Electronic Journal of Differential Equations, Vol. 2004(2004), No. 30, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PERIODIC DUFFING EQUATIONS WITH DELAY

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Abstract. Assuming *a priori* bounds on the mean of a T-periodic function p, we show that the Duffing equation

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau), x'(t-\tau)) = p(t),$$

with delay τ, admits a T-periodic solution.

1. Introduction

The existence of 2π-periodic solutions to the Duffing equation

$$x''(t) + g(x(t-\tau)) = p(t) \tag{1.1}$$

with delay $\tau \geq 0$ is a challenging problem of current interest. In [10] it is shown that such solutions exist for continuous 2π -periodic $p : \mathbb{R} \to \mathbb{R}$ of mean $\overline{p} = 0$ and continuous $g : \mathbb{R} \to \mathbb{R}$ for which there exist $A \in [0, 1/\pi^2[$ and $C \geq 0$ such that, for all $|x|$ large enough, one has simultaneously

$$xg(x) > 0\,, \tag{1.2}$$

$$|g(x)| \leq A|x| + C\,. \tag{1.3}$$

This result (like that found in [19] for a somewhat different equation with more complicated *a priori* bounds) was obtained by means of Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree (see [11] and [12]). It generalizes, for the case $\overline{p} = 0$, a similar result obtained in [9] for $\overline{p} \in \mathbb{R}$ where, for all $|x|$ large enough, condition (1.2) is replaced by

$$\operatorname{sgn}(x)(g(x) - \overline{p}) > 0 \tag{1.4}$$

and condition (1.3) by

$$|g(x)| \leq C \tag{1.5}$$

for some $C > 0$. In practice though, conditions (1.2) and (1.4) are often not met, as in the case of the classical forced pendulum equation where $\tau = 0$ and $g(x) = a \sin x$ ($a > 0$). The result presented in [19] rests on a complex inequality which is also not applicable to the forced pendulum equation (since it then takes the form $0 > 0$). In [5] it is shown by means of coincidence degree that equation (1.1) with $\tau = 0$ and $g' < 0$ (which also does not hold for the pendulum equation) possesses a unique

2000 *Mathematics Subject Classification.* 34K13.
Key words and phrases. Duffing equations, periodic solutions, delay equations,
a priori bounds, contraction principle.

Submitted January 22, 2004. Published March 3, 2004.

2π-periodic solution if and only if $\overline{p} \in g(\mathbb{R})$. As shown in [1], there are cases where the nonconservative forced pendulum equation with periodic forcing (and $\tau = 0$) admits no periodic solution. (See also [18].) The results obtained here on the existence of twice continuously differentiable periodic solutions to equations that generalize (1.1) are applicable to the forced pendulum equation. As we shall see in Theorem 4.5, the contraction principle yields a result which contains the following:

Theorem 1.1. *Given $A \in]0, 1/\sqrt{2}[$, let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|g(x_2) - g(x_1)| < A|x_2 - x_1| \tag{1.6}$$

for all $x_1, x_2 \in \mathbb{R}$ and let φ be the solution of mean zero of $x'' = p - \overline{p}$, where $p : \mathbb{R} \to \mathbb{R}$ is a continuous 2π-periodic function of mean $\overline{p}$. If

$$\inf_{r\in\mathbb{R}} \overline{g}(\varphi + r) + \lambda'\|\varphi\|_H \le \overline{p} \le \sup_{r\in\mathbb{R}} \overline{g}(\varphi + r) - \lambda'\|\varphi\|_H$$

where

$$\overline{g}(\varphi + r) = \frac{1}{2\pi}\int_{[0,2\pi]} g(\varphi(t) + r)\, dt, \quad \lambda' = \frac{\sqrt{2}A^2}{1 - \sqrt{2}A},$$
$$\|\varphi\|_H = \big[\frac{1}{2\pi}\int_{[0,2\pi]} (\varphi(t) + \varphi'(t))^2\, dt\big]^{1/2},$$

then the Duffing equation

$$x''(t) + g(x(t - \tau)) = p(t)$$

with delay $\tau \in \mathbb{R}$ admits a twice continuously differentiable 2π-periodic solution.

An equation like that of the conservative forced pendulum $x'' + a \sin x = p$ where $c = \tau = 0$ and $g(x) = a \sin x$ ($a > 0$) satisfies the Lipschitz condition (1.6) for $A = a$ and so, by the theorem above, one has the existence of 2π-periodic solutions of this equation whenever the stated *a priori* bounds on $\overline{p}$ are respected and $a \in]0, 1/\sqrt{2}[$.

In this paper, one exploits the argument presented in [2] for Josephson's equation

$$x'' + cx' + dx' \cos x + a \sin x = p$$

with $a, c, d \in \mathbb{R}$. Note that Josephson's equation does not satisfy the Lipschitz condition (2.2) on which rests this paper. As for results on the existence of almost periodic solutions to the Duffing equation, one could no doubt employ the techniques used in [3] for Josephson's equation. This is left for future work.

2. Preliminaries

For a given $T > 0$, let $C(T)$ be the class of all continuous real-valued T-periodic functions on $\mathbb{R}$ and $L^1(T)$ the set of all real-valued T-periodic functions on $\mathbb{R}$ the restriction of which to the segment $[0, T]$ are Lebesgue integrable functions. Let $C^1(T)$ be the class of all continuously differentiable functions in $C(T)$, $C^2(T)$ the class of all twice continuously differentiable functions in $C(T)$ and $L^2(T)$ the Hilbert space of all $x \in L^1(T)$ with usual finite norm

$$\|x\|_2 = \big[\frac{1}{T}\int_{[0,T]} |x(t)|^2\, dt\big]^{1/2}.$$

The inner product on $L^2(T)$ associated with this norm is given by

$$\langle x, y\rangle_2 = \frac{1}{T}\int_{[0,T]} x(t)y(t)\,dt.$$

For a given $c \in \mathbb{R}$, let L be the linear operator

$$Lx = x'' + cx'.$$

The theorem in the previous section will be extended to the case where $p \in L^1(T)$ and the function $g : \mathbb{R}^3 \to \mathbb{R}$ in the Duffing equation

$$Lx(t) + g(t-\tau, x(t-\tau), x'(t-\tau)) = p(t) \tag{2.1}$$

with delay $\tau \in \mathbb{R}$ is continuous and such that $g(t, x, y)$ is T-periodic in $t \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$, and satisfies the Lipschitz condition

$$|g(t, x_2, y_2) - g(t, x_1, y_1)| \le A|x_2 - x_1| + B|y_2 - y_1| \tag{2.2}$$

for all $t \in \mathbb{R}$, suitable $A, B \in [0, \infty[$ and all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. This condition implies the inequality

$$\sup\{|g(t, x, y)| : t \in \mathbb{R}\} \le A|x| + B|y| + C \tag{2.3}$$

for all $(x, y) \in \mathbb{R}^2$ and some $C \ge 0$ (put, for example, $C = \sup\{|g(t, 0, 0)| : t \in \mathbb{R}\}$). The inequality (2.3) is a natural generalization of (1.3).

