Electronic Journal of Differential Equations, Vol. 2004(2004), No. 43, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

REDUCIBILITY OF ZERO CURVATURE EQUATIONS

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Abstract. By introducing a natural reducibility definition for zero curvature equations, we give a Floquet representation for such systems and show applications to the reducibility problem for quasiperiodic 2-dimensional linear systems and to fiberwise linear dynamical systems on trivial vector bundles.

1. Introduction

The zero curvature equation is a differential equation on matrix functions in two variables associated to flat linear connections on vector bundles with 2-dimensional base space. This equation is found in various situations in geometry and dynamics, particularly in the theory of harmonic maps and integrable systems [2, 8] or in the theory of connections and its applications to Hamiltonian dynamics on vector bundles [1, 5, 6].

The solvability of the zero curvature equation is equivalent to the existence of a common solution for a pair of linear dynamical systems on matrix functions. Using this fact, we describe the set of solutions on the class of pairs of matrix functions with periodic coefficients in each one of its two variables and show that the problem is reduced to the solvability of a Lax type equation.

By introducing a natural concept of reducibility for zero curvature equations, as the existence of a global "gauge" transformation reducing the initial equation to another one with constant coefficients, we give a Floquet representation for such equations and discuss the reducibility problem for quasiperiodic linear dynamical systems and for fiberwise linear dynamical systems on vector bundles with 2-dimensional base space.

The reducibility problem for quasiperiodic systems, has been studied by some authors [3, 4], and conditions of analytical or arithmetical type have been given in order to solve this problem. Here, we clarify the reducibility property for such systems in terms of the geometrical meaning of the zero curvature equation.

The second application concerns the reducibility of fiberwise linear dynamical systems on vector bundles. We give conditions for such reducibility in terms of the

2000 *Mathematics Subject Classification.* 34A30, 34A26, 34C20, 37C55.
Key words and phrases. Zero-curvature equation, reducibility, Floquet representation, quasiperiodic linear systems, fiberwise linear dynamical system.

Submitted November 26, 2003. Published March 24, 2004.
Partially supported by grant 35212-E from the the Consejo Nacional de Ciencia y Tecnología de México.

solvability of zero curvature type equations and explain the kind of obstructions that can be found.

2. The zero curvature equation

We start with the following definition.

Definition 2.1. A pair (Θ_1, Θ_2) of $n \times n$ smooth complex matrix functions in variables $(s,t) \in \mathbb{R}^2$, is called a **compatible pair** if there exists a $n \times n$ smooth complex matrix function $G(s,t)$ satisfying simultaneously the matrix linear systems

$$\frac{\partial G}{\partial s}(s,t) = \Theta_1(s,t)G(s,t), \tag{2.1}$$

$$\frac{\partial G}{\partial t}(s,t) = \Theta_2(s,t)G(s,t) \tag{2.2}$$

with initial condition $G(0,0) = I$.

Note that any pair (K_1, K_2) of constant commuting matrices is a compatible pair. The common solution G in (2.1-2.2) takes the form

$$G(s,t) = \exp(K_1 s + K_2 t).$$

If (Θ_1, Θ_2) is a compatible pair, the property of second mixed derivatives

$$\frac{\partial^2 G}{\partial t \partial s} = \frac{\partial^2 G}{\partial s \partial t},$$

and expressions (2.1-2.2) imply

$$\frac{\partial \Theta_1}{\partial t} - \frac{\partial \Theta_2}{\partial s} + [\Theta_1, \Theta_2] = 0. \tag{2.3}$$

Equation (2.3) is called the **zero curvature equation**.

Conversely, we have the following proposition.

