\alpha \le C_M\big(\beta(\xi) - \beta(\xi')\big)(\xi - \xi')$$
where $\alpha = \begin{cases} 2 & \text{if } 1 < p \le 2, \\ p' & \text{if } p \ge 2. \end{cases}$

Proposition 3.3. *Assume in the case 1 that (H1), (H2) and (H3") hold, and that $p \ge 2d/(d+2)$. Then the solution of* (2.1) *is unique provided that $0 < \tau < \tau_1$, where τ_1 is a prescribed constant.*

Proof. Let w_1 and w_2 be two solutions of (3.1). Using the stability result which we will establish below (see theorem 4.1), we have
$$\|w_1\|_\infty + \|w_2\|_\infty \le M \quad \text{and} \quad \tau^{1/p}(\|w_1\|_{1,p} + \|w_2\|_{1,p}) \le K, \tag{3.5}$$
where M and K are positive constants which do not depend on N. Now, let us recall the relations verified by the p-Laplacian (see [8] or [12] for example). For every u and v in $W_0^{1,p}(\Omega)$, we have
$$\langle -\Delta_p u + \Delta_p v, u - v\rangle \ge C_p\|u - v\|_{1,p}^p \quad \text{if } p \ge 2 \tag{3.6}$$
$$\langle -\Delta_p u + \Delta_p v, u - v\rangle \ge C_p\frac{\|u - v\|_{1,p}^2}{(\|u\|_{1,p} + \|v\|_{1,p})^{2-p}} \quad \text{if } 1 < p \le 2 \tag{3.7}$$

(i) If $p \ge 2$, then from (3.3), (3.6), (H3"), Young's and Poincare's inequalities, we get
$$\begin{aligned}
&\lambda_1 C_p\tau\|w_1 - w_2\|_p^p + \int_\Omega \left(\beta(w_1) - \beta(w_2)\right)(w_1 - w_2)dx \\
&\le \frac{1}{p'C_M}\|\overline{f}(x,w_1) - \overline{f}(x,w_2)\|_{p'}^{p'} + \frac{\tau^p C_M^{p/p'}}{p}\|w_1 - w_2\|_p^p \\
&\le \frac{1}{p'}\int_\Omega \left(\beta(w_1) - \beta(w_2)\right)(w_1 - w_2)dx + \frac{\tau^p C_M^{p/p'}}{p}\|w_1 - w_2\|_p^p,
\end{aligned}$$
where λ_1 is the first eigenvalue of $-\Delta_p$. Then, from (H1), we obtain
$$\Big(\lambda_1 C_p\tau - \frac{\tau^p C_M^{p/p'}}{p}\Big)\|w_1 - w_2\|_p^p \le 0.$$
Therefore, when $0 < \tau < \big(\frac{p\lambda_1 C_p}{C_M^{p/p'}}\big)^{1/(p-1)}$, we get $w_1 = w_2$.

(ii) If $\frac{2d}{d+2} \le p \le 2$, then from (3.3), (3.7), (H3") and Young's inequality, we obtain
$$\begin{aligned}
&\tau^{2/p}\frac{C_p}{K^{2-p}}\|w_1 - w_2\|_{1,p}^2 + \int_\Omega \left(\beta(w_1) - \beta(w_2)\right)(w_1 - w_2)dx \\
&\le \frac{1}{2C_M}\|\overline{f}(x,w_1) - \overline{f}(x,w_2)\|_2^2 + \frac{\tau^2 C_M}{2}\|w_1 - w_2\|_2^2 \\
&\le \frac{1}{2}\int_\Omega \left(\beta(w_1) - \beta(w_2)\right)(w_1 - w_2)dx + \frac{\tau^2 C_M}{2}\|w_1 - w_2\|_2^2.
\end{aligned}$$

Since $p \geq \frac{2d}{d+2}$, we have $\|w_1 - w_2\|_2 \leq C'_p \|w_1 - w_2\|_{1,p}$. Then, from (H1), we obtain

$$\tau^{2/p}\left(\frac{C_p}{K^{2-p}} - \frac{1}{2}\tau^{2/p'} C_M {C'_p}^2\right)\|w_1 - w_2\|_{1,p}^2 \leq 0.$$

Therefore, when $0 < \tau < (\frac{2C_p}{C_M {C'_p}^2 K^{2-p}})^{p'/2}$, we obtain $w_1 = w_2$. □

Case 2. The function u_0 is in $L^2(\Omega)$.

