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NOTE ON A NON-OSCILLATION THEOREM OF ATKINSON

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Abstract. We present a general non-oscillation criterion for a second order two-term scalar nonlinear differential equation in the spirit of a classic result by Atkinson [1]. The presentation is simpler than most and can serve to unify many such criteria under one common theme that eliminates the need for specfic techniques in each of the classical cases (sublinear, linear, and superlinear). As is to be expected in a result of this kind, the applications are widespread and include, but are not limited to, linear, sublinear, superlinear differential equations as well as some transcendental cases and some possibly *mixed* cases.

1. Introduction

The study of oscillation and non-oscillation theory of second order ordinary differential equations has a very long history. The first theoretical results in this area were actually obtained by Sturm himself in his abstract [9] of his now classic 1836 memoir as a consequence of his memorable comparison theorems. Thousands of papers have been written since on all aspects of this theory ranging from difference equations to partial differential equations and even integral equations (cf., e.g., [7] for further details). In this note we revisit a classic theorem of F. V. Atkinson [1], probably the first such theorem that exhibits a necessary and sufficient condition for the existence of oscillatory solutions to (nonlinear) Emden-Fowler type equations.

We recall that a solution of a real second order differential equation is said to be *oscillatory* on a half-axis provided it has an infinite number of zeros on that semi-axis. By the standard existence and uniqueness theorem we see that there must be a sign change at a zero and zeros cannot accumulate on any finite interval. If the equation has at least one non-trivial solution with a finite number of zeros it is termed non-oscillatory. The question of interest here involves the determination of a general criterion that will ensure the non-oscillation of a two-(or more)-term nonlinear ordinary differential equation of the form (1.1) where the nonlinearity is of a general type. For the most part, work in this field has centered mostly on establishing necessary and sufficient conditions for the oscillation of all solutions of equations of the form $y''(x) + f(x)g(y) = 0, x \in [0, \infty)$ in the superlinear case after Atkinson's paper [1]. This was followed by an important though little referenced paper by G. Butler [3] who dropped the non-negativity assumption on $f(x)$ in

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the superlinear case. His necessary and sufficient condition was extended to a framework that includes both differential and difference equations in [7]. Recent work in this general area has centered around an extension of Atkinson's theorem [1] to delay equations and three-term equations with damping (cf., [8], [11], and [10]).

We present here a result on the existence of a positive solution of a nonlinear two-term scalar differential equation of the form

$$y''(x) + F(x, y(x)) = 0, \quad x \in [0, \infty) \tag{1.1}$$

where $F : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is always assumed to be continuous on its domain along with some basic conditions that ensure the existence and uniqueness of solutions to associated initial value problems. There is no loss of generality in assuming that (1.1) is defined on $\mathbb{R}^+$ rather than on a half-axis such as $[a, \infty)$ where $a > -\infty$, since a simple change of independent variable will transform the equation on $[a, \infty)$ back into (1.1).

Instead of specializing to various types of nonlinearities as is usually done we will proceed directly to the fully nonlinear case, without the additional standard asumption that the nonlinearity $F(x, y)$ in (1.1) is variables separable. Using a fixed point theorem we find, as a special case, a non-oscillation criterion that covers many of the different equations types observed in the literature (linear, sublinear, superlinear) as well as some rare transcendental cases and even mixed linear or semilinear cases. The advantage lies in a unified framework for different settings, one that provides a condition for the existence of a positive (and so necessarily non-oscillatory) asymptotically constant solution to the differential equation (1.1). Due to its generality, it is to be noted that when our results are specified to actual cases (such as a supelinear equation) our results are generally stronger than existing ones. But then we also require more than non-oscillation as a goal.