Proposition 2.2. *If the pair (Θ_1, Θ_2) satisfy the zero curvature equation, then (Θ_1, Θ_2) is a compatible pair.*

Proof. Let be $F^1(s,t)$ and $F^2(s,t)$ the matrix functions satisfying

$$\frac{\partial F^1}{\partial s}(s,t) = \Theta_1(s,t)F^1(s,t), \quad F^1(0,t) = I \quad \text{for } t \in \mathbb{R} \tag{2.4}$$

$$\frac{\partial F^2}{\partial t}(s,t) = \Theta_2(s,t)F^2(s,t), \quad F^2(s,0) = I \quad \text{for } s \in \mathbb{R} \tag{2.5}$$

and consider the smooth $n \times n$ matrix function

$$G(s,t) = F^1(s,t)F^2(0,t)$$

From the definition of G we verify directly that G satisfies (2.1). To prove that G also satisfies (2.2), consider the matrix function

$$H = \frac{\partial G}{\partial t} - \Theta_2 G$$

and verify that H satisfies the linear system (2.1)

$$\frac{\partial H}{\partial s} - \Theta_1 H = 0.$$

Since $H(0,t) = (\frac{\partial}{\partial t}G)(0,t) - \Theta_2(0,t)G(0,t) = (\frac{\partial}{\partial t}F^2)(0,t) - \Theta_2(0,t)F^2(0,t) = 0$, by uniqueness $H(s,t) = 0$ for all s, and then G satisfies (2.2). $\square$

Remark 2.3. From the previous proposition, we see that the common solution G has the representations

$$G(s,t) = F^1(s,t)F^2(0,t) = F^2(s,t)F^1(s,0) \tag{2.6}$$

The geometrical meaning of the zero curvature equation is shown through the following remarks.

Remark 2.4. The compatibility of the pair (Θ_1, Θ_2) is equivalent to the commutativity of the vector fields on $\mathbb{R}^2 \times \mathbb{C}^n$

$$X_1 = \frac{\partial}{\partial s} + \Theta_1 x \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t} + \Theta_2 x \frac{\partial}{\partial x}$$

Remark 2.5. On the trivial vector bundle $E = \mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$, every matrix pair (Θ_1, Θ_2) define a *linear connection* Γ on E with connection matrix 1-form

$$\Omega = -(\Theta_1 ds + \Theta_2 dt).$$

In terms of the linear connection Γ, the compatibility of the pair (Θ_1, Θ_2) means that its curvature matrix 2-form vanishes

$$\operatorname{Curv} \Omega \equiv d\Omega + \Omega \wedge \Omega = 0$$

Here, we use the matrix notation $(\Omega \wedge \Omega)_{ij} = \Omega_{is} \wedge \Omega_{sj}$ Moreover, the common solution G of (2.1) and (2.2) satisfies equation

$$dG + \Omega G = 0.$$

For more information on linear connections on vector bundles see [7].

Taking into account the equivalence between the compatibility of a matrix pair (Θ_1, Θ_2) and the solvability of the zero curvature equation, we have a full description of the solutions of (2.3) in the following terms.

Proposition 2.6. *The pair (Θ_1, Θ_2) is a solution of the zero curvature equation* (2.3)*, if and only if*

$$\Theta_1(s,t) = \Big(\frac{\partial F^2}{\partial s}(s,t) + F^2(s,t)L(s)\Big)(F^2(s,t))^{-1} \tag{2.7}$$

where $L(s)$ is some $n \times n$ smooth complex matrix function and $F^2(s,t)$ satisfies $(\frac{\partial}{\partial t}F^2)(s,t) = \Theta_2(s,t)F^2(s,t)$ with $F^2(s,0) = I$ for each $s \in \mathbb{R}$.

Proof. If (Θ_1, Θ_2) is a compatible pair, the matrix function Θ_1 satisfies the equation

$$\frac{\partial \Theta_1}{\partial t} = [\Theta_1, \Theta_2] + \frac{\partial \Theta_2}{\partial s}.$$

Solving this equation for Θ_1, we have that $\Theta_1 = \Theta^{\rm part} + \Theta^{\rm hom}$ where: $\Theta^{\rm part}(s,t) = \frac{\partial}{\partial s}F^2(s,t)(F^2(s,t))^{-1}$ and $\Theta^{\rm hom}(s,t) = F^2(s,t)L(s)(F^2(s,t))^{-1}$ for some matrix function $L(s)$. $\square$