Theorem 3.4. *We assume the hypotheses (H1')–(H3') and* $p \geq \frac{2d}{d+2}$*, then for each* $n = 1, \cdots, N$ *there exists a unique solution* U^n *of* (2.1) *in* $W_0^{1,p}(\Omega)$ *provided that* $0 < \tau < 1/C_2$.

The proofs of existence and uniqueness are the same as those of Theorem 3.2, with $h = \beta(U^{n-1})$ in $L^2(\Omega) \subset W^{-1,p'}(\Omega)$ for $p \geq 2d/(d+2)$. Therefore, we omit it.

4. Stability

Case 1. The function u_0 in $L^\infty(\Omega)$.

Theorem 4.1. *Assume (H1)–(H3). Then there exists* $C(T, u_0) > 0$ *depending on* T*,* u_0*,* β *,* g *and* Ω*, but not on* N*, such that for all* $n = 1, \cdots, N$*,*

$$\|U^n\|_\infty \leq C(T, u_0), \tag{4.1}$$

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p \leq C(T, u_0), \tag{4.2}$$

$$\sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C(T, u_0). \tag{4.3}$$

Proof. **(a)** From lemma 3.1, $U^n \in L^\infty(\Omega)$. Then, multiplying the first equation of (2.1) by $|\beta(U^n)|^k \beta(U^n)$, using Hölder's inequality and the hypotheses on f, we obtain

$$\|\beta(U^n)\|_{k+2}^{k+2} \leq \|\beta(U^n)\|_{k+2}^{k+1} \|\beta(U^{n-1})\|_{k+2} + C\tau \|\beta(U^n)\|_{k+1}^{k+1}.$$

Since $\|\beta(U^n)\|_{k+1} \leq C\|\beta(U^n)\|_{k+2}$, it follows that

$$|\beta(U^n)\|_{k+2} \leq \|\beta(U^{n-1})\|_{k+2} + C\tau,$$

and, by induction, we deduce that

$$\|\beta(U^n)\|_{k+2} \leq \|\beta(u_0)\|_{k+2} + NC\tau.$$

Finally, as $k \to \infty$, we obtain $\|U^n\|_\infty \leq C(T, u_0)$. Thus (4.1) is satisfied.
(b) Multiplying the first equation of (2.1) (with k instead of n) by U^k, and using (H2) and the relation

$$\int_\Omega \psi^*(\beta(U^k))dx - \int_\Omega \psi^*(\beta(U^{k-1}))dx \leq \int_\Omega \left(\beta(U^k) - \beta(U^{k-1})\right) U^k dx,$$

we obtain

$$\int_\Omega \psi^*(\beta(U^k))dx - \int_\Omega \psi^*(\beta(U^{k-1}))dx + \tau\|U^k\|_{1,p}^p \leq C_1\tau\|U^k\|_1. \tag{4.4}$$

Now, summing (4.4) from $k = 1$ to n, gives

$$\int_\Omega \psi^*(\beta(U^n))\,dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p \le C\tau \sum_{k=1}^n \|U^k\|_1 + \int_\Omega \psi^*(\beta(u_0))dx. \tag{4.5}$$

¿From (4.5) and Lemma 3.1, we deduce (4.2).

(c) Multiplying the first equation of (2.1) (with k instead of n) by $\beta(U^k)$ and using (H2), we have

$$\int_\Omega \big(\beta(U^k) - \beta(U^{k-1})\big)\beta(U^k)dx + \tau\langle -\Delta_p U^k, \beta(U^k)\rangle \le C_1\tau \int_\Omega |\beta(U^k)|dx. \tag{4.6}$$

With the aid of the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$, from (4.6) we obtain

$$\|\beta(U^k)\|_2^2 - \|\beta(U^{k-1})\|_2^2 + \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \le C\tau\|\beta(U^k)\|_1. \tag{4.7}$$

Now summing (4.7) from $k = 1$ to n, yields

$$\|\beta(U^n)\|_2^2 + \sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \le \|\beta(u_0)\|_2^2 + C\tau \sum_{k=1}^n \|\beta(U^k)\|_1. \tag{4.8}$$

Hence, by (4.8) and Lemma 3.1, we conclude (4.3). □

Case 2: The function u_0 is in $L^2(\Omega)$.