For related results dealing with fully nonlinear equations with damping (e.g., $F(x, y, y')$ in (1.1) see [12]). In [13] the author treats (1.1) under an assumption of convexity in the second variable on a majorant of F along with some additional conditions (e.g., $F(x, 0) = 0$, etc). As a result, Zhao concludes the existence of at least one positive solution on $(0, \infty)$ that is asymptotically linear as $x \to \infty$. For an equation of type (1.1) for the Laplacian in $\mathbb{R}^n$, see [5], where Atkinson's condition (3.3), below, is extended to this higher dimensional setting. Our conditions (2.1), (2.2), (2.3) in the sequel appear to be weaker than those presented in the literature and so the results may be of interest in the study of the existence of positive solutions to semilinear elliptic problems (cf., [5], [12], [13]). These papers make use of the Schauder-Tikhonov fixed point theorem and so generally there is no uniqueness of the fixed point, in contrast with our technique which does guarantee its uniqueness by virtue of the use of the contraction mapping principle.

2. The existence of a positive monotone solution

Our techniques invoke the fixed-point theorem of Banach-Cacciopoli (see [6]) and are based on the simple premise that, basically, in the variables separable case, the nonlinearity in the dependent variable y in (1.1) maps a given compact interval back into (and not necessarily onto) itself. This, along with the basic existence and uniqueness theorems appealed to above, will ensure that a non-oscillatory solution exists which, in fact, must be positive on the semi-axis $(0, \infty)$. In the linear case, that is where $F(x,y) = f(x)y$, this property results in the definition of a *disconjugate* equation by Sturm theory (that is, an equation in which every non-trivial solution has at most one zero).

Theorem 2.1. *Let $X = \{u \in C[0,\infty)| : 0 \le u(t) \le M, \text{ for all } t \ge 0\}$, where $M > 0$ is given but fixed. Assume that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and that for any $u \in X$,*

$$\int_0^\infty t\,F(t,u(t))\,dt \le M \tag{2.1}$$

and that there exists a function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that k is continuous and

$$\int_0^\infty t\,k(t)\,dt < 1, \tag{2.2}$$

such that for any $u, v \in \mathbb{R}^+$, we also have

$$|F(t,u) - F(t,v)| \le k(t)|u - v|, \quad t \ge 0. \tag{2.3}$$

Then (1.1) has a positive (and so non-oscillatory) monotone solution on $(0,\infty)$ such that $y(x) \to M$ as $x \to \infty$.

Proof. It is easy to see that the space X defined in the statement of the theorem is a closed subset of $C[0,\infty)$. It follows that $(X, \|\cdot\|_X)$ where $\|\cdot\|_X$ is defined as usual by the uniform norm, $\|u\|_X = \sup_{t\in[0,\infty)} u(t)$ is a Banach space. Following Atkinson (cf., [1]) we look for a uniformly bounded continuous solution of the nonlinear integral equation

$$y(x) = M - \int_x^\infty (t-x)F(t,y(t))\,dt \tag{2.4}$$

for $x \in [0,\infty)$. Clearly, the existence of such a solution implies that $y(x) \to M$ and $y'(x) \to 0$ as $x \to \infty$. Indeed, once such a solution is found in X we conclude that $y(x) \ge 0$ implies $y(x) > 0$ for all $x \in (0,\infty)$, on account of the tacit assumption of uniqueness of solutions of initial value problems associated with (1.1) and so (2.4).

We define a map on X as usual by

$$(Tu)(x) = M - \int_x^\infty (t-x)\,F(t,u(t))\,dt \tag{2.5}$$

where $u \in X$. Note that the right-side of (2.5) clearly converges for each $x \ge 0$, on account of (2.1). Indeed, for given $u \in X$ and $x \ge 0$,

$$0 \le \int_x^\infty (t-x)F(t,u(t))\,dt \le \int_0^\infty t\,F(t,u(t))\,dt \le M, \tag{2.6}$$

since $F(t,u(t)) \ge 0$ for such u (which implies that $(Tu)(x) \le M$) and the indefinite integral is a non-increasing function of x on $[0,\infty)$. Thus, $(Tu)(x) \ge 0$ for any $x \ge 0$. This shows that $TX \subseteq X$. Finally, we show that T is a contraction on X (and so in particular, T is continuous there). For $u, v \in X$,