The mean $\overline{x}$ of any $x \in L^1(T)$ is given by the Lebesgue integral

$$\overline{x} = \frac{1}{T}\int_{[0,T]} x(t)\,dt$$

and x can be identified with its Fourier series

$$\sum_{n\in\mathbb{Z}} \widehat{x}(n)e^{in\omega t},$$

where $i = \sqrt{-1}$, $\omega = 2\pi/T$ and

$$\widehat{x}(n) = \frac{1}{T}\int_{[0,T]} x(t)e^{-in\omega t}\,dt.$$

Hence, $\widehat{x}(0) = \overline{x}$ and, since x is real-valued, $\widehat{x}(-n)$ is the complex conjugate of $\widehat{x}(n)$. The class $P(T)$ of real *trigonometric polynomials* is the subset of all $x \in L^1(T)$ with $\widehat{x}(n) = 0$ for all but at most finitely many $n \in \mathbb{Z}$. Given $S \subset L^1(T)$, $\widetilde{S}$ denotes that subset of $L^1(T)$ given by

$$\widetilde{S} = \{x - \overline{x} : x \in S\}$$

and so one has $x = \widetilde{x} + \overline{x}$ for all $x \in L^1(T)$. If $x, y \in \widetilde{L^1}(T)$ are such that

$$\int_{[0,T]} x(t)q'(t)\,dt = -\int_{[0,T]} y(t)q(t)\,dt$$

for all $q \in \widetilde{P}(T)$, then y is the *weak derivative* of x (denoted x') and x can be taken continuous by means of $x = z - \overline{z}$ where

$$z(t) = \int_{[0,t]} y(t)\,dt$$

for all $t \in \mathbb{R}$. Similarly, if $x, z \in \widetilde{L^1}(T)$ are such that

$$\int_{[0,T]} x(t)q''(t)\,dt = \int_{[0,T]} z(t)q(t)\,dt$$

for all $q \in \widetilde{P}(T)$, then z is the *weak second derivative* of x (denoted x'') and x can be taken continuously differentiable.

Let H be the subspace of $L^2(T)$ consisting of all $x \in C(T)$ with weak derivative $x' \in \widetilde{L^2}(T)$. On H one has the inner product

$$\langle x, y\rangle_H = \langle x, y\rangle_2 + \langle x', y'\rangle_2$$

with associated norm

$$\|x\|_H = [\|x\|_2^2 + \|x'\|_2^2]^{1/2} = \|x + x'\|_2$$

and, on $\widetilde{H}$, the well-known Sobolev inequality (see for example [14])

$$\sup_{0\le t\le T} |x(t)|^2 \le \frac{T^2}{12}\|x'\|_2^2.$$

Consequently, strong convergence in $\widetilde{H}$ implies uniform convergence (to an element of $\widetilde{C}(T)$). Furthermore, $\widetilde{H}$ is complete, as is now shown.

Proposition 2.1. *$\widetilde{H}$ is a Hilbert space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\widetilde{H}$. Then there exists $y \in \widetilde{L^2}(T)$ such that $\|y - x_n'\|_2 \to 0$ (as $n \to \infty$) and, by above, there exists $x \in \widetilde{C}(T)$ such that $\|x - x_n\|_2 \to 0$ (as $n \to \infty$). For any $q \in \widetilde{P}(T)$, the relation $\langle x_n, q'\rangle_2 = -\langle x_n', q\rangle_2$ holds for all n, and so in the limit as $n \to \infty$, $\langle x, q'\rangle_2 = -\langle y, q\rangle_2$. From this follows that $y = x'$ in $\widetilde{L^2}(T)$. This shows that $x \in \widetilde{H}$. $\square$

Remark 2.2. Sobolev's inequality yields

$$|x(t)| \le \frac{T}{\sqrt{12}}\|x\|_H$$

for all $t \in \mathbb{R}$. Thus, a point evaluation is a bounded linear functional on $\widetilde{H}$ and so, if a sequence $\{x_n\}$ converges weakly in $\widetilde{H}$ to x_0 (denoted $x_n \rightharpoonup x_0$), then it converges pointwise to x_0. By the Banach-Steinhaus theorem, $\{x_n\}$ is bounded in $\widetilde{H}$ and so, by the above inequality, $\{x_n(t)\}$ is a uniformly bounded sequence of functions.

For $\psi \in H$, put

$$g_\tau[\psi](t) = g(t-\tau, \psi(t-\tau), \psi'(t-\tau)).$$

Proposition 2.3. *Given a continuous function $g : \mathbb{R}^3 \to \mathbb{R}$ for which $g(t, x, y)$ is T-periodic in $t \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$, let there exist $A, B, C \in [0, \infty[$ for which condition (2.3) is satisfied for all $t, x, y \in \mathbb{R}$. Then $g_\tau[\psi] \in L^2(T)$ for all $\psi \in H$.*

Proof. One has, for any $\psi \in H$,

$$\|g_\tau[\psi]\|_2 \le \|A|\psi| + B|\psi'| + C\|_2 \le C + \sqrt{A^2 + B^2}\|\psi\|_H < \infty$$

and so $g_\tau[\psi] \in L^2(T)$. $\square$

One now introduces a subspace of $\widetilde{H}$ used later in section 3. Let $\mathcal{H}$ be the subspace of H given by

$$\mathcal{H} = \{x \in C^1(T) : x' \in H\}$$

and on which is defined the inner product

$$\langle x, y\rangle_{\mathcal{H}} = \langle x', y'\rangle_H .$$

This inner product is associated with the norm $\|x\|_{\mathcal{H}}$ on $\widetilde{\mathcal{H}}$ given by $\|x\|_{\mathcal{H}} = \|x'\|_H$.

Proposition 2.4. *$\widetilde{\mathcal{H}}$ is a Hilbert space.*

This result is proved like Proposition 2.1 with the sequences $\{x_n\}$ and $\{x'_n\}$ replaced by $\{x'_n\}$ and $\{x''_n\}$ respectively.

Remark 2.5. By Wirtinger's inequality $\omega\|x\|_2 \le \|x'\|_2$ on $\widetilde{H}$ (see, for example [14]) one obtains

$$\omega\|x\|_H \le \|x\|_{\mathcal{H}}$$

for all $x \in \widetilde{\mathcal{H}}$. Furthermore, Sobolev's inequality yields

$$|x'(t)| \le \frac{T}{\sqrt{12}}\|x'\|_H$$

for all $x \in \widetilde{\mathcal{H}}$ and so if a sequence $\{x_n\}$ converges weakly in $\widetilde{\mathcal{H}}$ to x_0 (also denoted $x_n \rightharpoonup x_0$) then x'_n converges pointwise to x'_0. The sequence $\{x_n\}$ also converges weakly to x_0 in $\widetilde{H}$ since, for all $q \in \widetilde{P}(T)$, one has

$$\lim_{n\to\infty} \langle x_n, q''\rangle_H = -\lim_{n\to\infty} \langle x_n, q\rangle_{\mathcal{H}} = -\langle x_0, q\rangle_{\mathcal{H}} = -\langle x'_0, q'\rangle_H = \langle x_0, q''\rangle_H$$

and from this follows that x_n also converges pointwise (to x_0).