Theorem 4.2. *We assume hypotheses (H1')–(H3') and* $p \ge 2d/(d+2)$*. Then there exists a positive constant* $C(T, u_0)$ *such that, for all* $n = 1, \cdots, N$*,*

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \le C(T, u_0) \tag{4.9}$$

$$\max_{1\le k\le n} \|\beta(U^k)\|_2^2 + \sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \le C(T, u_0). \tag{4.10}$$

Proof. Since the proof is nearly the same as that of theorem 4.1, we just sketch it.

(a) As for (4.5) , we obtain

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \le C\tau \sum_{k=1}^n \|U^k\|_1 + \int_\Omega \psi^*(\beta(u_0))dx.$$

Thanks to Young's inequality, for all $\varepsilon > 0$ there exists $C_\varepsilon(T, u_0)$ such that

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \le \varepsilon\tau \sum_{k=1}^n \|U^k\|_p^p + C_\varepsilon(T, u_0).$$

Now for a suitable choice of ε, we have

$$\varepsilon\tau \sum_{k=1}^n \|U^k\|_p^p \le C_\varepsilon(T, u_0).$$

Therefore, (4.9) is satisfied.

(b) From (4.8), (H1') and (H2'), we obtain

$$\|\beta(U^n)\|_2^2 + \sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \le \|\beta(u_0)\|_2^2 + C\tau \sum_{k=1}^n \|\beta(U^k)\|_1$$

As in (a), we conclude (4.10). □

5. The semi-discrete dynamical system

In the remainder sections, we assume $u_0 \in L^2(\Omega)$ and the hypotheses (H1')–(H3'), we fix τ such that $0 < \tau < \min(1, \frac{1}{C_2})$, and assume that $p > 2d/(d+2)$. Theorem 3.4 allows us to define a map S_τ on $L^2(\Omega)$ by setting

$$S_\tau U^{n-1} = U^n.$$

Since S_τ is continuous, we have $S_\tau^n U^0 = U^n$.

Our aim is to study the discrete dynamical system associated with (2.1). We begin by showing the existence of absorbing balls in $L^\infty(\Omega)$. (We refer to [13] for the definition of absorbing sets and global attractor).

5.1. Absorbing sets in $L^\infty(\Omega)$.

Lemma 5.1. *If $p > 2d/(d+1)$, then there exists $n(d,p) \in \mathbb{N}^*$ depending on d and p, and $C > 0$ depending on d, Ω and the constants in (H1')–(H3') such that*

$$U^n \in L^\infty(\Omega) \quad \textit{for all } n \geq n(d,p), \tag{5.1}$$

$$\|U^{n(d,p)}\|_\infty \leq \frac{C}{\tau^{\alpha+\alpha^2+\cdots+\alpha^{n(d,p)}}}\big(\|u_0\|_2^{\alpha^{n(d,p)}} + 1\big), \tag{5.2}$$

where $\alpha = p'/p$. Moreover, if $d = 1, d = 2$ or $d < 2p$ then $n(d,p) = 1$.

Proof. The proof follows from a repeated application of lemma 2.1. We can write (2.1) as

$$-\tau\triangle_p(U^m) + F_m(x, U^m) = \beta(U^{m-1}) + C_6 sign(U^m) = T_m \quad \text{in } \Omega,$$
$$U^m = 0 \quad \text{on } \partial\Omega,$$

where $F_m(x,\xi) = \tau f(x, m\tau, \xi) + \beta(\xi) + C_6 sign(\xi)$. Note that by (H1') and (H2') we have $\xi F_m(x,\xi) \geq 0$ for all ξ and $T_m \in W^{-1,p'}(\Omega)$.