$$\begin{aligned}
|(Tu)(x) - (Tv)(x)| &\le \int_x^\infty (t-x)|F(t,u(t)) - F(t,v(t))|\,dt \\
&\le \int_x^\infty (t-x)k(t)|u(t) - v(t)|\,dt \\
&\le \|u - v\|_X \int_0^\infty t\,k(t)\,dt,
\end{aligned}$$

where we have used (2.3) and the fact that $\int_x^\infty (t-x)k(t)\,dt$ is a non-increasing function of x for $x \in [0,\infty)$, since $k(t) \geq 0$. From this we readily see that

$$\|Tu - Tv\|_X \leq \alpha \|u - v\|_X,$$

where $\alpha < 1$ is given by the left-side of (2.2). Hence T is a contraction on X and so T has a fixed point $u = y$ in X which must satisfy (2.4). The monotonicity is clear since all quantities in the integral in (2.5) are non-negative. This proves the theorem. □

For a pointwise criterion on $F(t,u)$ we can formulate the following result.

Corollary 2.2. *Let $M > 0$ be given. Assume that $F : \mathbb{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ and that there is a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_0^\infty t f(t)\,dt \leq 1, \tag{2.7}$$

and

$$F(t,u) \leq f(t)g(u), \quad t \geq 0, u \in \mathbb{R}^+, \tag{2.8}$$

for some function g where $g : [0,M] \to [0,M]$ is continuous on $[0,M]$. Assume further that there exists a function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that k is continuous and

$$\int_0^\infty t k(t)\,dt < 1,$$

such that for any $u, v \in \mathbb{R}^+$, we also have

$$|F(t,u) - F(t,v)| \leq k(t)|u - v|, \quad t \geq 0\,.$$

Then (1.1) has a positive (and so non-oscillatory) monotone solution on $(0,\infty)$ such that $y(x) \to M$ as $x \to \infty$.

Proof. Proceed as in the theorem up to (2.6). At this point, the estimate (2.8) is used in lieu of (2.1) to show that (2.1) may be replaced by (2.7) the remaining argument being similar. □

3. Applications

Although conditions (2.1), (2.3) and (2.2) are stronger than existing corresponding conditions for specific choices of the nonlinearity $F(x,y)$, Theorem 2.1 basically eliminates the distinction between the classical *sublinear* and *superlinear* cases determined by a growth condition on $F(x,\cdot)$, (cf., [10] for detailed definitions). We will show that Theorem 2.1 applies in various cases using the simple conditions enunciated there, bearing in mind that these hypotheses can be weakened considerably in specific cases.

Example 3.1 (A sublinear case)**.** Let $F(x,y) = f(x)g(y)$ where $g(y) = |y+1|^\nu$, and $0 < \nu < 1$. This case is characterized by the convergence of the reciprocal of $g(y)$ away from zero (cf., [11]). Let $f \in C[0,\infty), f(x) \geq 0$ on $[0,\infty)$. Then the function $k(t)$ in Theorem 2.1 may be chosen as $k(t) = f(t)\nu$ since g is Lipschitzian with Lipschitz constant at most ν on $[0,M] \equiv [0,1]$. An appropriate choice of $f(x)$ leads to a verification of each of (2.1) and (2.2) provided we choose $M = 1$ and

$$\int_0^\infty t f(t)\,dt \leq \frac{1}{2^\nu}. \tag{3.1}$$

In this case,

$$y'' + f(x)|y+1|^{\nu} = 0, \quad x \in [0, \infty),$$

has a unique positive solution $y(x) \to 1$ as $x \to \infty$. Of course, oscillations in the sublinear case have been completely characterized by Belohorec in [2] and others in recent times (cf., [11] for further details). Note that, in this case, the (necessary and sufficient) Belohorec condition

$$\int_0^{\infty} t^{\nu} f(t)\, dt < \infty$$

is automatically satisfied on account of (3.1).