Given $p \in L^1(T)$, there exists a unique function $\varphi \in \widetilde{C^1}(T) \subset \widetilde{H}$, with weak second derivative $\varphi'' \in \widetilde{L^1}(T)$, which satisfies the linear differential equation

$$Lx(t) = \widetilde{p}(t). \tag{2.4}$$

Furthermore $\varphi \in \widetilde{C^2}(T)$ whenever $p \in C(T)$. The substitution in (2.1) of $\varphi + x \in H$ in place of $x \in H$ yields the equivalent equation

$$Lx + g_\tau[\varphi + x] = \overline{p}. \tag{2.5}$$

The existence of a solution $x \in C^1(T)$ of (2.5) with weak second derivative $x'' \in \widetilde{L^2}(T)$ is equivalent to the existence of a scalar $r \in \mathbb{R}$ and of a function $x_r \in \widetilde{\mathcal{H}}$ such that $x = x_r$ is a solution of

$$Lx + g_\tau[\varphi + x + r] = \overline{p}$$

in $L^2(T)$. This, in turn, is equivalent to finding a scalar $r \in \mathbb{R}$ and a function $x_r \in \widetilde{\mathcal{H}}$ such that $x = x_r$ is simultaneously a solution of

$$Lx + \widetilde{g_\tau}[\varphi + x + r] = 0 \tag{2.6}$$

in $\widetilde{L^2}(T)$ and of

$$\overline{g_\tau}[\varphi + x + r] = \overline{p}. \tag{2.7}$$

To ease notation, let $g_{\tau,r}[x] = g_\tau[\varphi + x + r]$. Then equation (2.6) becomes

$$Lx + \widetilde{g_{\tau,r}}[x] = 0 \tag{2.8}$$

in $\widetilde{L^2}(T)$ while (2.7) takes the form

$$\overline{g_{\tau,r}}[x] = \overline{p}. \tag{2.9}$$

This reformulation of the original problem is modeled after that found in [7] for the forced pendulum equation.

3. The case $g = g(t, x, y)$

For a given $r \in \mathbb{R}$, equation (2.8) is equivalent to the system of equations

$$(n^2\omega^2 - in\omega c)\widehat{x}(n) = \widehat{g_{\tau,r}}[x](n)$$

for all $n \in \mathbb{Z}\backslash\{0\}$. Let $G_{\tau,r}$ be defined on $\widetilde{\mathcal{H}}$ by

$$G_{\tau,r}(x)(t) = \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{(n^2\omega^2 - in\omega c)} \widehat{g_{\tau,r}}[x](n)e^{in\omega t}. \tag{3.1}$$

The function

$$\gamma(t) = \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{(n^2\omega^2 - in\omega c)} e^{in\omega t}$$

lies in $\widetilde{H}$ and yields

$$G_{\tau,r}(x) = \gamma * g_{\tau,r}[x]$$

for all $x \in \widetilde{H}$, where the star represents the convolution operator.

3.1. **Contraction principle on $\widetilde{\mathcal{H}}$.** The Lipschitz condition (2.2) implies condition (2.3) and so, by Proposition 2.3 , $g_\tau[\psi] \in L^2(T)$ for all $\psi \in H$. Hence $G_{\tau,r} : \widetilde{H} \to \widetilde{\mathcal{H}} \subset \widetilde{H}$ and so $G_{\tau,r}$ maps $\widetilde{\mathcal{H}}$ into itself and $x = x_r \in \widetilde{\mathcal{H}}$ is a solution of (2.8) in the sense of $\widetilde{L^2}(T)$ whenever it is a fixed point of $G_{\tau,r}$ on $\widetilde{\mathcal{H}}$. The existence of such a fixed point is established by means of the contraction principle in the following theorem.

Theorem 3.1. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that $g(t, x, y)$ is T-periodic in $t \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$ and let $A, B \in [0, \infty[$ and $c \in \mathbb{R}$ be such that*

$$A^2 + B^2 < \omega^2 \Big(\frac{\omega^2 + \min\{1, c^2\}}{\omega^2 + 1}\Big) \tag{3.2}$$

and (2.2) is satisfied for all $t \in \mathbb{R}$ and all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then, given $p \in L^1(T)$ and $r \in \mathbb{R}$, there exists a unique continuously differentiable T-periodic function x_r of mean zero, with weak second derivative $x_r'' \in L^2(T)$, which is a solution of equation (2.8) in the sense of $L^2(T)$. Furthermore,

$$x_r = \lim_{n\to\infty} G^n_{\tau,r}(x) \tag{3.3}$$

in $L^2(T)$ for all $x \in L^2(T)$, and if $p \in C(T)$ then $x_r \in C^2(T)$.

Proof. For any $x, y \in \widetilde{\mathcal{H}}$ one has

$$\begin{aligned}
\|G_{\tau,r}(y) - G_{\tau,r}(x)\|^2_{\mathcal{H}} &= \sum_{n\in\mathbb{Z}\backslash\{0\}} \Big|\frac{(in\omega - n^2\omega^2)[\widehat{g_{\tau,r}}[y](n) - \widehat{g_{\tau,r}}[x](n)]}{n^2\omega^2 - in\omega c}\Big|^2 \\
&\le \sigma^2 \sum_{n\in\mathbb{Z}\backslash\{0\}} |\widehat{g_{\tau,r}}[y](n) - \widehat{g_{\tau,r}}[x](n)|^2 \\
&\le \sigma^2 \|g_{\tau,r}[y] - g_{\tau,r}[x]\|_2^2 \,,
\end{aligned}$$

where

$$\begin{aligned}\sigma &= \sup_{n\in\mathbb{Z}\setminus\{0\}} \Big[\frac{n^2\omega^2 + n^4\omega^4}{n^4\omega^4 + n^2\omega^2 c^2}\Big]^{1/2} \\ &= \begin{cases} 1 & \text{if } c^2 \geq 1 \\ \sqrt{\frac{\omega^2+1}{\omega^2+c^2}} & \text{if } c^2 < 1 \end{cases} \\ &= \sqrt{\frac{\omega^2+1}{\omega^2+\min\{1,c^2\}}}\end{aligned}$$

and so one obtains

$$\|G_{\tau,r}(y) - G_{\tau,r}(x)\|_{\mathcal{H}} \leq \frac{\sigma}{\omega}\sqrt{A^2+B^2}\|y-x\|_{\mathcal{H}}.$$

Hence, whenever (3.2) holds, $G_{\tau,r}$ is a contraction on $\widetilde{\mathcal{H}}$ and so admits a unique fixed point $x_r \in \widetilde{\mathcal{H}}$ given by the successive approximations (3.3) of Banach and Picard. $\square$

Since $\varphi'' \in L^1(T)$, the above result can be reformulated in terms of $z_r = \varphi + x_r + r$ as follows:

Corollary 3.2. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that $g(t,x,y)$ is T-periodic in $t \in \mathbb{R}$ for all $(x,y) \in \mathbb{R}^2$. Also, let $c \in \mathbb{R}$ and $A, B \in [0,\infty[$ be such that (2.2) and (3.2) hold for all $t \in \mathbb{R}$. Then, given $r \in \mathbb{R}$ and $p \in L^1(T)$, there exists a function $z_r \in C^1(T)$ of mean r, with weak second derivative $z_r'' \in L^1(T)$, such that for some constant $k_r \in \mathbb{R}$, $x = z_r$ is a solution in the sense of $L^1(T)$ of the Duffing equation*

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau), x'(t-\tau)) = p(t) + k_r$$

with delay $\tau \in \mathbb{R}$. Furthermore, if $p \in C(T)$ then $z_r \in C^2(T)$.