Now, applying lemma 2.1, we can find an increasing sequence $\big(\alpha(m)\big)_{m\geq 1}$ such that

$$\alpha(m) \geq p', \quad \frac{1}{\alpha(m+1)} = \frac{1}{(p-1)\alpha(m)} - \frac{1}{d}, \tag{5.3}$$

$$\|U^m\|_{\alpha(m)} \leq \frac{C_m}{\tau^{\alpha+\alpha^2+\cdots+\alpha^m}}\big(\|u_0\|_2^{\alpha^m} + 1\big). \tag{5.4}$$

We shall stop the iteration on m once we have $\alpha(m-1) > d/p$. Indeed, if $q > d/p$, then there exists $r > d/(p-1)$ such that $L^q(\Omega) \subset W^{-1,r}(\Omega)$. Then we have $T_m \in W^{-1,r}(\Omega)$ and thus $U^m \in L^\infty(\Omega)$. $n(d,p)$ will be the first integer m such that $\alpha(m-1) > d/p$. Then (5.2) follows from (5.4) and lemma 2.1. □

Remark 5.2. (i) If $d = 1$ or $d = 2$, then for all $q > 1$, we have $L^2(\Omega) \subset W^{-1,q}(\Omega)$, in particular for $q > \frac{d}{p-1}$. If $d \geq 3$ and $d < 2p$, we can choose $q > 1$ to be such that $\frac{d}{p-1} < q < \frac{2d}{d-2}$. In the two cases, $T_1 \in W^{-1,q}(\Omega)$ for some $q > \frac{d}{p-1}$ and, from lemma 2.1, $U^1 \in L^\infty(\Omega)$. We have then $n(d,p) = 1$.
(ii) If $\alpha(m) \leq \frac{d}{p}$ for all m, then $l = \lim_{m\to\infty} \alpha(m)$ exists and equals $\frac{2-p}{p-1}d$. Consequently, for $p > 2d/(d+1)$, we have $l < p'$, which contradicts the fact that $\alpha(m) \geq p'$. Hence, the existence of $n(d,p)$ is justified.

In the remaining of this article, we set $n_0 = n(d,p)$ and $C_1 = C\big(\|u_0\|_2^{\alpha^{n_0}} + 1\big)$.

Lemma 5.3. *Let k be such that $1 < k < q-1$ and $k \leq 1 + \frac{1}{n_0}$. Then, there exist $\gamma > 0, \delta > 0$ depending on the data of (H1')–(H3') and $\mu > 0$ depending on $n_0, q, \gamma, \delta, k$ such that, for all $n \geq n_0$, we have*

$$\|\beta(U^n)\|_\infty \leq \big(\frac{\delta}{\gamma}\big)^{1/(q-1)} + \frac{C_1 + \mu}{\big(\tau^\beta(n-n_0+1)\big)^{1/(k-1)}},$$

where $\beta = \begin{cases} 1 & \text{if } \alpha \leq 1, \\ \alpha^{n_0} & \text{if } \alpha \geq 1. \end{cases}$

Proof. From lemma 5.1, for $n \geq n_0$, we have

$$U^n \in L^\infty(\Omega) \text{ and } \|U^{n_0}\|_\infty \leq C_1/\tau^{\alpha+\alpha^2+\dots+\alpha^{n_0}}.$$

Multiplying the first equation of (2.1) by $|\beta(U^n)|^m\beta(U^n)$ for some positive integer m, we derive from (H1') and (H2'), after dropping some positive terms, that

$$\|\beta(U^n)\|_{m+2}^{m+2} \leq \int_\Omega |\beta(U^n)|^{m+1}\beta(U^{n-1})dx + C\tau\ |\beta(U^n)\|_{m+1}^{m+1} - C\tau\|\beta(U^n)\|_{m+q}^{m+q}.$$

By setting

$$y_m^n = \|\beta(U^n)\|_{m+2} \text{ and } z_n = \|\beta(U^n)\|_\infty,$$

and using Hölder's inequality, we deduce the existence of two constants $\gamma > 0, \delta > 0$ (not depending on m nor on U^n) such that

$$y_m^n + \gamma\tau(y_m^n)^{q-1} \leq \delta\tau + y_m^{n-1}.$$

As m approaches infinity, we then obtain

$$z_n + \gamma\tau z_n^{q-1} \leq \delta\tau + z_{n-1},$$

with $z_{n_0} \leq C_1/\tau^{\alpha+\alpha^2+\dots+\alpha^{n_0}}$.
(i) If $\alpha \leq 1$, then $\alpha + \alpha^2 + \dots + \alpha^{n_0} \leq n_0$. So, we have

$$z_{n_0} \leq C_1/\tau^{n_0},$$
$$z_n + \gamma\tau z_n^{q-1} \leq \delta\tau + z_{n-1}.$$