Example 3.2 (A linear case)**.** Consider

$$y'' + f(x)y = 0, \quad x \in [0, \infty),$$

where f is chosen so that

$$\int_0^{\infty} t\, f(t)\, dt < 1. \tag{3.2}$$

The choice $M = 1$ and $k(t) = f(t)$ in Theorem 2.1 gives the basic result that our linear equation is non-oscillatory (in fact, disconjugate) since it has an asymptotically constant solution $y(x) \to 1$ as $x \to \infty$. In the linear case, this conclusion is classical. For example, using [4, p. 255, Exercise 3] we see that (3.2) implies that the linear equation is disconjugate. Furthermore, [6, p. 35, Exercise 6.3] indicates that our equation must have asymptotically constant solutions under the milder assumption that the integral appearing in (3.2) is merely finite.

Example 3.3 (A superlinear case)**.** Once again, we let $F(x, y) = f(x)g(y)$ where, say, $g(y) = y^{2n-1}$, and $n > 1$ is an integer. This case, generally motivated by the convergence of the reciprocal of the integral of $g(y)$ at infinity was inspired by Atkinson's paper [1], thus leading to hundreds of papers in the subject. In this case, note that g is a *self-map* (i.e., $g([0, 1]) = [0, 1]$) and g is Lipschitzian with Lipschitz constant equal to $2n - 1$. It then follows from our theorem that if f is chosen so that

$$\int_0^{\infty} t\, f(t)\, dt < \frac{1}{2n-1},$$

then $y'' + f(x)y^{2n-1} = 0$ will have a positive solution on $(0, \infty)$ (and so is non-oscillatory there) since it will be asymptotically constant: $y(x) \to 1$ as $x \to \infty$. Of course, our condition is stronger than Atkinson's original necessary and sufficient condition for non-oscillation, namely

$$\int_0^{\infty} tf(t)\, dt < \infty, \tag{3.3}$$

which implies the same result (of non-oscillation but not necessarily of positivity on $(0, \infty)$).

Example 3.4 (A transcendental case)**.** The following example is rarely covered in the literature since the reciprocal of the function $g(y) = \sin(\frac{\pi}{2}y)$ does not converge at all at infinity (to either a finite or extended real number) due to oscillations in the nonlinearity, nor does it converge at 0 being infinite there. Thus, it is neither sublinear nor superlinear. Fixing this g we choose $f \in C[0, \infty)$, $f(x) \geq 0$ so that

$$\int_0^{\infty} tf(t)\, dt < \frac{2}{\pi}.$$

Note that g is, once again, a self-map on $[0,1]$. In this case, for our choice of f, we see that use of the theorem shows that the equation

$$y'' + f(x)\,\sin(\frac{\pi}{2}y) = 0, \quad x \in [0,\infty),$$

admits a positive asymptotically constant solution $y(x) \to 1$ as $x \to \infty$.

Example 3.5 (A mixed case)**.** Our final example covers a mixed situation, one where $F(x,y)$ is a sum of a linear and nonlinear term in the second variable. It appears that the literature involving such problems is scarce as well. We choose

$$F(x,y) = \frac{e^{-x}y}{4} + \frac{e^{-2x}}{1+y}, \quad (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

The choice $M = 1$ in the theorem gives $\int_0^\infty t\,F(t,u(t))\,dt \le \frac{1}{2}$ for $u \in X$. In fact, (2.3) is satisfied provided we put

$$k(t) = \frac{e^{-t}}{4} + \frac{e^{-2t}}{2}.$$

It follows that (2.2) is verified as well and so the nonlinear equation

$$y'' + \frac{e^{-x}}{4}y + \frac{e^{-2x}}{1+y} = 0, \quad x \in [0,\infty),$$

has a positive solution that tends asymptotically to one as $x \to \infty$.

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April 27, 2004: Corrigendum

In the proof of Theorem 2.1, replace "Banach space" by "complete metric space". Thanks, Angelo Mingarelli.

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