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RATE OF CONVERGENCE FOR SOLUTIONS TO DIRICHLET PROBLEMS OF QUASILINEAR EQUATIONS

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Abstract. We obtain rates of convergence for solutions to Dirichlet problems of quasilinear elliptic (possibly degenerate) equations in slab-like domains. The rates found depend on the convergence of the boundary data and of the coefficients of the operator. These results are obtained by constructing appropriate barrier functions based on the structure of the operator and on the convergence of the boundary data.

1. Introduction and statement of Main Results

Let Ω be a slab-like domain in $\mathbb{R}^n$ $(n \geq 2)$ defined by

$$\Omega = \{(\mathbf{x}, y) \in \mathbb{R}^n : |y| < M,\ |\mathbf{x}| > N_1\}$$

where $\mathbf{x} = (x_1, \dots, x_{n-1})$, N_1 and M are fixed positive constants. For a continuous function ϕ on $\partial\Omega$, we consider a Dirichlet problem

$$\begin{aligned} Qf &= 0 \quad \text{in } \Omega, \\ f &= \phi \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.1}$$

where Q is a second-order quasilinear operator of the form

$$Qf = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y, f, Df) D_{ij} f + B(\mathbf{x}, y, f, Df)\,. \tag{1.2}$$

Here $(a_{ij}(\mathbf{x}, y, t, P))$ is a positive semi-definite matrix in which each entry (and B) is a C^1 function on $\mathbf{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

We shall investigate the asymptotic behavior of bounded solutions of (1.1). That is, if there is a function $\Phi \in C(S^{n-1} \times [-M, M])$ and a decreasing function $g_1(t)$, such that $g_1(t) \to 0$ as $t \to \infty$, and that

$$|\phi(\mathbf{x}, \pm M) - \Phi(\mathbf{x}/|\mathbf{x}|, \pm M)| \leq g_1(|\mathbf{x}|) \quad \text{for } |\mathbf{x}| > N_1\,. \tag{1.3}$$

We want to see how fast $f(\mathbf{x}, y)$ approaches a limiting function. Specifically, we want to find a function $k(\mathbf{x}/|\mathbf{x}|, y)$ and a decreasing function $d(t)$ such that

$$|f(\mathbf{x}, y) - k(\mathbf{x}/|\mathbf{x}|, y)| \leq d(|\mathbf{x}|) \quad \text{for } (\mathbf{x}, y) \in \Omega. \tag{1.4}$$

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Apparently the function $d(t)$ can not approach zero faster than $g_1(t)$. In general, $d(t)$ will approach zero slower than $g_1(t)$ as illustrated in the following example.

Example 1.1 ([12, example 3])**.** Let

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 > 1,\ |x_3| < 1\},$$
$$Qu = (1/3)\Delta u,$$
$$\phi(x_1, x_2, \pm 1) = \frac{2x_2(x_1^2 + x_1^2 + 1)^{1/2}}{x_1^2 + x_2^2} - \frac{x_2}{(x_1^2 + x_2^2 + 1)^{3/2}}.$$

Then

$$f(x_1, x_2, x_3) = \frac{2x_2\sqrt{x_1^2 + x_2^2 + x_3^2}}{x_1^2 + x_2^2} - \frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}.$$

is a bounded solution to $Qf = 0$ in Ω, $f = \phi$ on $\partial\Omega$ (see [4, pp. 165-1666]). When

$$\Phi(\omega) = 2\omega_2 \quad \text{for} \quad \omega = (\omega_1, \omega_2) \in S^1,$$

a short calculation shows that $|\phi(\mathbf{x}, \pm 1) - \Phi(\mathbf{x}/|\mathbf{x}|)| = O(|\mathbf{x}|^{-4})$ as $|\mathbf{x}| \to \infty$. From the results in [11] or [13], we see that $k(\mathbf{x}/|\mathbf{x}|, y)$ in (1.4) must be $\Phi(\mathbf{x}/|\mathbf{x}|)$. However we can calculate that

$$|f(\mathbf{x}, y) - \Phi(\mathbf{x}/|\mathbf{x}|)| = O(|\mathbf{x}|^{-2}) \quad \text{as } |\mathbf{x}| \to \infty.$$

Thus in this case $g_1(t)$ behaves like t^{-4} and $d(t)$ behaves like t^{-2}. That is, $d(t)$ approaches zero much slower than $g_1(t)$.

Although $d(t)$ can not go to zero faster than $g_1(t)$ in general, there are a lot of cases that $d(t)$ will go to zero at the same rate as $g_1(t)$ (the best case we can expect). When $g_1(t)$ has one of the special forms like $t^{-\alpha}$, $e^{-t^{\alpha}}$, and when the lower order term B is zero, in [12], it is proved that $d(t)$ can be chosen as a function of the same form as $g_1(t)$. Thus in this case $d(t)$ and $g_1(t)$ go to zero in the same rate. In [14], when the lower order term B and boundary limit Φ are smooth enough, $d(t)$ also approaches zero in the same rate as $g_1(t)$ if $g_1(t)$ approaches zero slower than $t^{-1/2}$, or t^{-1}, or t^{-2} (depending on the structure of the operator and smoothness of the data).

In this paper, we want to investigate when $d(t)$ will go to zero in the same rate as $g(t)$ even when $g(t)$ approaches zero faster than t^{-2} and the lower order term B is not zero. From above example, we see that it is clear some condition on Φ is necessary even for the Laplace operator Q. Comparing to the assumptions used in [14], we mainly add a new assumption that $\Phi(\omega, y)$ and $k(\omega, y)$ are independent of ω. We will obtain fast rate of convergence for bounded solutions of (1.1) that improves the results in [14].