Consider now, the class of matrix pairs with periodic entries in the variables s and t

$$\Theta_i(s,t) = \Theta_i(s+2\pi,t) = \Theta_i(s,t+2\pi), \quad i = 1,2. \tag{2.8}$$

In this class, the solvability of (2.3) is reduced to the solvability of a Lax equation, in the following terms

Proposition 2.7. *Let be $\Theta_2(s,t)$ a $n\times n$ smooth matrix function satisfying the periodicity conditions*

$$\Theta_2(s+2\pi,t)=\Theta_2(s,t+2\pi)=\Theta_2(s,t)$$

and denote by $m_2(s)$ the monodromy matrix corresponding to the linear periodic system on $\mathbb{R}^n$

$$\frac{dx}{dt}=\Theta_2(s,t)x$$

Then, the pair (Θ_1,Θ_2) with Θ_1 as in (2.7), is a solution of (2.3) with periodic entries if and only if there exists a $n\times n$ smooth matrix function $L(s)$ satisfying,

$$L(s)=L(s+2\pi) \tag{2.9}$$

$$\frac{dm_2}{ds}=[L(s),m_2]. \tag{2.10}$$

Remark 2.8. Condition (2.10) implies the invariance of the spectrum of the monodromy matrix $m_2(s)$.

3. Reducibility

Let be $R(s,t)$ a $n\times n$ non singular complex matrix function with $R(0,0)=I$ and G the common solution to the equations (2.1)-(2.2). Then the matrix function

$$Y(s,t)=R(s,t)G(s,t)$$

is a simultaneous solution of

$$\frac{\partial Y}{\partial s}=(\frac{\partial R}{\partial s}+R\Theta_1)R^{-1}Y \tag{3.1}$$

$$\frac{\partial Y}{\partial t}=(\frac{\partial R}{\partial t}+R\Theta_2)R^{-1})Y \tag{3.2}$$

with $Y(0,0)=I$, and the pair

$$((\frac{\partial R}{\partial s}+R\Theta_1)R^{-1},(\frac{\partial R}{\partial t}+R\Theta_2)R^{-1}) \tag{3.3}$$

becomes a compatible pair.

Remark 3.1. In particular if we take $R=G^{-1}$ the transformed system (3.1)–(3.2) takes the form

$$\frac{\partial Y}{\partial s}=0$$

$$\frac{\partial Y}{\partial t}=0$$

and the solutions of the linear dynamical systems

$$\frac{dx}{ds}=\Theta_1(s,t)x,$$

$$\frac{dx}{dt}=\Theta_2(s,t)x$$

under the change of variable $y=R(s,t)x$ are transformed respectively into the straight lines $y(s)=x(0,t)$ and $y(t)=x(s,0)$.

Definition 3.2. Let $\mathcal{B}$ a given class of bounded complex functions on $\mathbb{R}^2$. A compatible complex matrix pair (Θ_1, Θ_2) is called **$\mathcal{B}$-reducible** if there exists a $n \times n$ smooth non singular complex matrix function $R(s,t)$ with $R(0,0) = I$ such that R, R^{-1} and dR have entries belonging to $\mathcal{B}$ and satisfy

$$(\frac{\partial R}{\partial s} + R\Theta_1)R^{-1} = K_1 \tag{3.4}$$

$$(\frac{\partial R}{\partial t} + R\Theta_2)R^{-1} = K_2. \tag{3.5}$$

where K_1, K_2 are constant matrices.The matrix function $R(s,t)$ is called the **reducibility matrix function**. If R is a real matrix, the pair is called **real reducible**.