Then we can apply [4, Lemma 7.1] to obtain

$$z_n \leq \big(\frac{\delta}{\gamma}\big)^{1/(q-1)} + \frac{C_1+\mu}{\big(\tau(n-n_0+1)\big)^{\frac{1}{k-1}}} \equiv c_\alpha(n).$$

(ii) If $\alpha \geq 1$, then $\alpha + \alpha^2 + \dots + \alpha^{n_0} \leq n_0\alpha^{n_0}$. By setting $\tau_1 = \tau^{\alpha^{n_0}}$, we have

$$z_{n_0} \leq C_1/\tau_1^{n_0},$$
$$z_n + \gamma'\tau_1 z_n^{q-1} \leq \delta'\tau_1 + z_{n-1},$$

where $\gamma' = \tau^{1-\alpha^{n_0}}\gamma$ and $\delta' = \tau^{1-\alpha^{n_0}}\delta$. Then, once again, we can apply [4, lemma 7.1] to obtain

$$z_n \leq \big(\frac{\delta}{\gamma}\big)^{1/(q-1)} + \frac{C_1+\mu}{\big(\tau_1(n-n_0+1)\big)^{\frac{1}{k-1}}} \equiv c_\alpha(n).$$

□

Remark 5.4. In the case $\alpha \geq 1$, a slight modification has to be introduced in the proof of [4, lemma 7.1], since μ depends on δ' and γ' and hence on τ. In fact, with the same notation, it suffices to choose μ such that

$$\gamma\left(\frac{\delta}{\gamma}\right)^{1-\frac{k}{q-1}}\mu^{k-1} \geq 2^{\frac{1}{k-1}}/(k-1).$$

and to remark that $\gamma' \geq \gamma$.

Consequently, lemma 5.3 implies that there exist absorbing sets in $L^q(\Omega)$ for all $q \in [1,\infty]$. Indeed, this is due to the fact that

$$\|U^n\|_\infty \leq \max\big(\beta^{-1}(c_\alpha(n)), |\beta^{-1}(-c_\alpha(n))|\big),$$

for all $n \geq n_0$, with $c_\alpha(n) \to \big(\frac{\delta}{\gamma}\big)^{1/(q-1)}$ as $n \to \infty$.

5.2. **Absorbing sets in $W_0^{1,p}(\Omega)$, existence of the the global attractor.** Multiplying equation (2.1) by $\delta_n = U^n - U^{n-1}$, we obtain

$$\begin{aligned}\langle\frac{\beta(U^n)-\beta(U^{n-1})}{\tau},\delta_n\rangle + \int_\Omega |\nabla U^n|^{p-2}\nabla U^n.(\nabla U^n - \nabla U^{n-1})dx\\ + \langle f(x,n\tau,U^n),\delta_n\rangle = 0. \end{aligned}\quad (5.5)$$

By setting

$$F_\beta(u) = \int_0^u \big(f(x,n\tau,w) + C_2\beta(w)\big)dw,$$

we deduce from (H3') that $F'_\beta(u)(u-v) \geq F_\beta(u) - F_\beta(v)$, and then

$$\begin{aligned}\langle f(x,n\tau,U^n),\delta_n\rangle &= \langle f(x,n\tau,U^n) + C_2\beta(U^n),\delta_n\rangle - C_2\langle\beta(U^n),\delta_n\rangle\\ &\geq \int_\Omega \big(F_\beta(U^n) - F_\beta(U^{n-1})\big)\,dx - C_2\langle\beta(U^n),\delta_n\rangle.\end{aligned}$$