The spatial decay estimates for solutions of partial differential equations have applications in fluid mechanics, extensible films and Saint-Venant's principle of elasticity theory. For extensive reviews of the research in this area, we refer the readers to [5, 6, 7]. Here we just mention some of the closely related results. In [1], an exponential decay estimate was obtained when Ω is a cylinder, B is a quadratic function of Df and $\phi = 0$; In [8], an exponential decay estimate for energy function was considered when $n = 2$, $\phi = 0$. In [9], an exponential decay estimate for energy function was obtained for equations modelling the constant mean curvature equation on a strip (n=2) with $\phi = 0$. In [10], Phragmén-Lindelöf type results were obtained for equation modelling constant mean curvature equation on a semi-infinite strip with $\phi = 0$; and finally in [14], for general boundary data ϕ, the rates of convergence for solutions of (1.1) were obtained in terms of the structure of Q and the rate of convergence (1.3). The result in this paper will do better in either dealing with general boundary data, or general equation, or obtaining better estimates on the rate of convergence.

Now we state the assumptions to be used in this paper. We assume that the coefficients of Q are normalized so that

$$\mathrm{Trace}(a_{ij}) = \sum_{i=1}^{n} a_{ii} = 1 \tag{1.5}$$

We assume $\phi(\mathbf{x}, y)$ has a limit in the following sense.

(C1) There exists a function $\Phi(y)$ defined on $[-M, M]$ and a decreasing function $g_1(t)$, $g_1(t) \to 0$ as $t \to +\infty$, such that

$$|\phi(\mathbf{x}, \pm M) - \Phi(\pm M)| \le g_1(|\mathbf{x}|) \quad \text{for } |\mathbf{x}| \ge N_1. \tag{1.6}$$

We assume the term a_{nn} satisfies the assumption.

(C2) For any fixed positive numbers a, b, there is a positive number $\mu(a, b)$ such that

$$a_{nn}(\mathbf{x}, y, z, \mathbf{v}) \ge \mu(a, b) \tag{1.7}$$

for all $(\mathbf{x}, y) \in \Omega$, $z \in R$, $\mathbf{v} \in \mathbb{R}^n$ with $|z| \le a$, $|\mathbf{v}| \le b$.

We assume that the term $B(\mathbf{x}, y, z, \mathbf{p}, q)$ satisfies:

(C3) There is a C^1 function $E(y, z, q)$ on $[-M, M] \times R^2$ and for each fixed bounded set D in R^2, there are positive constants C, $\alpha_0 \ge 1$ and a decreasing function $g_2(t)$, $g_2(t) \to 0$ as $t \to +\infty$, satisfying

$$\Big|\frac{B(\mathbf{x}, y, z, \mathbf{p}, q)}{a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q)} - E(y, z, q)\Big| \le g_2(|\mathbf{x}|) + C|\mathbf{p}|^{\alpha_0}$$

for $(\mathbf{x}, y) \in \overline{\Omega}$, $(z, q) \in D$ and $|\mathbf{p}| \le 1$.

We assume, as in [14], that an ODE involving E is solvable.

(C4) There is a function $k(y) \in C^1([-M, M]) \cap C^2((-M, M))$, such that

$$k''(y) + E(y, k, k') = 0 \quad \text{on } |y| \le M, \quad k(\pm M) = \Phi(\pm M). \tag{1.8}$$

(C5) $E(y, z, q)$ is non-increasing on z.

Then we have the following theorem on the rate of convergence.

Theorem 1.2. *Assume (C1)–(C5) and that* $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *is a bounded solution of (1.1). Then for each integer* J*, there is a number* C_J*, such that*

$$\big|f(\mathbf{x}, y) - k(y)\big| \le C_J g_1(\frac{1}{2^J}|\mathbf{x}|) + C_J g_2(\frac{1}{2^J}|\mathbf{x}|) + \frac{C_J}{|\mathbf{x}|^{J\beta}} \quad \text{on } \Omega. \tag{1.9}$$

where $\beta = \min\{\alpha_0, 2\}$.

As an application of this result, we give the following example.

Example 1.3. Consider the Dirichlet problem for the prescribed mean curvature equation

$$\sum_{i,j=1}^{n} \frac{(1+|Df|^2)\delta_{ij} - D_i f D_j f}{n+(n-1)|Df|^2} D_{ij} f = n\Lambda \frac{(\mathbf{x},y)(1+|Df|^2)^{3/2}}{n+(n-1)|Df|^2} \quad \text{in } \Omega$$
$$f(\mathbf{x}, \pm M) = \phi(\mathbf{x}, \pm M) \quad for \quad |\mathbf{x}| > N_1.$$

If there are functions $\Lambda_0(y)$, $\Phi(y)$ and $k(y)$ satisfying that for $|\mathbf{x}| > N_1$, $|y| \le M$,

$$|\phi(\mathbf{x}, \pm M) - \Phi(\pm M)| \le g_1(|\mathbf{x}|), \quad |\Lambda(\mathbf{x}, y) - \Lambda_0(y)| \le g_2^*(|\mathbf{x}|),$$
$$k'' - n\Lambda_0(y)(1+(k')^2)^{3/2} = 0 \quad \text{on } |y| < M, \ k(\pm M) = \Phi(\pm M),$$

then in $(C2)$, we can choose $\mu(a, b) = \frac{1}{n+(n-1)b^2}$. In $(C3)$ we can choose $E(y, z, q) = n\Lambda_0(1+q^2)^{3/2}$, $g_2(t) = c^* g_2^*(t)$ and $\alpha_0 = 2$. Then from (1.9), for a bounded solution $f(\mathbf{x}, y)$, we have that for any integer J, there is a constant C_J such that

$$|f(\mathbf{x}, y) - k(y)| \le C_J g_1(\frac{1}{2^J}|\mathbf{x}|) + C_J g_2(\frac{1}{2^J}|\mathbf{x}|) + \frac{C_J}{|\mathbf{x}|^{2J}} \quad \text{on } \Omega. \tag{1.10}$$

The main idea in the proof of the theorem is to use the barriers constructed in [14] repeatedly. The construction in [14] was adapted from [11] which in turn was inspired on [2] and [15].