If the pair (Θ_1, Θ_2) is reducible, then equations (3.4)–(3.5) can be solved for the matrix function R,

$$\frac{\partial R}{\partial s} = K_1 R - R\Theta_1$$

$$\frac{\partial R}{\partial t} = K_2 R - R\Theta_2$$

obtaining the expressions

$$R(s,t) = \exp(K_1 s) n_1(t) (F^1(s,t))^{-1}$$

and

$$R(s,t) = \exp(K_2 t) n_2(s) (F^2(s,t))^{-1}$$

where $n_1(t) = R(0,t)$ and $n_2(s) = R(s,0)$. In terms of the common solution (2.6), we have

$$R(0,t) = n_1(t) = \exp(K_2 t)(F^2(0,t))^{-1}$$

and R takes the form

$$R(s,t) = \exp(K_2 t + K_1 s) G(s,t)^{-1}. \tag{3.6}$$

Proposition 3.3 (Floquet representation)**.** *A compatible pair (Θ_1, Θ_2) is $\mathcal{B}$ - reducible, if and only if the common matrix solution $G(s,t)$ of (2.1-2.2), has the form*

$$G(s,t) = R^{-1}(s,t) \exp(K_1 s + K_2 t) \tag{3.7}$$

with R, R^{-1} and dR with coefficients in $\mathcal{B}$ and K_1, K_2 commuting constant matrices.

Now, let us consider the reducibility problem in the class of matrix functions pairs with periodic entrees in both variables (s,t) as in (2.8). Here, the reducibility matrix R must satisfy the conditions

$$R(s + 2\pi, t) = R(s, t + 2\pi) = R(s,t).$$

In this case, any compatible periodic matrix pair can always be reduced to a constant pair system. To show that, take $F^1(s,t)$ and $F^2(s,t)$ as in (2.4) and (2.5) and define the reducibility matrix function with the expression

$$R(s,t) = \exp(K_1 s + K_2 t) G^{-1}(s,t)$$

with $K_1 = \frac{1}{2\pi} \ln F^1(2\pi, 0)$, $K_2 = \frac{1}{2\pi} \ln F^2(0, 2\pi)$.

If the matrix functions $F^1(2\pi,0)$ and $F^2(0,2\pi)$ do not possess a real logarithm, we can take the square $(F^1(2\pi,0))^2$ and $(F^2(0,2\pi))^2$ and define

$$K_1 = \frac{1}{4\pi}\ln(F^1(2\pi,0))^2,$$
$$K_2 = \frac{1}{4\pi}\ln(F^1(2\pi,0))^2.$$

In this case the matrix function $R(s,t) = \exp(sK_1 + tK_2)G(s,t)^{-1}$ satisfies the conditions

$$R(s+4\pi,t) = R(s,t+4\pi) = R(s,t)$$

and we have real-reducibility but relative to a bigger class of functions.

Proposition 3.4. *Any compatible pair of matrix functions* (Θ_1,Θ_2) *satisfying* (2.8) *is reducible in the same class.*

Remark 3.5. In geometric terms, we can say that for any flat linear connection on the trivial vector bundle $\mathbb{T}^2\times\mathbb{R}^n \to \mathbb{T}^2$ with base space the 2-torus $\mathbb{T}^2$, its connection matrix 1-form $\Omega = -\Theta_1 ds - \Theta_2 dt$, can be transformed under a "gauge" transformation into a global constant matrix 1-form.

3.1. **Reducibility of quasiperiodic linear systems.** As an application of the reducibility criterion on the class of periodic matrix pairs, we consider now the reducibility problem for quasiperiodic linear systems.

We recall that a quasiperiodic linear system is by definition a time dependent linear system of the form

$$\frac{dx}{dt} = V(\alpha_1(t),\alpha_2(t))x, \tag{3.8}$$

where $V(\alpha_1,\alpha_2)$ is a $n\times n$ real matrix function satisfying

$$V(\alpha_1+2\pi,\alpha_2) = V(\alpha_1,\alpha_2+2\pi) = V(\alpha_1+2\pi,\alpha_2+2\pi),$$

with $\alpha(t) = (\alpha_1(t),\alpha_2(t))$ and $\alpha_1(t) = \omega_1 t + \alpha_1^0$, $\alpha_2(t) = \omega_2 t + \alpha_2^0$ for $(\alpha_1^0,\alpha_2^0)\in\mathbb{R}^2$. The vector (ω_1,ω_2) is called the *frequency vector*.