The constant k_r is given by

$$k_r = \frac{1}{T}\int_{[0,T]} [g(t, z_r(t), z_r'(t)) - p(t)]\,dt. \tag{3.4}$$

The conservative pendulum equation

$$x'' + a\sin x = p(t) \tag{3.5}$$

with $a \in \mathbb{R}$ and continuous T-periodic forcing $p(t)$, can be used to show that (2.8) may admit no solution that also solves (2.9) (i.e. for which $k_r \neq 0$ for all $r \in \mathbb{R}$). For example, if the constant term in the Fourier series of the forcing $p(t)$ is too great, then the pendulum will wind indefinitely about its fixed point, and so no periodic motion will be possible. (See also [1].)

When $\overline{p} \in \{\overline{g_{\tau,r}}[z_r] : r \in \mathbb{R}\}$, equation (2.9) is satisfied for some $r \in \mathbb{R}$ and one obtains the following resolution of the original equation (2.1).

Corollary 3.3. *If, in the context of the theorem, one has $\overline{p} = \overline{g_{\tau,r}}[z_r]$ for some $r \in \mathbb{R}$, then there exists a continuously differentiable function with weak second derivative in $L^1(T)$, which is a solution of equation (2.1) in the sense of $L^1(T)$. Furthermore, if $p \in C(T)$ then the solution is twice continuously differentiable.*

One now searches for conditions that will permit the existence of some $r \in \mathbb{R}$ such that $k_r = 0$ (i.e. such that x_r solves not only (2.8) but also (2.9)).

Lemma 3.4. *In the context of the previous theorem, if $r \to r_0$ in $\mathbb{R}$, then $x_r \rightharpoonup x_{r_0}$ in $\widetilde{\mathcal{H}}$.*

Proof. By the contraction principle, the set $\{x_r : r \in \mathbb{R}\}$ lies in a (weakly compact) ball in $\widetilde{\mathcal{H}}$ and so there exists a subsequence $\{x_{r_n}\}_{n=1}^{\infty}$ such that $r_n \to r_0$ and x_{r_n} converges weakly in $\widetilde{\mathcal{H}}$ to an element x_0 as $n \to \infty$. Thus, by Remark 2.5 both $\{x_{r_n}\}_{n=1}^{\infty}$ and $\{x'_{r_n}\}_{n=1}^{\infty}$ are uniformly bounded sequences which converge pointwise to x_0 and x'_0, respectively. Hence, for all $q \in \widetilde{P}(T)$, one has by Lebesgue's dominated convergence theorem

$$\lim_{n\to\infty} \langle Lx_{r_n}, q\rangle_2 = -\lim_{n\to\infty} \langle x'_{r_n} + cx_{r_n}, q'\rangle_2 = -\langle x'_0 + cx_0, q'\rangle_2 = \langle Lx_0, q\rangle_2$$

and

$$\lim_{n\to\infty} \langle g_{\tau,r_n}[x_{r_n}], q\rangle_2 = \langle g_{\tau,r_0}[x_0], q\rangle_2 .$$

Since $\langle Lx_{r_n} + g_{\tau,r_n}[x_{r_n}], q\rangle_2 = 0$ for all $n \in \mathbb{N}$, then in the limit as $n \to \infty$ one obtains

$$\langle Lx_0 + g_{\tau,r_0}[x_0], q\rangle_2 = 0$$

for all $q \in \widetilde{P}(T)$. By uniqueness, $x = x_{r_0}$ is the only solution in $\widetilde{\mathcal{H}}$ of

$$\langle Lx + g_{\tau,r_0}[x], q\rangle_2 = 0$$

and so one has $x_0 = x_{r_0}$. □

The intermediate value theorem can now be applied to justify the existence of a solution of equation (2.9) (i.e. the existence of $r \in \mathbb{R}$ for which $k_r = 0$). Thus, one has the following corollary to Theorem 3.1.

Corollary 3.5. *In the context of the theorem, equation (2.1) admits a continuously differentiable solution with weak second derivative in $L^1(T)$ if and only if*

$$\inf_{r\in\mathbb{R}} \overline{g}(t, \varphi + x_r + r, \varphi' + x'_r) \le \overline{p} \le \sup_{r\in\mathbb{R}} \overline{g}(t, \varphi + x_r + r, \varphi' + x'_r). \tag{3.6}$$

Furthermore, if p is continuous, then the solution is twice continuously differentiable.

The bounds in (3.6) being difficult to calculate in most cases, *a priori* bounds for $\overline{p}$ that imply condition (3.6) will now be obtained.

3.2. ***A priori* bounds for $\overline{p}$.** By (2.2) one has

$$\begin{aligned}
|\overline{g_{\tau,r}}[x_r] - \overline{g_{\tau,r}}[G^n_{\tau,r}(x)]| &\le \frac{1}{T}\int_{[0,T]} |g_{\tau,r}[x_r](t) - g_{\tau,r}[G^n_{\tau,r}(x)](t)|\,dt \\
&\le \frac{1}{T}\int_{[0,T]} A|x_r(t) - [G^n_{\tau,r}(x)](t)|\,dt \\
&\quad + \frac{1}{T}\int_{[0,T]} B|x'_r(t) - [G^n_{\tau,r}(x)]'(t)|\,dt \\
&\le \sqrt{A^2+B^2}\|x_r - G^n_{\tau,r}(x)\|_H \\
&\le \sqrt{A^2+B^2}\sum_{k=n}^{\infty}\|G^{k+1}_{\tau,r}(x) - G^k_{\tau,r}(x)\|_H \\
&\le \sqrt{A^2+B^2}\|G_{\tau,r}(x) - x\|_H \sum_{k=n}^{\infty}(\frac{\beta}{\omega})^k \\
&= \frac{(\beta/\omega)^n}{1-\beta/\omega}\sqrt{A^2+B^2}\|G_{\tau,r}(x) - x\|_H\,,
\end{aligned}$$

where

$$\beta = \sigma\sqrt{A^2+B^2} = \sqrt{\frac{(A^2+B^2)(\omega^2+1)}{\omega^2+\min\{1,c^2\}}}\,. \tag{3.7}$$

Hence (3.6) holds whenever, for an $x \in \widetilde{\mathcal{H}}$ and some $n \in \mathbb{N}$, one has

$$\begin{aligned}
&\inf_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(x)] + \lambda_n\|G_{\tau,r}(x) - x\|_H \\
&\le \overline{p} \\
&\le \sup_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(x)] - \lambda_n\|G_{\tau,r}(x) - x\|_H\,,
\end{aligned}$$

where

$$\lambda_n = \frac{(\beta/\omega)^n}{1-\beta/\omega}\sqrt{(A^2+B^2)}\,.$$

For $x = -\varphi$ one has $G_{\tau,r}(-\varphi) = 0$ and so (3.6) holds whenever, for some $n \in \mathbb{N}$, one has

$$\inf_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(-\varphi)] + \lambda_n\|\varphi\|_H \le \overline{p} \le \sup_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(-\varphi)] - \lambda_n\|\varphi\|_H\,, \tag{3.8}$$

where

$$\overline{g_{\tau,r}}[G^n_{\tau,r}(-\varphi)] = \frac{1}{T}\int_{[0,T]} g_{\tau,r}[G^n_{\tau,r}(-\varphi)]\,dt\,.$$

The following statement subsumes what has been proved for the case $n = 1$.