2. The barrier functions

From (C2), for fixed positive numbers K_0 and K_1, there is a constant c_1, $0 < c_1 < 1$, such that

$$a_{nn}(\mathbf{x}, y, t, \mathbf{v}) \ge c_1 \tag{2.1}$$

for $(\mathbf{x}, y) \in \Omega$, $t \in R$ with $|t| \le 40K_0 + 20$, $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| \le K_1 + 2$. We define a new operator on functions $u(\mathbf{x}, y) \in C^2(\Omega)$ with parameters $t \in R$ and $\mathbf{v} \in \mathbb{R}^n$ by

$$Q_1 u = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y, t, Du + \mathbf{v}) D_{ij} u.$$

Then we can prove that there are positive decreasing functions $\chi(t)$ (depending on c_1 only), $h_a(t)$ and a positive increasing function $A(t)$ (depending on c_1 and M only) such that for any number K, $0 < K \le 3K_0 + 1$, there is a number H_0 (depending only on K_0, M and c_1), such that when $H > H_0$

$$0 < \chi(H) < 1; \quad \frac{22MH}{c_1} \le A(H)e^{\chi(H)} \le \frac{66MH}{c_1}; \tag{2.2}$$

and the function

$$z = \gamma + A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - |\mathbf{x} - \mathbf{x}_0|^2\}^{1/2} \tag{2.3}$$

satisfies the following conditions for $|t| \le 40K_0 + 20$, $|\mathbf{v}| \le K_1 + 1$, $0 \le \gamma < 1$:

$$Q_1 z \le \frac{-3c_1}{22eMH} \quad \text{in } \Omega_{\mathbf{x}_0,H,K} \cap \Omega \tag{2.4}$$

$$\gamma \le z \le \gamma + \frac{4M}{H} + 4K \quad \text{on } \overline{\Omega}_{\mathbf{x}_0,H,K} \tag{2.5}$$

$$z \ge \gamma + K \quad \text{on } \partial\Omega_{\mathbf{x}_0,H,K} \cap \{|y| < M\} \tag{2.6}$$

$$z(\mathbf{x}_0, y) \le \gamma + \frac{2M}{H} \quad \text{for } |y| \le M \tag{2.7}$$

$$|D_x z(\mathbf{x}, y)| \le 2\big(\frac{c_1 K}{M}\big)^{1/2}\frac{1}{\sqrt{H}}, \quad |D_y z(\mathbf{x}, y)| \le \frac{1}{H} \quad \text{on } \Omega_{\mathbf{x}_0,H,K}, \tag{2.8}$$

where

$$\Omega_{\mathbf{x}_0,H,K} = \big\{(\mathbf{x}, y) : |y| < M, |\mathbf{x} - \mathbf{x}_0| < \sqrt{\frac{2K}{A(H)e^{\chi(H)}}} h_a^{-1}(y + M)\big\}. \tag{2.9}$$

To make this paper self-contained, we include the following section.

3. Construction of barrier functions [14]

Set $\Phi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$ and $\Phi_1(\rho) = 11/c_1$ if $\rho \ge 1$, and define a function

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3 \Phi_1(\rho)} \quad \text{for } \alpha > 0.$$

It is clear that $\chi(\alpha)$ is a decreasing function with range $(0, \infty)$. Let η be the inverse of χ. Then η is a positive, decreasing function with range $(0, \infty)$.

Let $c_2 = 11/c_1$. For $\alpha > 1$, we have

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3 \Phi_1(\rho)} = \int_\alpha^\infty \frac{d\rho}{c_2 \rho^3} = \frac{1}{2c_2}\alpha^{-2}. \tag{3.1}$$

Thus

$$\eta(\beta) = (2c_2\beta)^{-1/2} \quad \text{for } 0 < \beta < (2c_2)^{-1}. \tag{3.2}$$

Let $H \ge 2$. Since $\eta(\chi(H)) = H$ and η is decreasing, we have $\eta(\beta) > H$ for $0 < \beta < \chi(H)$. We define a function

$$A(H) = 2M\Big(\int_1^{e^{\chi(H)}} \eta(\ln t)dt\Big)^{-1}. \tag{3.3}$$

For the rest of this article, we set $a = A(H)$ and define

$$h_a(r) = \int_r^{ae^{\chi(H)}} \eta(\ln \frac{t}{a})\, dt \quad \text{for } a \le r \le ae^{\chi(H)}. \tag{3.4}$$

Then

$$h_a(ae^{\chi(H)}) = 0, \quad h_a(a) = h_{A(H)}(A(H)) = 2M. \tag{3.5}$$

For $a < r \le ae^{\chi(H)}$,

$$\begin{gathered} h_a'(r) = -\eta(\ln \frac{r}{a}) < 0, \quad |h_a'(r)| > H, \\ h_a''(r) = \frac{1}{r}(\eta(\ln \frac{r}{a}))^3 \Phi_1(\eta(\ln \frac{r}{a})). \end{gathered} \tag{3.6}$$

Thus for $a < r \le ae^{\chi(H)}$,

$$\frac{h_a''(r)}{(h_a'(r))^2} = -\frac{h_a'(r)}{r}\Phi_1(-h_a'(r)). \tag{3.7}$$

Let h_a^{-1} be the inverse of h_a. Then h_a^{-1} is decreasing and

$$h_a^{-1}(0) = A(H)e^{\chi(H)}, \quad h_a^{-1}(2M) = A(H). \tag{3.8}$$

Further for $-M \le y \le M$,

$$(h_a^{-1})'(y+M) = \frac{1}{h_a'(h_a^{-1}(y+M))}$$

$$\begin{aligned}(h_a^{-1})''(y+M) &= (\frac{1}{h_a'(h_a^{-1}(y+M))})' \\ &= -\frac{h_a''(h_a^{-1}(y+M))(h_a^{-1})'(y+M)}{(h_a'(h_a^{-1}(y+M)))^2} \\ &= -\frac{h_a''(h_a^{-1}(y+M))}{(h_a'(h_a^{-1}(y+M)))^3} \\ &= \frac{1}{h_a^{-1}(y+M)}\Phi_1(-h_a'(h_a^{-1}(y+M))).\end{aligned}$$