System (3.8) is called *non-resonant* if $k_1\omega_1 + k_2\omega_2 = 0$ with $k_1,k_2\in\mathbb{Z}$ implies $k_1 = k_2 = 0$. On the contrary, the system is called *resonant*.

For each value of the initial vector (α_1^0,α_2^0) the quasiperiodic system (3.8) is a time dependent linear dynamical system. This system is periodic if and only if the system is resonant.

Definition 3.6. The quasiperiodic system (3.8) is called **reducible** if there exists a non singular complex matrix function $R(\alpha_1,\alpha_2)$ satisfying

$$R(\alpha_1+2\pi,\alpha_2) = R(\alpha_1,\alpha_2+2\pi) = R(\alpha_1,\alpha_2) \tag{3.9}$$

and for all (α_1^0,α_2^0), the time dependent change of coordinates

$$y = R(\omega_1 t + \alpha_1^0, \omega_2 t + \alpha_2^0)x,$$

transforms system (3.8) into a system of the form

$$\frac{dy}{dt} = By \tag{3.10}$$

with B a constant matrix.

According to the previous definition, the reducibility of the quasiperiodic system implies the simultaneous reducibility of a family of time dependent linear systems into a single linear system with constant coefficients.

To discuss the reducibility of (3.8), we introduce the autonomous system on $\mathbb{R}^2 \times \mathbb{R}^n$

$$\begin{aligned}\frac{d\alpha_1}{dt} &= \omega_1\\ \frac{d\alpha_2}{dt} &= \omega_2\\ \frac{dx}{dt} &= V(\alpha_1, \alpha_2)x\end{aligned}$$

with associated vector field X on $\mathbb{R}^2 \times \mathbb{R}^n$ given by

$$X = \omega_1 \frac{\partial}{\partial \alpha_1} + \omega_2 \frac{\partial}{\partial \alpha_2} + V(\alpha_1, \alpha_2)x\frac{\partial}{\partial x} \tag{3.11}$$

We call the vector field X a *quasiperiodic vector field*.

If the quasiperiodic system (3.8) is non-resonant we will say that vector field X is *non-resonant*. Analogously, if the quasiperiodic system is resonant the corresponding quasiperiodic vector field is called *resonant*.

The following lemma will be useful in the proof of our main theorem about reducibility for quasiperiodic linear systems.

Lemma 3.7. *Assume that quasiperiodic vector field (3.11) is the sum $X = Y + Z$ of two commuting resonant quasiperiodic vector fields Y, Z having independent frequency vectors (μ_1, μ_2) and (ρ_1, ρ_2),*

$$\begin{aligned}Y &= \rho_1 \frac{\partial}{\partial \alpha_1} + \rho_2 \frac{\partial}{\partial \alpha_2} + A(\alpha_1, \alpha_2)x\frac{\partial}{\partial x},\\ Z &= \mu_1 \frac{\partial}{\partial \alpha_1} + \mu_2 \frac{\partial}{\partial \alpha_2} + B(\alpha_1, \alpha_2)x\frac{\partial}{\partial x}.\end{aligned}$$

Then, there exists two resonant and commuting vector fields $\overline{X}$ and $\overline{Y}$ of the form

$$\begin{aligned}\overline{X} &= \omega_1 \frac{\partial}{\partial \alpha_1} + W^1(\alpha_1, \alpha_2)x\frac{\partial}{\partial x},\\ \overline{Y} &= \omega_2 \frac{\partial}{\partial \alpha_2} + W^2(\alpha_1, \alpha_2)x\frac{\partial}{\partial x},\end{aligned}$$

such that $X = \overline{X} + \overline{Y}$ and $[\overline{X}, \overline{Y}] = 0$.