Now, using (H1'), we get $\psi'(v)(u-v) \leq \psi(u) - \psi(v)$. Therefore,

$$\begin{aligned}&\int_\Omega \beta(U^n)(U^n - U^{n-1})dx\\ &= \int_\Omega \big(\beta(U^n) - \beta(U^{n-1})\big)(U^n - U^{n-1})dx + \int_\Omega \beta(U^{n-1})(U^n - U^{n-1})dx\\ &\leq \int_\Omega \big(\beta(U^n) - \beta(U^{n-1})\big)(U^n - U^{n-1})dx + \int_\Omega \big(\psi(U^n) - \psi(U^{n-1})\big)\,dx.\end{aligned}$$

With the aid of the inequality

$$|a|^{p-2}a.(a-b) \geq \frac{1}{p}|a|^p - \frac{1}{p}|b|^p, \quad (5.6)$$

we obtain

$$\int_\Omega |\nabla U^n|^{p-2}\nabla U^n.(\nabla U^n - \nabla U^{n-1})dx \geq \frac{1}{p}\|U^n\|_{1,p}^p - \frac{1}{p}\|U^{n-1}\|_{1,p}^p. \quad (5.7)$$

Since $\tau < 1/C_2$, from (5.5) we obtain

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F_\beta(U^n)dx \leq C_2\int_\Omega \big(\psi(U^n) - \psi(U^{n-1})\big)\,dx + \int_\Omega F_\beta(U^{n-1})dx. \quad (5.8)$$

Now, setting $F(u) = \int_0^u f(x, n\tau, w)dw$ yields

$$\int_\Omega F_\beta(u)dx = \int_\Omega F(u)dx + C_2 \int_\Omega \psi(u)dx.$$

Hence, from (5.8), we get

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx \le \frac{1}{p}\|U^{n-1}\|_{1,p}^p + \int_\Omega F(U^{n-1})dx.$$

By setting

$$y_n = \frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx,$$

we get $y_n \le y_{n-1}$. And by choosing $N\tau = 1$, using the boundedness of U^n and the stability analysis, there exists $n_\tau > 0$ such that

$$\tau \sum_{n=n_0}^{n_0+N} y^n \le a_1, \quad \text{for all } n \ge n_\tau.$$

Then we apply the discrete version of *the uniform Gronwall lemma* [4, Lemma 7.5] with $h_n = 0$ to obtain

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx \le C \quad \text{for all } n \ge n_\tau.$$

On the other hand, since U^n is bounded, we deduce that $\|U^n\|_{1,p} \le C$. We have then proved the following result.

Proposition 5.5. *If $\tau < 1/C_2$, there exist absorbing sets in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. More precisely, for any $u_0 \in L^2(\Omega)$, there exists a positive integer n_τ such that*

$$\|U^n\|_\infty + \|U^n\|_{1,p} \le C, \quad \forall n \ge n_\tau, \tag{5.9}$$

where C does not depend on τ.

For the nonlinear map S_τ to satisfy the properties of the semi-group, namely $S_\tau^{n+p} = S_\tau^n o S_\tau^p$, we need (2.1) to be autonomous. So, we assume that $f(x,t,\xi) \equiv f(x,\xi)$. Thus, S_τ defines a semi-group from $L^2(\Omega)$ into itself and possesses an absorbing ball B in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Setting

$$\mathcal{A}_\tau = \bigcap_{n\in\mathbb{N}} \overline{\cup_{m\ge n} S_\tau^m(B)},$$

$\mathcal{A}_\tau$ is a compact subset of $L^2(\Omega)$ which attracts all solutions. That means that for all $u_0 \in L^2(\Omega)$,

$$\operatorname{dist}\big(\mathcal{A}_\tau, S_\tau^n u_0\big) \mapsto 0 \quad \text{as } n \mapsto \infty.$$

Therefore, we have proved the following result.

Theorem 5.6. *Assuming that $u_0 \in L^2(\Omega)$ and (H1')–(H3'), the discrete problem (2.1) has an associated solution semi-group S_τ that maps $L^2(\Omega)$ into $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. This semi-group has a compact attractor which is bounded in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.*

6. Additional regularity for the attractor

In this section, we shall show supplementary regularity estimates on the solutions of problem (2.1) in the particular case where $\beta(\xi) = \xi$. We obtain therefore more regularity for the attractor obtained in section 5. The assumptions are similar to those used for the continuous problem in [7]; namely $u_o \in L^2(\Omega)$ and f verifying the following assumption

- (H4') $f(x,t,\xi) = g(\xi) - h(x)$ where $h \in L^\infty(\Omega)$ and g satisfying the conditions (H1')–(H3').