Thus

$$(h_a^{-1})''(y+M)h_a^{-1}(y+M) = \Phi_1(-h_a'(h_a^{-1}(y+M))). \tag{3.9}$$

Now we choose an $H_0 > 2$ such that for $H \ge H_0$,

$$H_0 > \frac{1}{\sqrt{2c_2}} + 3M + 4 + \frac{24nc_1K_0}{M}, \quad \sqrt{\frac{4K_0}{A(H)e^{\chi(H)}}} \le \frac{1}{\sqrt{2}}. \tag{3.10}$$

For $H > H_0$, by (3.1), (3.2), we have

$$\begin{aligned}A(H)^{-1} &= (2M)^{-1}\int_1^{e^{\chi(H)}} \eta(\ln t)\,dt \\ &= (2M)^{-1}\int_0^{\chi(H)} \eta(m)e^m\,dm \\ &= (2M)^{-1}\int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}\,dm\,.\end{aligned}$$

From

$$\frac{1}{\sqrt{2c_2}}\int_0^{\chi(H)} \frac{1}{\sqrt{m}}dm \le \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}dm \le \frac{e^{\chi(H)}}{\sqrt{2c_2}}\int_0^{\chi(H)} \frac{1}{\sqrt{m}}dm,$$

we have

$$\frac{1}{c_2H} = \frac{2\sqrt{\chi(H)}}{\sqrt{2c_2}} \le \int_0^{\chi(H)} \frac{e^m}{\sqrt{2c_2m}}dm \le \frac{2e^{\chi(H)}\sqrt{\chi(H)}}{\sqrt{2c_2}} = \frac{e^{\frac{1}{2c_2H^2}}}{c_2H}.$$

Thus

$$2Mc_2H \ge A(H) \ge 2Mc_2He^{-\chi(H)} = 2Mc_2He^{-\frac{1}{2c_2H^2}}. \tag{3.11}$$

For $\mathbf{x}_0 \in \mathbb{R}^{n-1}$, a constant γ with $0 \le \gamma < 1$ and a fixed constant K with $0 < K \le 3K_0 + 1$, we define a domain $\Omega_{\mathbf{x}_0,H,K}$ in $(\mathbf{x}, y)$ space by (2.9) and define a function $z = w_{\mathbf{x}_0,\gamma,H}(\mathbf{x}, y)$ by (2.3). Since $h_a^{-1}(y+M) \ge 0$ for $|y| \le M$, $(\mathbf{x}_0, y) \in \Omega_{\mathbf{x}_0,H,K}$

for $|y| < M$. Further it is clear that the function $z = w(\mathbf{x}, y) = w_{\mathbf{x}_0,\gamma,H}(\mathbf{x}, y)$ is well defined on $\Omega_{\mathbf{x}_0,H,K}$.

Now we verify (2.5). Since h_a^{-1} is a decreasing function, $h_a^{-1}(y+M) \le h_a^{-1}(0) = A(H)e^{\chi(H)}$ for $y \ge -M$. Thus

$$z \ge \gamma + A(H)e^{\chi(H)} - h_a^{-1}(y+M) \ge \gamma.$$

From (2.9) and (3.8), we have that on $\Omega_{\mathbf{x}_0,H,K}$,

$$\begin{aligned}
z &= \gamma + A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2} \\
&\le \gamma + A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - \frac{2K}{A(H)e^{\chi(H)}}(h_a^{-1}(y+M))^2\}^{1/2} \\
&= \gamma + A(H)e^{\chi(H)} - h_a^{-1}(y+M)(1 - \frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\
&\le \gamma + A(H)e^{\chi(H)} - h_a^{-1}(2M)(1 - \frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\
&= \gamma + A(H)e^{\chi(H)} - A(H)(1 - \frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\
&\le \gamma + A(H)(1 + e\chi(H)) - A(H)(1 - 2\frac{2K}{A(H)e^{\chi(H)}}).
\end{aligned}$$

Since $e^t \le 1 + et$ for $0 < t < 1$, and $\sqrt{1-t} \ge 1 - 2t$ for $0 \le t \le \frac{1}{2}$, the above expression is equal to

$$\gamma + eA(H)\chi(H) + \frac{4K}{e^{\chi(H)}} \le \gamma + e\frac{2Mc_2H}{2c_2H^2}) + 4K \le \gamma + \frac{4M}{H} + 4K\,.$$

This because of (3.1), (3.10) and (3.11). This is (2.5).

For (2.6), on $\partial\Omega_{\mathbf{x}_0,H,K} \cap \{(\mathbf{x}, y) : |y| < M\}$,

$$|\mathbf{x}-\mathbf{x}_0| = \sqrt{\frac{2K}{A(H)e^{\chi(H)}}}h^{-1}(y+M).$$

Then from (3.8), we have

$$\begin{aligned}
z &= \gamma + A(H)e^{\chi(H)} - \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2} \\
&= \gamma + A(H)e^{\chi(H)} - h_a^{-1}(y+M)(1 - \frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\
&\ge \gamma + A(H)e^{\chi(H)} - A(H)e^{\chi(H)}(1 - \frac{2K}{A(H)e^{\chi(H)}})^{1/2} \\
&\ge \gamma + A(H)e^{\chi(H)}(1 - (1 - \frac{2K}{2A(H)e^{\chi(H)}})) \\
&= \gamma + K.
\end{aligned}$$

Here we have used (3.10) and the fact that $\sqrt{1-t} \le 1 - \frac{1}{2}t$ for $0 < t < 1$.