Proof. Let be $\Delta = \rho_1\mu_2 - \mu_1\rho_2$. By assumption $\Delta \neq 0$. Taking $\omega_1 = \mu_1 + \rho_1$, $\omega_2 = \mu_2 + \rho_2$ and $W^1 = uA + vB$ and $W^2 = qA + rB$ with $u = \frac{\omega_1\mu_2}{\Delta}$, $v = \frac{-\omega_1\rho_2}{\Delta}$, $q = \frac{-\omega_2\mu_1}{\Delta}$ and $r = \frac{\omega_2\rho_1}{\Delta}$, we obtain the expressions of W^1, W^2 with the required properties. □

Now, suppose the quasiperiodic vector field X can be represented in the form

$$X = \omega_1 \frac{\partial}{\partial \alpha_1} + \omega_2 \frac{\partial}{\partial \alpha_2} + (W^1(\alpha_1, \alpha_2) + W^2(\alpha_1, \alpha_2))x\frac{\partial}{\partial x}$$

where the resonant vector fields

$$\omega_1 \frac{\partial}{\partial \alpha_1} + W^1(\alpha_1, \alpha_2)x\frac{\partial}{\partial x} \quad \text{and} \quad \omega_2 \frac{\partial}{\partial \alpha_2} + W^2(\alpha_1, \alpha_2)x\frac{\partial}{\partial x}$$

commute. From remark 2.4, the pair $(\frac{1}{\omega_1}W^1, \frac{1}{\omega_2}W^2)$ is compatible and there exists a smooth matrix function $G(\alpha_1, \alpha_2)$ such that

$$\begin{gathered}\frac{\partial G}{\partial \alpha_1}(\alpha_1, \alpha_2) = \frac{1}{\omega_1}W^1(\alpha_1, \alpha_2)G(\alpha_1, \alpha_2),\\ \frac{\partial G}{\partial \alpha_2}(\alpha_1, \alpha_2) = \frac{1}{\omega_2}W^2(\alpha_1, \alpha_2)G(\alpha_1, \alpha_2)\\ G(0,0) = I.\end{gathered}$$

Taking $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and $\alpha_1(t) = \omega_1 t + \alpha_1^0$, $\alpha_2(t) = \omega_2 t + \alpha_2^0$ for $(\alpha_1^0, \alpha_2^0) \in \mathbb{R}^2$, we note that

$$G(\alpha(t)) = G(\omega_1 t + \alpha_1^0, \omega_2 t + \alpha_2^0)$$

satisfies

$$\frac{d}{dt}(G(\alpha(t)) = V(\alpha(t))G(\alpha(t)).$$

From proposition 3.4, we know that a necessary and sufficient condition for the real reducibility of the pair $(\frac{1}{\omega_1}W^1, \frac{1}{\omega_2}W^2)$ is that matrices $G(2\pi, 0)$ and $G(0, 2\pi)$ have a real logarithm. Suppose this is the case and denote by

$$\begin{aligned}K_1 &= \frac{1}{2\pi}\ln G(2\pi, 0),\\ K_2 &= \frac{1}{2\pi}\ln G(0, 2\pi).\end{aligned}$$

Then, the non-singular matrix function

$$R(\alpha_1, \alpha_2) = \exp(\alpha_1 K_1 + \alpha_2 K_2)G(\alpha_1, \alpha_2)^{-1}$$

defines for each choice of the initial condition (α_1^0, α_2^0), the linear change of coordinates

$$y = R((\alpha_1(t), \alpha_2(t))x, \tag{3.12}$$

which transforms the quasiperiodic system (3.8) into the system with constant coefficients

$$\frac{dy}{dt} = (\omega_1 K_1 + \omega_2 K_2)y.$$

Conversely, if the quasiperiodic system (3.8) is reducible to the system $\frac{dy}{dt} = By$ under the quasiperiodic linear transformation (3.12), then

$$V = R^{-1}(BR - \omega_1\frac{\partial R}{\partial \alpha_1} - \omega_2\frac{\partial R}{\partial \alpha_1})$$

and the vector fields on $\mathbb{T}^2 \times \mathbb{R}^n$ given by

$$\begin{aligned}Q^1 &= \omega_1\frac{\partial}{\partial \alpha_1} + \frac{1}{2}R^{-1}(BR - 2\omega_1\frac{\partial R}{\partial \alpha_1})x\frac{\partial}{\partial x},\\ Q^2 &= \omega_2\frac{\partial}{\partial \alpha_2} + \frac{1}{2}R^{-1}(BR - 2\omega_2\frac{\partial R}{\partial \alpha_2})x\frac{\partial}{\partial x},\end{aligned}$$

satisfy $X = Q^1 + Q^2$ with $[Q^1, Q^2] = 0$.

