Electronic Journal of Differential Equations, Vol. 2004(2004), No. 46, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

AN APPLICATION OF THE DUAL VARIATIONAL PRINCIPLE TO A HAMILTONIAN SYSTEM WITH DISCONTINUOUS NONLINEARITIES

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Abstract. In this article, we study the existence of solutions to the Hamiltonian elliptic system with discontinuous nonlinearities

$$\begin{aligned} -\Delta u &= au + bv + f(x,v), \\ -\Delta v &= cu + av + g(x,u) \end{aligned}$$

on a bounded subset of $\mathbb{R}^n$, with zero Dirichlet boundary conditions. The functions f and g have a finite number of jumping discontinuities.

1. Introduction

Differential equations with discontinuous nonlinearities play an important role in modelling problems in mathematical physics and in different applications in other fields. In this work, we give some contributions to the study of systems of equations with discontinuous nonlinearities. Our goal concerns finding non-trivial solutions to the system

$$\begin{gathered} -\Delta u = au + bv + f(x,v) \quad \text{in } \Omega \\ -\Delta v = cu + av + g(x,u) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a smooth bounded domain, $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ is a matrix of real entries, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are functions with a finite number of jumping discontinuities whose properties will be detailed later.

In the scalar case, problems with discontinuous nonlinearities have been studied in the previous decades; see for example [3, 4, 6, 7, 8] and the references therein. These problems model many physical phenomena, [9, 10], such those of plasma physics in the case of the distribution of temperature in an electric arc for which the constitutive law contains a discontinuity [2, 1]. Many physical interesting cases also arise when the discontinuity is of the form $h(s-\theta)f(s)$ where $h(s) = 0$, for

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2000 *Mathematics Subject Classification.* 35J50, 37K05, 34A34.
Key words and phrases. Hamiltonian systems, discontinuous nonlinearities, dual variational principle.

Submitted January 26, 2004. Published March 31, 2004.
Research partially supported by the Milênio Project/CNPq/Brazil.

$s \leq 0$ and $h(s) = 1$, for $s > 0$, is the Heaviside function, and f is a continuous function.

Form the mathematical point of view we observe that the classical variational method can not be used directly to these problems since the Euler-Lagrange functionals associated to them are not of class C^1. In these cases, Convex Analysis and the Dual Variational Method are powerful tools to be employed.

Motivated by the aforementioned facts in the scalar case, more specifically, by the papers [3] and [4], we study a class of hamiltonian systems with discontinuous nonlinearities such those introduced in (1.1). With this intention, we had to develop some specific techniques and procedures in order to apply the Dual Variational Method to attack the system case problems.

This article is schemed as follows: In Section 2 we state some preliminaries definitions and results in order to state the main results (Theorems 3.1 and 3.2) in Section 3. In Section 4 we prove Theorem 3.1, and in Section 5 we give an application of this theorem. Finally in Section 6 we prove Theorem 3.2.

2. Preliminaries

To state our main theorem we need to establish some notation. Let us consider a function $f : \mathbb{R} \to \mathbb{R}$ discontinuous only at $x = \theta$, but such that the limit from the right $f(\theta^+)$ and from the left $f(\theta^-)$ exist, where

$$f(\theta^\pm) = \lim_{x \to \theta^\pm} f(x).$$

Suppose that there is $m > 0$ such that the function $ms + f(s)$ is strictly increasing. We set the interval

$$T_{f,\theta} = [f(\theta^-), f(\theta^+)]$$

and define the following multivalued function induced by "filling in" the jump of the discontinuous function $ms + f(s)$ at $x = \theta$:

$$\hat{f}_m(s) = \begin{cases} f(s) + ms, & s \neq \theta \\ T_{f,\theta} + m\theta, & s = \theta. \end{cases}$$

Now it is possible to define a single valued function (the "inverse" of $\hat{f}_m$)

$$\overline{f}_m(t) = \begin{cases} \theta, & \text{if } t \in T_{f,\theta} + m\theta \\ s, \quad \text{with } ms + f(s) = t, & \text{if } t \notin T_{f,\theta} + m\theta \end{cases}$$

such that $\overline{f}_m(t) = s$ if and only if $t \in \hat{f}_m(s)$. It is easy to check that $\overline{f}_m \in C(\mathbb{R})$.