For (2.7), since $h_a^{-1}(r)$ and η are decreasing functions, we have

$$\begin{aligned}\frac{-1}{h_a'(h_a^{-1}(y+M))} &= \frac{1}{\eta(\ln(\frac{1}{a}h_a^{-1}(y+M)))}\\ &\le \frac{1}{\eta(\ln e^{\chi(H)})}\\ &= \frac{1}{\eta(\chi(H))} = \frac{1}{H}, \quad \text{for } |y| \le -M.\end{aligned} \tag{3.12}$$

Then by (2.3), we have

$$\frac{\partial z}{\partial y}(\mathbf{x}_0, y) = \frac{-1}{h_a'(h_a^{-1}(y+M))} = \frac{1}{H}, \quad \text{for } |y| \le -M.$$

Then (2.7) follows from this and

$$z(\mathbf{x}_0, -M) = \gamma + A(H)e^{\chi(H)} - h_a^{-1}(0) = \gamma + A(H)e^{\chi(H)} - A(H)e^{\chi(H)} = \gamma.$$

For (2.4) and (2.8), set $S = \{(h_a^{-1}(y+M))^2 - |\mathbf{x}-\mathbf{x}_0|^2\}^{1/2}$, we have that for $1 \le i \le n-1$,

$$\frac{\partial z}{\partial x_i} = \frac{1}{S}(x_i - x_{0i}), \quad \frac{\partial z}{\partial y} = -\frac{1}{S}h_a^{-1}(h_a^{-1})'.$$

By (3.10) and (3.11), on $\Omega_{\mathbf{x}_0,H,K}$, we have

$$\begin{gathered}\frac{1}{2}h_a^{-1}(y+M) \le S \le h_a^{-1}(y+M),\\ \frac{|\mathbf{x}-\mathbf{x}_0|}{S} \le 2(\frac{2K}{A(H)e^{\chi(H)}})^{1/2} \le 2(\frac{2K}{2Mc_2H})^{1/2}.\end{gathered}$$

Thus, by (3.12), we have

$$|\frac{\partial z}{\partial x_i}| \le 2(\frac{c_1K}{MH})^{1/2}, \quad |\frac{\partial z}{\partial y}| \le \frac{h_a^{-1}(y+M)}{S|h_a'(h_a^{-1}(y+M)|} \le \frac{2}{H}. \tag{3.13}$$

This gives (2.8). Hence from (3.10), for any positive semi-definite matrix (d_{ij}) with $trace(d_{ij}) = 1$ (hence all eigenvalues of (d_{ij}) are less than or equal to 1), we have

$$|\sum_{i,j=1}^{n} d_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}| \le |Dz|^2 \le 1 \tag{3.14}$$

For t with $|t| \le 40K_0 + 20$, $|\mathbf{v}| \le K_1$, from (3.14), we have $|Dz| + |\mathbf{v}| \le K_1 + 1$. Then from (1.5) and (2.1), for $S = \{(h_a^{-1}(y+M))^2 - |\mathbf{x} - \mathbf{x}_0|^2\}^{1/2}$, we have

$$\begin{aligned}
Q_1 z &= \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y, t, Dz + \mathbf{v}) D_{ij} z \\
&= \frac{1}{S}\sum_{i=1}^{n-1} a_{ii} + \frac{1}{S^3}\sum_{i,j=1}^{n-1} a_{ij}(x_i - x_i^0)(x_j - x_j^0) - \frac{1}{S^3}\sum_{i=1}^{n-1} a_{in}(x_i - x_i^0) h_a^{-1}(h_a^{-1})' \\
&\quad - \frac{1}{S} a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'') + \frac{1}{S^3} a_{nn}(h_a^{-1})^2((h_a^{-1})')^2 \\
&= \frac{1}{S}\{1 - a_{nn} + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn}((h_a^{-1})^2 + h_a^{-1}(h_a^{-1})'')\} \\
&\le \frac{1}{S}\{1 + \sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} - a_{nn} h_a^{-1}(h_a^{-1})''\}\,.
\end{aligned}$$

(By (2.1), (3.9)), (3.11) and (3.14) this expression is less than or equal to

$$\frac{-9}{S} \le \frac{-9}{h_a^{-1}(y+M)} \le \frac{-9}{A(H)e^{\chi(H)}} \le \frac{-9}{2Mc_2He^{\frac{1}{2c_2H^2}}} \le \frac{-3c_1}{22eMH}$$

which implies (2.4).

4. Proof of Main Theorem

For the proof of Theorem 1.2, we need the following result on E and k.

Proposition 4.1 ([13, Lemma 4.1])**.** *Under assumptions (C4) and C5), there exist numbers $\gamma_1 > 0$ and c_3 (depending only on k, E), such that for any constant δ_1 with $|\delta_1| < \min\{\gamma_1, 1\}$, there is a (unique) function $k_\lambda(y) = k_\lambda(y)$ in $C^1([-M, M]) \cap C^2((-M, M))$ satisfying*

$$k_\lambda''(y) + E(y, k_\lambda(y), k_\lambda'(y)) = -\frac{3}{4c_3}\delta_1, \quad k_\lambda(\pm M) = k(\pm M),$$

and on $|y| \le M$,

$$|k(y) - k_\lambda(y)| \le |\delta_1|, \quad |k'(y) - k_\lambda'(y)| \le |\delta_1|, \quad |k''(y) - k_\lambda''(y)| \le |\delta_1|.$$

Proof of Theorem 1.2. We assume that there exists a non-increasing function $g(t)$ such that

$$|f(\mathbf{x}, y) - k(y)| \le g(|\mathbf{x}|) \quad \text{for } (\mathbf{x}, y) \in \Omega. \tag{4.1}$$

(since $f(\mathbf{x}, y)$ and $k(y)$ are bounded, (4.1) holds for $g(t)$ to be some appropriate constant. $g(t)$ will also take other forms as we shall explain later).