From the above considerations, we state the following result.

Theorem 3.8. *The quasiperiodic system (3.8) is reducible if and only if the corresponding quasiperiodic vector field (3.11) is the sum of two commuting resonant quasiperiodic vector fields.*

This theorem can be rewritten in terms of the zero curvature equation as follows.

Corollary 3.9. *The quasiperiodic system (3.8) is reducible if and only if there exists a matrix function* $\widetilde{V}(\alpha_1, \alpha_2)$ *such that*

$$\widetilde{V}(\alpha_1 + 2\pi, \alpha_2) = \widetilde{V}(\alpha_1, \alpha_2 + 2\pi) = \widetilde{V}(\alpha_1, \alpha_2),$$
$$\omega \cdot \frac{\partial \widetilde{V}}{\partial \alpha} - \widetilde{\omega} \cdot \frac{\partial V}{\partial \alpha} + [\widetilde{V}, V] = 0,$$

where $\widetilde{\omega} = (\omega_1, -\omega_2)$.

Proof. The necessity follows if we define $\widetilde{V} = \widetilde{\omega} \cdot (\frac{\partial}{\partial \alpha} R^{-1})R$. The sufficiency is proved expressing the quasiperiodic vector field X as the sum of the commuting resonant vector fields

$$Q_1 = \omega_1 \frac{\partial}{\partial \alpha_1} + \frac{1}{2}(V - \widetilde{V})x\frac{\partial}{\partial x},$$
$$Q_2 = \omega_2 \frac{\partial}{\partial \alpha_2} + \frac{1}{2}(V + \widetilde{V})x\frac{\partial}{\partial x}.$$

□

3.2. **Reducibility of fiberwise linear systems.** On the trivial vector bundle $E = \mathbb{R}^2 \times \mathbb{R}^n$ with coordinates $\xi = (s, t) \in \mathbb{R}^2$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, dynamical systems of the form

$$\frac{d\xi}{d\tau} = v(\xi), \tag{3.13}$$
$$\frac{dx}{d\tau} = V(\xi)x, \tag{3.14}$$

are called **fiberwise linear dynamical systems**.

We can associate to system (3.13)–(3.14), any pair (V, W) of matrix functions such that W has the form

$$W = (\frac{\partial F}{\partial t} + FL(t))F^{-1}, \tag{3.15}$$

where F is the fundamental matrix for $\frac{\partial}{\partial s}F = V(s,t)F$, and $F(0,t) = I$ for all t. In such a situation, (V, W) satisfies the zero curvature equation and there exists a matrix function G such that

$$dG = (V ds + W dt)G,$$
$$G(0,0) = I.$$

Under the gauge transformation

$$(s,t,x) \to (s,t,G^{-1}(s,t)x) \tag{3.16}$$

the system (3.13)-(3.14) is transformed into the system

$$\frac{d\xi}{d\tau} = v(\xi),$$
$$\frac{dy}{d\tau} = G^{-1}(i_v\Omega + V)Gy$$

where $\Omega = -(V ds + W dt)$ and $i_v\Omega$ denotes the interior product of the 1-form Ω with the vector field v. Moreover, if

$$d(G^{-1}(i_v\Omega + V)G) = 0,$$

or equivalently, if

$$L_v\Omega + dV + [\Omega, V] = 0,$$

where L_v denotes the ordinary Lie derivative operator on differential forms , applied to Ω entry by entry, then the gauge transformation (3.16) transforms the original system (3.13)–(3.14) into a system of the form

$$\frac{d\xi}{d\tau} = v(\xi), \tag{3.17}$$

$$\frac{dy}{d\tau} = Ky, \tag{3.18}$$

with $K = (i_v\Omega + V)(0,0)$.