Theorem 3.6. *Let $p : \mathbb{R} \to \mathbb{R}$ with mean $\overline{p}$ be a T-periodic function which is Lebesgue integrable on $[0,T]$ and, for a given $c \in \mathbb{R}$, let φ be the continuously differentiable solution of mean zero of the linear equation $x'' + cx' = p - \overline{p}$. Also let $A, B \in [0,\infty[$ be such that*

$$\beta = \sqrt{\frac{(A^2+B^2)(\omega^2+1)}{(\omega^2+\min\{1,c^2\})}} < \omega\,,$$

where $\omega = 2\pi/T$ and let $g : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function for which $g(t,x,y)$ is T-periodic in $t \in \mathbb{R}$ for all $(x,y) \in \mathbb{R}^2$ and such that

$$|g(t,x_2,y_2) - g(t,x_1,y_1)| \le A|x_2 - x_1| + B|y_2 - y_1|$$

for all $t \in \mathbb{R}$ and all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. If

$$\inf_{r\in\mathbb{R}}[\overline{g}(t,\varphi + r,\varphi') + \lambda\|\varphi\|_H] \le \overline{p} \le \sup_{r\in\mathbb{R}}[\overline{g}(t,\varphi + r,\varphi') - \lambda\|\varphi\|_H]\,,$$

where

$$\begin{gathered}\overline{g}(t,\varphi + r,\varphi') = \frac{1}{T}\int_{[0,T]} g(t,\varphi(t) + r,\varphi'(t))\,dt\,,\\ \|\varphi\|_H = [\frac{1}{T}\int_{[0,T]} |\varphi(t) + \varphi'(t)|^2\,dt]^{1/2}\,,\\ \lambda = \frac{\beta}{\omega - \beta}\sqrt{(A^2 + B^2)}\,.\end{gathered}$$

Then the Duffing equation

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau), x'(t-\tau)) = p(t)$$

with delay $\tau \in \mathbb{R}$ admits a continuously differentiable T -periodic solution with weak second derivative which is Lebesgue integrable on $[0,T]$. Furthermore, if p is continuous, then the solution is twice continuously differentiable.

Example 3.7. For $\alpha \in \mathbb{R}$, the Duffing equation with delay $\tau \in \mathbb{R}$

$$x''(t) + \frac{1}{3}\cos^2(t-\tau)\ln(1 + x^2(t-\tau) + (x')^2(t-\tau)) = \alpha + \sin t$$

is such that $c = 0$, $T = 2\pi$ (and so $\omega = 1$), $\varphi(t) = -\sin t$ and

$$g(t,x,y) = \frac{1}{3}\cos^2(t)\ln(1 + x^2 + y^2).$$

Since

$$|\ln(1 + x_2^2 + y_2^2) - \ln(1 + x_1^2 + y_1^2)| \le |x_2 - x_1| + |y_2 - y_1|$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, then

$$|g(t,x_2,y_2) - g(t,x_1,y_1)| \le \frac{1}{3}[|x_2 - x_1| + |y_2 - y_1|]$$

and so one has (2.2) for $A = B = \frac{1}{3}$. Furthermore $\beta = 2/3 < 1$, $\|\varphi\|_H = 1$, $\lambda = 2\sqrt{2}/3$,

$$\begin{aligned}\inf_{r\in\mathbb{R}}\overline{g}(t,\varphi + r,\varphi') &= \inf_{r\in\mathbb{R}}\frac{1}{6\pi}\int_{[0,2\pi]}(\cos^2 t)\ln(1 + (r - \sin t)^2 + \cos^2 t)\,dt\\ &\le \inf_{r\in\mathbb{R}}\frac{1}{6\pi}\int_{[0,2\pi]}(\cos^2 t)\ln(1 + (|r| + |\sin t|)^2 + \cos^2 t)\,dt\\ &= \frac{1}{6\pi}\ln 2\int_{[0,2\pi]}\cos^2 t\,dt\\ &= \frac{1}{6}\ln 2\end{aligned}$$

and

$$\sup_{r\in\mathbb{R}}\overline{g}(t,\varphi + r,\varphi') = \sup_{r\in\mathbb{R}}\frac{1}{6\pi}\int_{[0,2\pi]}\ln(1 + (r - \sin t)^2 + \cos^2 t) = \infty.$$

Hence, by the previous theorem, the Duffing equation admits a 2π -periodic solution whenever

$$\frac{1}{6}\ln 2+\frac{2\sqrt{2}}{3}\leq \alpha<\infty .$$

The example above does not fulfill condition (1.4) and so the results in [10] and [19] (as well as those in [4], [6], [8] [13] and [15] for the case $\tau=0$) do not apply.

4. The case $g = g(t, x)$

In this case condition (2.2) becomes

$$|g(t,x_2)-g(t,x_1)|\leq A|x_2-x_1| \tag{4.1}$$

for suitable $A\geq 0$ and all $t,x_{1,}x_2\in\mathbb{R}$, and so (2.3) becomes

$$\sup\{|g(t,x)|:t\in\mathbb{R}\}\leq A|x|+C \tag{4.2}$$

for some $C\geq 0$ and all $x\in\mathbb{R}$., One can take $C=\sup\{|g(t,0)|:t\in\mathbb{R}\}$, for example.

4.1. **Contraction principle on $\widetilde{H}$.**

By Proposition 2.3, one has $G_{\tau,r}:\widetilde{H}\to\widetilde{\mathcal{H}}\subset\widetilde{H}$ and so $x=x_r\in\widetilde{H}$ is a solution of (2.8) in the sense of $\widetilde{L^2}(T)$ if and only if it is a fixed point of $G_{\tau,r}$ on $\widetilde{H}$. Thus, the following analog of Theorem 3.1 for the case $g=g(t,x)$ does not require the space $\widetilde{\mathcal{H}}$ for its proof. The less restrictive space $\widetilde{H}$ is sufficient and this results in a somewhat different (and more useful) inequality than (3.2).

Theorem 4.1. *Let $g:\mathbb{R}^2\to\mathbb{R}$ be a continuous function such that $g(t,x)$ is T-periodic in $t\in\mathbb{R}$ for all $x\in\mathbb{R}$ and let $A\geq 0$ and $c\in\mathbb{R}$ be such that*

$$A^2<\omega^2\big(\frac{\omega^2+c^2}{\omega^2+1}\big) \tag{4.3}$$

holds and (4.1) is satisfied for all $t,x_1,x_2\in\mathbb{R}$. Then, given $p\in L^1(T)$ and $r\in\mathbb{R}$, there exists a unique continuously differentiable T-periodic function x_r of mean zero, with weak second derivative $x_r''\in L^2(T)$, which is a solution of equation (2.8) in the sense of $L^2(T)$. Furthermore,

$$x_r=\lim_{n\to\infty}G^n_{\tau,r}(x)$$

in $L^2(T)$ for all $x\in L^2(T)$, and if $p\in C(T)$ then $x_r\in C^2(T)$.