For a small positive number δ_1 (to be chosen later), let $k_\lambda(y)$ be the function defined in the Proposition. We will use the barrier function $u(\mathbf{x}, y) + k_\lambda(y)$. Let

$$K_0 = \sup\{|f(\mathbf{x}, y)| : (\mathbf{x}, y) \in \Omega\} + \sup\{|k(y)| : |y| \le M\} + g(0),$$
$$K_1 = 2\sup\{|k'(y)| : |y| \le M\} + 10\,.$$

and c_1 be a number such that

$$c_1 \le a_{nn}(\mathbf{x}, y, t, \mathbf{v}) \tag{4.2}$$

for $(\mathbf{x}, y) \in \Omega$, $|t| \le 40K_0 + 20$, $\mathbf{v} \in R^n$ with $|\mathbf{v}| \le K_1 + 2$.

Since $g_1(t) \to 0$ as $t \to \infty$, there is a number H_1 such that $g_1(\frac{1}{2}|\mathbf{x}|) \le \frac{1}{2}$ for $|\mathbf{x}| \ge H_1$ and $H_1 > 800MK_0/c_1$. We fixed an $\mathbf{x}_0$ with $|\mathbf{x}_0| \ge H_0 + H_1$ (H_0 is given in (2.3)). Let $u(\mathbf{x}, y) = z(\mathbf{x}, y)$ defined on $\Omega_{\mathbf{x}_0,H,K}$ be given by (2.3) with the choice of parameters:

$$\gamma = g_1(\frac{1}{2}|\mathbf{x}_0|) + \delta_1, \quad H = \frac{c_1|\mathbf{x}_0|^2}{800MK}, \quad K = 2g(\frac{1}{2}|\mathbf{x}_0|).$$

¿From (2.2), (2.8), $h_a^{-1}(y + M) \le A(H)e^{\chi(H)}$ and the choice of H, there is a number c_6 independent of δ_1, such that on $\Omega_{\mathbf{x}_0,H,K}$,

$$|D_x u| \le \frac{c_6}{|\mathbf{x}_0|} g(\frac{1}{2}|\mathbf{x}_0|), \quad |D_y u| \le \frac{c_6}{|\mathbf{x}_0|^2} g(\frac{1}{2}|\mathbf{x}_0|),$$

and

$$|\mathbf{x} - \mathbf{x}_0| \le \sqrt{\frac{2K}{A(H)e^{\chi(H)}} h_a^{-1}(y + M)} \le \sqrt{2KA(H)e^{\chi(H)}} \le \frac{1}{2}|\mathbf{x}_0|. \tag{4.3}$$

Then for $|\mathbf{x}_0|$ large, on $\Omega_{\mathbf{x}_0,H,K}$, (where c_9 is independent of δ_1)

$$\frac{1}{2}|\mathbf{x}_0| \le |\mathbf{x}| \le \frac{3}{2}|\mathbf{x}_0|, \quad |D_x u|^{\alpha_0} \le c_9 \frac{(g(\frac{1}{2}|\mathbf{x}_0|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}}, \quad |D_y u| \le c_9 \frac{g(\frac{1}{2}|\mathbf{x}_0|)}{|\mathbf{x}|^2}. \tag{4.4}$$

Then from (2.5), for $|\mathbf{x}_0| \ge H_0 + H_1$ and any positive constant b, $b < 10K_0 + 1$, on $\Omega_{\mathbf{x}_0,H,K}$, we have

$$u(\mathbf{x}, y) + k_\lambda(y) + b \le 40K_0 + 20, |Du(\mathbf{x}, y)| + |k'_\lambda(y)| \le K_1 + 1.$$

Set

$$M_3 = \sup\big\{\frac{\partial E}{\partial q}(y, z, q) : |y| \le M,\ |z| \le 40K_0 + 20,\ |q| \le K_1 + 1\big\}$$

¿From (2.4), for $0 < b < 10K_0 + 1$, we have (note that k_λ depends on y only)

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))D_{ij}(u + k_\lambda)\\
&+ B(\mathbf{x}, y, u + k_\lambda + b, D(u + Dk_\lambda))\\
&< a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))k''_\lambda(y) + B(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda)).
\end{aligned}$$

By (C3) and(C5), Proposition 4.1, (4.2), (4.4) and $b > 0$, $u > 0$, we have

$$\begin{aligned}
&a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))k_\lambda''(y) + B(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda)) \\
&= a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))\big\{k_\lambda''(y) + \frac{B(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))}{a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))}\big\} \\
&= a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))\Big\{k_\lambda''(y) + E(y, k_\lambda, k_\lambda') + E(y, k_\lambda, D_y u + k_\lambda') \\
&\quad - E(y, k_\lambda, k_\lambda') + E(y, k_\lambda + u + b, D_y u + k_\lambda') - E(y, k_\lambda, D_y u + k_\lambda') \\
&\quad + \frac{B(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))}{a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + Dk_\lambda))} - E(y, k_\lambda + u + b, D_y u + k_\lambda')\Big\} \\
&\le a_{nn}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))\big\{ - \frac{3}{4c_3}\delta_1 + M_3|D_y u| + g_2(|\mathbf{x}|) + C|D_x u|^{\alpha_0}\big\} \\
&\le -\frac{3c_1}{4c_3}\delta_1 + M_3|D_y u| + C|D_x u|^{\alpha_0} + g_2(|\mathbf{x}|) \\
&\le \frac{Cc_9(g(\frac{1}{2}|\mathbf{x}_0|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}} + \frac{c_9 M_3 g(\frac{1}{2}|\mathbf{x}_0|)}{|\mathbf{x}|^2} + g_2(|\mathbf{x}|) - \frac{3c_1}{4c_3}\delta_1
\end{aligned}$$