We summarize the above considerations in the following proposition.

Proposition 3.10. *The fiberwise linear system* (3.13)-(3.14) *is reducible if there exists a matrix function $L(t)$ such that the matrix 1-form*

$$\Omega = -(V ds + (\frac{\partial}{\partial t}F + FL(t))F^{-1}dt),$$

where F is the fundamental matrix of $\frac{\partial}{\partial s}F = VF$ and $F(0,t) = I$, satisfies the equation

$$L_v\Omega + dV + [\Omega, V] = 0. \tag{3.19}$$

In this case, the transformation $(s,t,x) \to (s,t,G^{-1}(s,t)x)$, where $dG = \Omega G$ and $G(0,0) = I$, reduces the system (3.13-3.14) to (3.17-3.18).

In the previous proposition, the choice of the matrix function $L(t)$ allow us to look for gauge transformations satisfying the conditions imposed by the geometry of the base or by a given structure on the fibers. Sometimes, obstructions can appear and the existence of a globally defined reducibility matrix is not possible. To show such obstructions, we present the following example on a trivial vector bundle having the cylinder as base space.

On the trivial vector bundle $E = (S \times \mathbb{R}) \times \mathbb{R}^2$ with base the cylinder $S \times \mathbb{R}$, take coordinates (θ, s) for $S \times \mathbb{R}$, and $x = (x_1, x_2)$ for $\mathbb{R}^2$. Consider the fiberwise linear system

$$\begin{aligned} \frac{d\theta}{d\tau} &= 1 \\ \frac{ds}{d\tau} &= 0 \\ \frac{dx}{d\tau} &= V(\theta, s)x, \end{aligned} \tag{3.20}$$

where $V(\theta + 2\pi, s) = V(\theta, s)$.

Suppose the existence of a reducibility matrix function $R(\theta, s)$ satisfying

$$R(\theta + 2\pi, s) = R(\theta, s)$$

and such that the change of coordinates $y = R(\theta, s)x$ transforms the system (3.20) into a system of the form

$$\begin{aligned}\frac{d\theta}{d\tau} &= 1\\ \frac{ds}{d\tau} &= 0\\ \frac{dx}{d\tau} &= K_1 x.\end{aligned}$$

In this case we have

$$V = R^{-1}(K_1 R - \frac{\partial}{\partial\theta} R).$$

Now, take any constant matrix K_2 commuting with K_1. Then, the matrix function

$$W = R^{-1}(K_2 R - \frac{\partial}{\partial\theta} R)$$

should form with V a compatible pair (V, W) satisfying the condition

$$W(\theta + 2\pi, s) = W(\theta, s).$$

Moreover, W has the form

$$W(\theta, s) = (\frac{\partial F}{\partial s}(\theta, s) + F(\theta, s)L(s))(F(\theta, s))^{-1},$$

where $F(\theta, s)$ is the fundamental matrix of the periodic linear system

$$\frac{dx}{d\theta} = V(\theta, s)x \tag{3.21}$$

and the matrix function $L(s)$ satisfies the expression

$$\frac{dm}{ds} = [L(s), m] \tag{3.22}$$

for the monodromy matrix function

$$m(s) = F(2\pi, s).$$

Now, suppose that V is the matrix function

$$V(\theta, s) = \begin{pmatrix} s & 0 \\ s\cos\theta & s \end{pmatrix}.$$

For that choice of V, the fundamental solution of (3.21) is

$$F = \exp(\theta s)\begin{pmatrix} 1 & 0 \\ s\sin\theta & 1 \end{pmatrix}$$

and the monodromy matrix $m = F(2\pi, s)$ takes the form

$$m(s) = \exp(2\pi s)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The spectrum of $m(s)$ is not constant and therefore it cannot be a solution of any equation of Lax type of the form (3.22). Consequently, the system is not reducible.

Acknowledgement. The author express his gratitude to Dr. Yuri Vorobjev and to M. C. Guillermo Dávila for their interest, encouragement and fruitful discussions.

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