Proof. Proceeding as in the proof of Theorem 3.1, one obtains for all $x,y\in\widetilde{H}$,

$$\|G_{\tau,r}(y)-G_{\tau,r}(x)\|_H\leq\beta'\|y-x\|_H$$

where

$$\beta'=\sigma'A=\frac{A}{\omega}\sqrt{\frac{\omega^2+1}{\omega^2+c^2}}<1\,.$$

Hence $G_{\tau,r}$ is a contraction on $\widetilde{H}$ and so admits a unique fixed point $x_r\in\widetilde{H}$ given by the successive approximations of Banach and Picard. ☐

As was done for Corollary 3.2, one can also reformulate the result above in terms of $z_r=\varphi+x_r+r$ where φ is the unique T -periodic solution of mean zero of (2.4) with $\varphi''\in L^1(T)$.

Corollary 4.2. *Let $c \in \mathbb{R}$ and $A \geq 0$ be such that (4.1) and (4.3) hold and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $g(t,x)$ is T-periodic in $t \in \mathbb{R}$ for all $x \in \mathbb{R}$. Then, given $r \in \mathbb{R}$ and $p \in L^1(T)$ of mean $\overline{p}$, there exists $z_r \in C^1(T)$ of mean r with weak second derivative $z_r'' \in L^1(T)$ such that, for some $k_r \in \mathbb{R}$, $x = z_r$ is a solution in the sense of $L^1(T)$ of the Duffing equation*

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau)) = p(t) + k_r$$

with delay $\tau \in \mathbb{R}$. Furthermore, if $p \in C(T)$ then $z_r \in C^2(T)$.

The constant k_r is again given by (3.4). The following lemma permits one to deduce, under condition (3.6), the existence of some $r \in \mathbb{R}$ such that $k_r = 0$ (i.e. such that x_r is also a solution of (2.9)).

Lemma 4.3. *In the context of the previous theorem, if $r \to r_0$ in $\mathbb{R}$ then $x_r \rightharpoonup x_{r_0}$ in $\widetilde{H}$.*

The proof is like that of Lemma 3.4. The intermediate value theorem now yields the following corollary to Theorem 3.6.

Corollary 4.4. *In the context of the previous theorem, equation (2.1) admits a continuously differentiable solution with weak second derivative in $L^1(T)$ if and only if (3.6) is satisfied. Furthermore, if p is continuous, then the solution is twice continuously differentiable.*

Clearly condition (3.6) holds whenever one has $\operatorname{sgn}(x)(g(t,x) - \overline{p}) > 0$ for all $t \in \mathbb{R}$ and all $|x|$ large enough. This is essentially condition (1.4) found in [9].

4.2. ***A priori* bounds for $\overline{p}$.** Proceeding as in section 3.2, one has, for all $x \in \widetilde{H}$ and all $n \in \mathbb{N}$,

$$|\overline{g_{\tau,r}}[x_r] - \overline{g_{\tau,r}}[G^n_{\tau,r}(x)]| \leq \frac{A(\beta')^n}{1-\beta'} \|G_{\tau,r}(x) - x\|_2 \,,$$

where

$$\beta' = A\sigma' = \frac{A}{\omega}\sqrt{\frac{\omega^2+1}{\omega^2+c^2}} \,.$$

Hence condition (3.6) holds whenever

$$\inf_{r\in\mathbb{R}} \overline{g_{\tau,r}}(G^n_{\tau,r}(x)) + \lambda'_n \|G_{\tau,r}(x) - x\|_2 \leq \overline{p} \leq \sup_{r\in\mathbb{R}} \overline{g_{\tau,r}}(G^n_{\tau,r}(x)) - \lambda'_n \|G_{\tau,r}(x) - x\|_2$$

for some $x \in \widetilde{H}$ and an $n \in \mathbb{N}$, where

$$\lambda'_n = \frac{A(\beta')^n}{1-\beta'} \,.$$

The case $x = -\varphi$ and $n = 1$ yields the following:

Theorem 4.5. *Let $p : \mathbb{R} \to \mathbb{R}$ with mean $\overline{p}$ be a T-periodic function which is Lebesgue integrable on $[0,T]$ and φ be the continuously differentiable solution of mean zero of the linear equation $x'' + cx' = p - \overline{p}$ for a given $c \in \mathbb{R}$. Also let $A \in [0,\infty[$ be such that*

$$\beta' = \frac{A}{\omega}\sqrt{\frac{\omega^2+1}{\omega^2+c^2}} < 1 \,,$$

where $\omega = 2\pi/T$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function for which $g(t,x)$ is T-periodic in $t \in \mathbb{R}$ for all $x \in \mathbb{R}$ and such that

$$|g(t,x_2) - g(t,x_1)| \le A|x_2 - x_1|$$

for all $t, x_1, x_2 \in \mathbb{R}$. If

$$\inf_{r\in\mathbb{R}}[\overline{g}(t,\varphi + r) + \lambda'\|\varphi\|_2] \le \overline{p} \le \sup_{r\in\mathbb{R}}[\overline{g}(t,\varphi + r) - \lambda'\|\varphi\|_2]\,,$$

where

$$\|\varphi\|_2 = [\frac{1}{T}\int_{[0,T]} |\varphi(t)|^2\,dt]^{\frac{1}{2}} \quad \text{and} \quad \lambda' = \frac{A\beta'}{1-\beta'}$$

then the equation

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau)) = p(t)$$

with delay $\tau \in \mathbb{R}$ admits a continuously differentiable T-periodic solution with weak second derivative which is Lebesgue integrable on $[0,T]$. Furthermore, if p is continuous, then the solution is twice continuously differentiable.

To show that the results of this section can be applied to cases not covered by section 3.2, consider the following example where $\beta > 1$ and $\beta' < 1$.

Example 4.6. *For $\alpha \in \mathbb{R}$ the equation with delay $\tau \in \mathbb{R}$*

$$x''(t) + 2x'(t) + \sqrt{2}\cos^2(t-\tau)\ln(1 + x^2(t-\tau)) = \alpha + \sin t - 2\cos t$$

is such that $c = 2$, $T = 2\pi$ (and so $\omega = 1$), $\varphi(t) = -\sin t$ and

$$g(t,x) = \sqrt{2}\cos^2(t)\ln(1+x^2).$$

Hence

$$|g(t,x_2) - g(t,x_1)| \le \sqrt{2}|x_2 - x_1|$$

and so one has (4.1) for $A = \sqrt{2}$. Furthermore $\beta' = 2/\sqrt{5} < 1$ (and $\beta = \sqrt{2} > 1$), $\|\varphi\|_2 = 1/\sqrt{2}$, $\lambda' = 2\sqrt{2}/(\sqrt{5}-2)$,

$$\begin{aligned}
\inf_{r\in\mathbb{R}} \overline{g}(t,\varphi + r) &= \inf_{r\in\mathbb{R}} \frac{\sqrt{2}}{2\pi}\int_{[0,2\pi]} \cos^2(t)\ln(1 + (r - \sin t)^2)\,dt \\
&\le \inf_{r\in\mathbb{R}} \frac{\sqrt{2}}{2\pi}\int_{[0,2\pi]} \cos^2(t)\ln(1 + (|r| + |\sin t|)^2)\,dt \\
&< \frac{\sqrt{2}}{2\pi}\ln 2\int_{[0,2\pi]} \cos^2(t)\,dt \\
&= \frac{1}{\sqrt{2}}\ln 2
\end{aligned}$$

and

$$\sup_{r\in\mathbb{R}} \overline{g}(t,\varphi + r) = \sup_{r\in\mathbb{R}} \frac{\sqrt{2}}{2\pi}\int_{[0,2\pi]} \cos^2(t)\ln(1 + (r - \sin t)^2)\,dt = \infty\,.$$