Set $d = \frac{1}{2}|\mathbf{x}_0|$, we have that on $\Omega_{\mathbf{x}_0,H,K}$, $|\mathbf{x}| \ge d$ (by (4.4)). Now we fixed a d such that $d > H_2 \ge H_0 + H_1$ and choose δ_1 by

$$\frac{3c_1}{4c_3}\delta_1 = \frac{Cc_9(g(\frac{1}{2}|\mathbf{x}_0|))^{\alpha_0}}{d^{\alpha_0}} + \frac{c_9 M_3 g(\frac{1}{2}|\mathbf{x}_0|)}{d^2} + g_2(d). \tag{4.5}$$

Then on $\Omega_{\mathbf{x}_0,H,K}$, we have (since $|\mathbf{x}| \ge d$ and g_2 is non-increasing)

$$\begin{aligned}
&\sum_{i,j=1}^n a_{ij}(\mathbf{x}, y, u + k_\lambda + b, D(u + k_\lambda))D_{ij}(u + k_\lambda) \\
&+ B(\mathbf{x}, y, , u + k_\lambda + b, D(u + k_\lambda)) \\
&< \frac{Cc_9(g(\frac{1}{2}|\mathbf{x}_0|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}} + \frac{c_9 M_3 g(\frac{1}{2}|\mathbf{x}_0|)}{|\mathbf{x}|^2} + g_2(|\mathbf{x}|) - \frac{3c_1}{4c_3}\delta_1 \le 0.
\end{aligned} \tag{4.6}$$

For such an $\mathbf{x}_0$, on $\Omega_{\mathbf{x}_0,H,K} \cap \Omega$, we will compare the function $f(\mathbf{x}, y)$ with the function $u(\mathbf{x}, y) + k_\lambda(y)$. On $\partial\Omega_{\mathbf{x}_0,H,K} \cap \Omega$, (2.6) and Proposition 4.1 imply

$$\begin{aligned}
u(\mathbf{x}, y) + k_\lambda(y) &\ge \gamma + K + k_\lambda(y) \\
&\ge g_1(\frac{1}{2}|\mathbf{x}_0|) + \delta_1 + 2g(\frac{1}{2}|\mathbf{x}_0|) + k_\lambda(y) \\
&\ge 2g(\frac{1}{2}|\mathbf{x}_0|) + k(y) \ge f(\mathbf{x}, y)
\end{aligned}$$

(by (4.1) and $|\mathbf{x}| \ge \frac{1}{2}|\mathbf{x}_0|$ on $\Omega_{\mathbf{x}_0,H,K}$). On $\Omega_{\mathbf{x}_0,H,K} \cap \partial\Omega$, $y = \pm M$ and $\Phi(\pm M) = k(\pm M) = k_\lambda(\pm M)$. Then from (**C1**) and (2.5), we have

$$\begin{aligned}
\phi(\mathbf{x}, \pm M) &= \phi(\mathbf{x}, \pm M) - \Phi(\pm M) + \Phi(\pm M) \le g_1(|\mathbf{x}|) + k(\pm M) \\
&\le g_1(\frac{1}{2}|\mathbf{x}_0|) + \delta_1 + k_\lambda(\pm M) = \gamma + k_\lambda(\pm M) \\
&\le u(\mathbf{x}, \pm M) + k_\lambda(\pm M).
\end{aligned}$$

Let

$$\Omega_1 = \{(\mathbf{x}, y) \in \Omega_{\mathbf{x}_0,H,K} \cap \Omega : f(\mathbf{x}, y) > u(\mathbf{x}, y) + k_\lambda(y)\}.$$

Since $f(\mathbf{x},y)\le u(\mathbf{x},y)+k_\lambda$ on $\partial(\Omega_{\mathbf{x}_0,H,K}\cap\Omega)$, Ω_1 is in the interior of $\Omega_{\mathbf{x}_0,H,K}$. If $(\mathbf{x}_2,y_2)\in\Omega_1$, we let $b=f(\mathbf{x}_2,y_2)-(u(\mathbf{x}_2,y_2)+k_\lambda(y_2))$, then $0<b<f(\mathbf{x}_2,y_2)\le K_0<10K_0+1$. Using this b in (4.6) and evaluating the formula at $(\mathbf{x}_2,y_2)$, we get

$$\sum_{i,j=1}^n a_{ij}(\mathbf{x}_2,y_2,f(\mathbf{x}_2,y_2),D(u+k_\lambda))D_{ij}(u+k_\lambda)+B(\mathbf{x}_2,y_2,f(\mathbf{x}_2,y_2),D(u+k_\lambda))<0.$$

Since $(\mathbf{x}_2,y_2)\in\Omega_1$ is arbitrary, on Ω_1, we have

$$\sum_{i,j=1}^n a_{ij}(\mathbf{x},y,f(\mathbf{x},y),D(u+k_\lambda))D_{ij}(u+k_\lambda)+B(\mathbf{x},y,f(\mathbf{x},y),D(u+k_\lambda))<0. \quad (4.7)$$

Now we can apply a comparison principle [3, Theorem10.1] to conclude that Ω_1 is empty. Thus

$$f(\mathbf{x},y)\le u(\mathbf{x},y)+k_\lambda(y)\quad \text{on } \Omega_{\mathbf{x}_0,H,K}\cap\Omega.$$

In particular, from (2.7)

$$f(\mathbf{x}_0,y)\le u(\mathbf{x}_0,y)+k_\lambda(y)\le\gamma+\frac{2M}{H}+k_\lambda(y).$$