The problem (2.1) becomes

$$\delta_n - \triangle_p U^n + g(U^n) = f, \tag{6.1}$$

where $\delta_n = \frac{U^n - U^{n-1}}{\tau}$. First, we state the following lemma which we shall use to prove the main result of this section.

Lemma 6.1. *There exists a positive constant C such that for all $n_0 \geq n_\tau$, and all N in $\mathbb{N}$, we have*

$$\tau \sum_{n=n_0}^{n_0+N} \|\delta_n\|_2^2 \leq C. \tag{6.2}$$

Proof. Multiplying (6.1) by δ_n, using (5.7), (5.9), (H4') and Young's inequality, we get after some simple calculations

$$\frac{1}{4}\tau\|\delta_n\|_2^2 + \frac{1}{p}\|U^n\|_{1,p}^p - \frac{1}{p}\|U^{n-1}\|_{1,p}^p \leq C\tau. \tag{6.3}$$

Summing (6.3) from $n = n_0$ to $n = n_0 + N$, yields

$$\frac{1}{4}\tau \sum_{n=n_0}^{n_0+N} \|\delta_n\|_2^2 + \frac{1}{p}\|U^{n_0+N}\|_{1,p}^p \leq \frac{1}{p}\|U^{n_0}\|_{1,p}^p + CN\tau. \tag{6.4}$$

Now, if $n_0 \geq n_\tau$, U^{n_0} is in an $W_0^{1,p}(\Omega)$-absorbing ball, and Choosing $N\tau = 1$, we therefore obtain (6.2) from (6.4). $\square$

Theorem 6.2. *For all $n \geq n_\tau$, we have $\|\delta_n\|_2 \leq C$, where C is a positive constant.*

Proof. By subtracting equation (6.1) with $n-1$ instead of n, from equation (6.1) and multiplying the difference by δ_n, we deduce from the monotonicity of the p-Laplacian operator, Young's inequality and (H3') that

$$\frac{1}{2}\|\delta_n\|_2^2 \leq \frac{1}{2}\|\delta_{n-1}\|_2^2 + C\tau\|\delta_n\|_2^2.$$

Setting $y_n = \frac{1}{2}\|\delta_n\|_2^2$ and $h_n = C\|\delta_n\|_2^2$, and using [4, lemma 7.5] and lemma 6.1, we deduce that

$$y_{n+N} \leq \frac{C}{N\tau} + C.$$

If $n \geq n_\tau$ and $N\tau = 1$, then we get the desired estimate. $\square$

Using this theorem, we have the following regularizing estimates.

Corollary 6.3. *If $p > 2d/(d+2)$ and $p \neq 2$, then there exists some σ, $0 < \sigma < 1$, such that*

$$\|U^n\|_{B_\infty^{1+\sigma,p}(\Omega)} \leq C \text{ for all } n \geq n_\tau,$$

where $B_\infty^{\alpha,p}(\Omega)$ denotes a Besov space defined by real interpolation method.

If $p = 2$*, then* $\|U^n\|_{W^{2,2}(\Omega)} \leq C$ *for all* $n \geq n_\tau$*.*

Proof. (i) If $2d/(d+2) < p < 2$ then there exists some σ', $0 < \sigma' < 1$ such that

$$L^2(\Omega) \hookrightarrow W^{-\sigma',p'}(\Omega) \tag{6.5}$$

By (6.1), (6.5), (H4') and theorem 6.2 we get

$$\| - \Delta_p U^n\|_{B_\infty^{-\sigma',p'}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

Therefore, Simon's regularity result in [12] yields

$$\|U^n\|_{B_\infty^{1+(1-\sigma')(p-1)^2,p}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

(ii) If $p > 2$, then, by (6.1), (H4') and theorem 6.2, we get $\| - \Delta_p U^n\|_{p'} \leq C$ for all $n \geq n_\tau$. Therefore, Simon's regularity result in [12] yields

$$\|U^n\|_{B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

(iii) For $p = 2$, see [4]. □

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