Hence, by the previous theorem, the delay equation admits a 2π-periodic solution whenever

$$\frac{1}{\sqrt{2}}\ln 2 + \frac{2}{(\sqrt{5}-2)} \le \alpha < \infty\,.$$

5. The case of bounded $g = g(t,x)$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a bounded continuous function such that $g(t,x)$ is T-periodic in $t \in \mathbb{R}$ for all $x \in \mathbb{R}$. Condition (4.2) then reduces to

$$\sup\{|g(t,x)| : t \in \mathbb{R}\} \le C \tag{5.1}$$

for some $C \ge 0$ and all $x \in \mathbb{R}$.

5.1. **Contraction principle on $\widetilde{L^2}(T)$.** Suppose that one has simultaneously conditions (4.1) and (5.1). Since g is bounded and continuous, one has $g_{\tau,r}[x] \in L^2(T)$ for all $x \in L^2(T)$ and so (3.1) defines $G_{\tau,r}$ as a function from $\widetilde{L^2}(T)$ to $\widetilde{\mathcal{H}} \subset \widetilde{L^2}(T)$. Hence a fixed point of $G_{\tau,r}$ on $\widetilde{L^2}(T)$ lies in $\widetilde{\mathcal{H}}$. Proceeding as in the proof of Theorem 3.1, one obtains for all $x, y \in \widetilde{L^2}(T)$,

$$\|G_{\tau,r}(y) - G_{\tau,r}(x)\|_2^2 \le (\beta'')^2 \|y - x\|_2^2$$

where

$$\beta'' = \frac{A}{\omega\sqrt{\omega^2 + c^2}}\,.$$

Hence, when $\beta'' < 1$, $G_{\tau,r}$ is a contraction on $\widetilde{L^2}(T)$ and so admits a unique fixed point $x_r \in \widetilde{\mathcal{H}}$ given by the successive approximations of Banach and Picard. This proves the following statement.

Theorem 5.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a bounded continuous function such that $g(t,x)$ is T-periodic in $t \in \mathbb{R}$ for all $x \in \mathbb{R}$ and let $A \ge 0$ and $c \in \mathbb{R}$ be such that*

$$A < \omega\sqrt{\omega^2 + c^2} \tag{5.2}$$

holds and (4.1) is satisfied for all $t, x_1, x_2 \in \mathbb{R}$. Then, given $p \in L^1(T)$ and $r \in \mathbb{R}$, there exists a unique continuously differentiable T -periodic function x_r of mean zero, with weak second derivative $x_r'' \in L^2(T)$, which is a solution of equation (2.8) in the sense of $L^2(T)$. Furthermore,

$$x_r = \lim_{n\to\infty} G^n_{\tau,r}(x)$$

in $L^2(T)$ for all $x \in L^2(T)$, and if $p \in C(T)$ then $x_r \in C^2(T)$.

Condition (5.2) is an improvement over (4.3) which, in turn, is an improvement over (3.2) with $B = 0$. There are important cases where g satisfies both the Lipschitz condition (4.1) and the boundedness condition (5.1). For example, the nonconservative pendulum equation

$$x'' + cx' + a\sin x = p \tag{5.3}$$

with $a, c \in \mathbb{R}$ and forcing $p \in L^1(T)$ is such that these two conditions hold for $A = C = |a|$. Hence (5.2) becomes

$$|a| < \omega\sqrt{\omega^2 + c^2}$$

and so in this manner one obtains the result stated in [18] (and proved in [7] and [14]) to the effect that the equation

$$x'' + a\sin x = p(t) + (a\overline{\sin} x - \overline{p})$$

admits a twice continuously differentiable T-periodic solution of mean zero whenever (5.2) holds.

The intermediate value theorem now yields the following statement.

Corollary 5.2. *In the context of the previous theorem, equation (2.1) admits a continuously differentiable solution with weak second derivative in $L^1(T)$ if and only if (3.6) is satisfied. Furthermore, if p is continuous, then the solution is twice continuously differentiable.*

The pendulum equation (5.3) is such that the function $g(t,x) = a\sin x$ is 2π-periodic in x for all $t \in \mathbb{R}$. This fact yields results for (5.3) that are stronger than the above corollary and that have been given in [18] (and generalized in [2]). Moreover, we point out that when (5.2) holds the techniques employed in [18] yield the added result that the map $r \to x_r$ from $\mathbb{R}$ to $C^2(T)$ is analytic. On the other hand, [16] provides an upper bound, as a function of $\overline{p}$, for the number of T-periodic solutions of (5.3) when $aT^2 < 18\sqrt{3}$, $c = 0$ and $p \in L^1(T)$. No condition other than (3.6) is imposed on p. Clearly the condition $aT^2 < 18\sqrt{3}$ in [16] is more restrictive than the inequality $aT^2 < 4\pi^2$ imposed in [2] and [18] for the case $c = 0$. Similar results, applicable to the conservative pendulum equation, are obtained in [17] for equation (2.1) with $\tau = 0$, $c = 0$ and $g(t,x) = \frac{\partial}{\partial x}V(t,x)$ where the potential $V(t,x)$ must satisfy certain subquadraticity conditions. No similar result is obtained here. On the other hand, the elementary contraction principle allows one in the following section to replace condition (3.6) by more manageable *a priori* bounds.

Example 5.3. Given a continuous 2π-periodic function $p : \mathbb{R} \to \mathbb{R}$, the equation with delay $\tau \in \mathbb{R}$

$$x''(t) + x'(t) + \sqrt{\frac{3}{2}}\cos(t-\tau)\sin x(t-\tau) = p(t)$$

is such that $c = 1$, $T = 2\pi$ (and so $\omega = 1$), $g(t,x) = \sqrt{3/2}\cos t \sin x$, and so one has (4.1) and (5.2) for $A = \sqrt{3/2}$. One notes that $\beta'' = \sqrt{3}/2 < 1$ and $\beta' = \sqrt{3/2} > 1$, and so (5.2) is satisfied while (4.3) is not. Thus, by the previous corollary, the delay equation admits a twice continuously differentiable 2π-periodic solution if and only if (3.6) holds.