Then from Proposition 2 and the choices of γ, H and δ_1, we have

$$\begin{aligned}
f(\mathbf{x}_0,y)&\le g_1(\frac{1}{2}|\mathbf{x}_0|)+\delta_1+\frac{c_{11}g(\frac12|\mathbf{x}_0|)}{|\mathbf{x}_0|^2}+k_\lambda(y)\\
&\le g_1(\frac{1}{2}|\mathbf{x}_0|)+\frac{c_{11}g(\frac12|\mathbf{x}_0|)}{|\mathbf{x}_0|^2}+k(y)+2\delta_1
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{x}_0,y)-k(y)&\le g_1(\frac{1}{2}|\mathbf{x}_0|)+\frac{c_{11}g(\frac12|\mathbf{x}_0|)}{|\mathbf{x}_0|^2}+2\delta_1\\
&=g_1(\frac{1}{2}|\mathbf{x}_0|)+\frac{c_{11}g(\frac12|\mathbf{x}_0|)}{|\mathbf{x}_0|^2}+\frac{c_{12}(g(\frac12|\mathbf{x}_0|))^{\alpha_0}}{d^{\alpha_0}}+\frac{c_{12}g(\frac12|\mathbf{x}_0|)}{d^2}+g_2(d)
\end{aligned}$$

However $d=\frac12|\mathbf{x}_0|$, thus

$$f(\mathbf{x}_0,y)-k(y)\le g_1(\frac{1}{2}|\mathbf{x}_0|)+\frac{c_{13}(g(\frac12|\mathbf{x}_0|))^{\alpha_0}}{|\mathbf{x}_0|^{\alpha_0}}+\frac{c_{13}g(\frac12|\mathbf{x}_0|)}{|\mathbf{x}_0|^2}+g_2(\frac{1}{2}|\mathbf{x}_0|).$$

Since d is arbitrary as long as it is greater than H_2, $\mathbf{x}_0$ is arbitrary as long as $|\mathbf{x}_0|$ is greater than $2H_2$. Then we have that when $|\mathbf{x}|$ is large,

$$f(\mathbf{x},y)-k(y)\le g_1(\frac{1}{2}|\mathbf{x}|)+\frac{c_{13}(g(\frac12|\mathbf{x}|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}}+\frac{c_{13}g(\frac12|\mathbf{x}|)}{|\mathbf{x}|^2}+g_2(\frac{1}{2}|\mathbf{x}|).$$

Similarly using $-u(\mathbf{x},y)+k_\lambda(y)$ (now with $\delta_1<0$ and choosen appropriately), we can prove that

$$f(\mathbf{x},y)-k(y)\ge-\big\{g_1(\frac{1}{2}|\mathbf{x}|)+\frac{c_{13}(g(\frac12|\mathbf{x}|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}}+\frac{c_{13}g(\frac12|\mathbf{x}|)}{|\mathbf{x}|^2}+g_2(\frac{1}{2}|\mathbf{x}|)\big\}.$$

Thus on Ω, we have

$$|f(\mathbf{x},y)-k(y)|\le g_1(\frac{1}{2}|\mathbf{x}|)+\frac{c_{13}(g(\frac12|\mathbf{x}|))^{\alpha_0}}{|\mathbf{x}|^{\alpha_0}}+\frac{c_{13}g(\frac12|\mathbf{x}|)}{|\mathbf{x}|^2}+g_2(\frac{1}{2}|\mathbf{x}|).$$

Since $\alpha_0 \geq 1$ and $g(t)$ is non-increasing, we have $g(t)^{\alpha_0} \leq g(1)^{\alpha_0-1}g(t)$, thus for some constant c_{14}, we have ($\beta = \min\{\alpha_0, 2\}$)

$$|f(\mathbf{x}, y) - k(y)| \leq g_1(\frac{1}{2}|\mathbf{x}|) + g_2(\frac{1}{2}|\mathbf{x}|) + \frac{c_{14}}{|\mathbf{x}|^\beta} g(\frac{1}{2}|\mathbf{x}|) \quad \text{on } \Omega. \tag{4.8}$$

Now we choose $g(t) = K_0$ in (4.1), then (4.8) becomes

$$|f(\mathbf{x}, y) - k(y)| \leq g_1(\frac{1}{2}|\mathbf{x}|) + g_2(\frac{1}{2}|\mathbf{x}|) + \frac{c_{14}K_0}{|\mathbf{x}|^\beta} \quad \text{on } \Omega. \tag{4.9}$$

shen we choose

$$g(t) = g_1(\frac{1}{2}t) + g_2(\frac{1}{2}t) + \frac{c_{14}K_0}{t^\beta},$$

that (4.9) implies (4.1) is still true. Then for this new $g(t)$, we can apply (4.8) to obtain

$$|f(\mathbf{x}, y) - k(y)| \leq c_{15}g_1(\frac{1}{4}|\mathbf{x}|) + c_{15}g_2(\frac{1}{4}|\mathbf{x}|) + \frac{c_{15}}{|\mathbf{x}|^{2\beta}} \quad \text{on } \Omega. \tag{4.10}$$

for some constant c_{15} (since g_1, g_2 are non-increasing). Then once again, in (4.1), we can reset the function $g(t)$ as

$$g(t) = c_{15}g_1(\frac{1}{4}t) + c_{15}g_2(\frac{1}{4}t) + \frac{c_{15}}{t^{2\beta}}. \tag{4.11}$$

and apply (4.8) to conclude that there is a constant c_{16} such that

$$|f(\mathbf{x}, y) - k(y)| \leq c_{16}g_1(\frac{1}{8}|\mathbf{x}|) + c_{16}g_2(\frac{1}{8}|\mathbf{x}|) + \frac{c_{16}}{|\mathbf{x}|^{3\beta}} \quad \text{on } \Omega. \tag{4.12}$$

We can repeat this procedure to conclude that for any integer J, there is a number C_J, such that

$$|f(\mathbf{x}, y) - k(y)| \leq C_J g_1(\frac{1}{2^J}|\mathbf{x}|) + C_J g_2(\frac{1}{2^J}|\mathbf{x}|) + \frac{C_J}{|\mathbf{x}|^{J\beta}} \quad \text{on } \Omega. \tag{4.13}$$

□

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