5.2. ***A priori bounds for $\overline{p}$.*** Proceeding as in section 4.2, one has

$$|\overline{g_{\tau,r}}[x_r] - \overline{g_{\tau,r}}[G^n_{\tau,r}(x)]| \le \frac{A(\beta'')^n}{1-\beta''}\|G_{\tau,r}(x) - x\|_2$$

for all $x \in \widetilde{L^2}(T)$ and all $n \in \mathbb{N}$. Hence condition (3.6) holds whenever

$$\inf_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(x)] + \lambda''_n\|G_{\tau,r}(x) - x\|_2 \le \overline{p} \le \sup_{r\in\mathbb{R}} \overline{g_{\tau,r}}[G^n_{\tau,r}(x)] - \lambda''_n\|G_{\tau,r}(x) - x\|_2 \tag{5.4}$$

for some $x \in \widetilde{L^2}(T)$ and an $n \in \mathbb{N}$, where

$$\lambda''_n = \frac{A(\beta'')^n}{1-\beta''}.$$

Thus the case $x = -\varphi$ and $n = 1$ yields the following result.

Theorem 5.4. *Let $p : \mathbb{R} \to \mathbb{R}$ with mean $\overline{p}$ be a T-periodic function which is Lebesgue integrable on $[0,T]$ and φ be the continuously differentiable solution of mean zero of the linear equation $x'' + cx' = p - \overline{p}$ for a given $c \in \mathbb{R}$. Also let $A \in [0,\infty[$ be such that*

$$\beta'' = \frac{A}{\omega\sqrt{\omega^2 + c^2}} < 1$$

where $\omega = 2\pi/T$ *and let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be a bounded continuous function for which* $g(t,x)$ *is* T *-periodic in* $t \in \mathbb{R}$ *for all* $x \in \mathbb{R}$ *and such that*

$$|g(t,x_2) - g(t,x_2)| \leq A|x_2 - x_1|$$

for all $t, x_1, x_2 \in \mathbb{R}$. *If*

$$\inf_{r\in\mathbb{R}} \overline{g}(t,\varphi+r) + \lambda''\|\varphi\|_2 \leq \overline{p} \leq \sup_{r\in\mathbb{R}} \overline{g}(t,\varphi+r) - \lambda''\|\varphi\|_2$$

where

$$\|\varphi\|_2 = [\frac{1}{T}\int_{[0,T]} |\varphi(t)|^2\,dt]^{1/2} \quad \textit{and} \quad \lambda'' = \frac{A\beta''}{1-\beta''}$$

then the equation

$$x''(t) + cx'(t) + g(t-\tau, x(t-\tau)) = p(t)$$

with delay $\tau \in \mathbb{R}$ *admits a continuously differentiable* T *-periodic solution with weak second derivative which is Lebesgue integrable on* $[0,T]$. *Furthermore, if* p *is continuous, then the solution is twice continuously differentiable.*

The above theorem can now be applied to the forced pendulum equation.

Example 5.5. *Given* $T > 0$ *and* $\omega = 2\pi/T$, *suppose that the nonconservative pendulum equation*

$$x''(t) + cx'(t) + a\sin x(t) = \alpha + b\sin\omega t$$

with forcing $p(t) = \alpha + b\sin\omega t$ *is such that* $|a| < \omega\sqrt{\omega^2+c^2}$. *Then*

$$\varphi(t) = -\frac{b\omega}{c^2\omega^2+\omega^4}[c\cos\omega t + \omega\sin\omega t]$$

is the T*-periodic solution of mean zero of the linear equation*

$$x''(t) + cx'(t) = b\sin\omega t.$$

One has

$$\|\varphi\|_2 = \frac{|b|}{\sqrt{2}\sqrt{\omega^2+c^2}}$$

and, by Maclaurin's series expansion for trigonometric functions,

$$\begin{aligned}
\overline{\sin}\varphi &= \overline{\sin}(-\frac{be^{i\omega t}}{2(c\omega+i\omega^2)} - \frac{be^{-i\omega t}}{2(c\omega-i\omega^2)}) \\
&= \frac{1}{T}\int_0^T \sin(-\frac{be^{i\omega t}}{2(c\omega+i\omega^2)})\cos(\frac{be^{-i\omega t}}{2(c\omega-i\omega^2)})\,dt \\
&\quad + \frac{1}{T}\int_0^T \cos(\frac{be^{i\omega t}}{2(c\omega+i\omega^2)})\sin(-\frac{be^{-i\omega t}}{2(c\omega-i\omega^2)})\,dt \\
&= 0\,.
\end{aligned}$$

Thus,

$$\inf_{r\in\mathbb{R}} a\overline{\sin}(\varphi+r) = \inf_{r\in\mathbb{R}} a[\overline{\cos}\varphi\sin r + \overline{\sin}\varphi\cos r] = \inf_{r\in\mathbb{R}} a\overline{\cos}\varphi\sin r = -|a\overline{\cos}\varphi|$$

and

$$\sup_{r\in\mathbb{R}} a\overline{\sin}(\varphi+r) = \sup_{r\in\mathbb{R}} a[\overline{\cos}\varphi\sin r + \overline{\sin}\varphi\cos r] = \sup_{r\in\mathbb{R}} a\overline{\cos}\varphi\sin r = |a\overline{\cos}\varphi|$$

and so, by the previous theorem, one is assured of the existence of a twice continuously differentiable T-periodic solution whenever α satisfies

$$-|a\overline{\cos}\varphi| + \lambda'' \frac{|ab|}{\sqrt{2}\sqrt{\omega^2+c^2}} < \alpha < |a\overline{\cos}\varphi| - \lambda'' \frac{|ab|}{\sqrt{2}\sqrt{\omega^2+c^2}}$$

where

$$\lambda'' = \sqrt{\frac{a^2}{\omega\sqrt{\omega^2+c^2} - |a|}}$$

and

$$\begin{aligned}
\overline{\cos}\varphi &= \overline{\cos}\Big(-\frac{be^{i\omega t}}{2(c\omega+i\omega^2)} - \frac{be^{-i\omega t}}{2(c\omega-i\omega^2)}\Big)\\
&= \frac{1}{T}\int_0^T \cos\Big(\frac{be^{i\omega t}}{2(c\omega+i\omega^2)}\Big)\cos\Big(\frac{be^{-i\omega t}}{2(c\omega-i\omega^2)}\Big)\,dt\\
&\quad -\frac{1}{T}\int_0^T \sin\Big(\frac{be^{i\omega t}}{2(c\omega+i\omega^2)}\Big)\sin\Big(\frac{be^{-i\omega t}}{2(c\omega-i\omega^2)}\Big)\,dt\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2^n(n!))^2}\Big(\frac{b^2}{c^2\omega^2+\omega^4}\Big)^n.
\end{aligned}$$

In many cases the *a priori* bounds of the previous theorem are never satisfied. For example, the equation with delay $\tau \in \mathbb{R}$

$$x''(t) + x'(t) + \sqrt{\frac{3}{2}}\cos(t-\tau)\sin x(t-\tau) = \alpha - \sin t + \cos t \tag{5.5}$$

is such that $\varphi(t) = \sin t$. Hence

$$\overline{g}(t, \varphi + r) = \frac{\sqrt{3/2}}{2\pi}\int_0^{2\pi} \cos t \sin(r + \sin t)\,dt = 0$$

and so

$$\inf_{r\in\mathbb{R}} \overline{g}(t, \varphi + r) = \sup_{r\in\mathbb{R}} \overline{g}(t, \varphi + r) = 0.$$

Thus, the *a priori* bounds of the previous theorem are not satisfied here. So one needs to apply (5.4) for different choices than $x = -\varphi$ and/or $n = 1$.

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