Let us go back to system (1.1). Add $\begin{pmatrix} mu \\ nv \end{pmatrix}$, $m, n > 0$, to both sides of (1.1), so that the right hand side functions become

$$f_m(v) := f(v) + mv, \quad g_n(u) := g(u) + nu$$

which are strictly increasing. Following the previous discussion, we may define the functions $\hat{f}_m$, $\hat{g}_n$, and their respective "inverses", $\overline{f}_m, \overline{g}_n$. For simplicity, we denote $f^{-1} = \overline{f}_m$ and $g^{-1} = \overline{g}_n$.

Remarks. (i) The results proved in this paper also work for functions with a finite number of jumping discontinuities. Moreover, it completes the study made in [3] and [4] for the elliptic systems case.
(ii) For simplicity we shall deal with the autonomous case, when $f(x,v) = f(v)$ and $g(x,u) = g(u)$. The stated general case follows with simple changes.

Let us denote

$$\begin{aligned} &p(\lambda, A) = (\lambda - a)^2 - bc \\ &J_{m,n} = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix} \\ &A_{m,n} = A - J_{m,n} \end{aligned}$$

We denote by λ_j the eigenvalues of the eigenvalue problem $(-\Delta, H_0^1(\Omega))$, subjected to Dirichlet boundary condition, and by φ_j its corresponding eigenfunctions.

We define system (1.1) as *resonant* if $p(\lambda_1, A) = 0$ and *nonresonant* otherwise.

2.1. **The Linear Case.** To apply the dual variational method to system (1.1), we have to study, for $m, n > 0$ to be picked up later, the system

$$\begin{gathered} -\Delta u = au + (b-m)v + f(x) \quad \text{in } \Omega \\ -\Delta v = (c-n)u + av + g(x) \quad \text{in } \Omega \\ u = v = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

where $f, g \in L^2(\Omega)$.

Theorem 2.1. *For small $m, n > 0$, system (2.1) has a unique solution $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ for each pair $(f,g) \in L^2(\Omega) \times L^2(\Omega)$.*

Proof. The Laplacean operator $-\Delta$ subjected to Dirichlet boundary conditions is invertible and

$$(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega) \subset H_0^1(\Omega).$$

Thus, it is possible to define the operator

$$\begin{aligned} T_{A_{m,n}} : \quad L^2(\Omega) \times L^2(\Omega) &\to H_0^1(\Omega) \times H_0^1(\Omega) \\ F = (f,g)) &\mapsto T_{A_{m,n}} F \end{aligned}$$

where

$$-T_{A_{m,n}} F = \begin{pmatrix} a(-\Delta)^{-1} & (m-b)(-\Delta)^{-1} \\ (n-c)(-\Delta)^{-1} & a(-\Delta)^{-1} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

Now, system (2.1) is equivalent to

$$\begin{gathered} U + T_{A_{m,n}} U = G, \quad \text{in } \Omega \\ U = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where $U = (u,v)$ and $G = ((-\Delta)^{-1} f, (-\Delta)^{-1} g)$.

The proof of the proposition relies on the following remarks:

$T_{A_{m,n}}$ is a continuous operator, since $(-\Delta)^{-1}$ is.
By the compact Sobolev embedding, the operator $T_{A_{m,n}} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is compact.
$I + T_{A_{m,n}}$ is a one-to-one operator for small m and n.

Indeed, suppose that $(u, v) \neq (0, 0)$ satisfy (2.1) with $f = g = 0$. Then

$$\Big(\int_\Omega u\varphi_j dx, \int_\Omega v\varphi_j dx \Big) := (X, Y) \neq (0, 0)$$

for some j. Hence, multiplying both equations in (2.1) by φ_j and integrating by parts we achieve

$$(A_{m,n} - \lambda_j I) \begin{pmatrix} X \\ Y \end{pmatrix} = 0.$$

Since $(X, Y) \neq (0, 0)$, we have that $p(\lambda_j, A_{m,n}) = 0$. However it is possible to find small m and n, such that $p(\lambda_j, A_{m,n}) \neq 0$. Hence $(u, v) = (0, 0)$.

The proposition follows from the Fredholm Alternative. $\square$

The solution operator. The above proposition allows us defining the continuous operator (In fact, a compact operator)

$$\begin{array}{rccc} B: & L^2(\Omega) \times L^2(\Omega) & \to & L^2(\Omega) \times L^2(\Omega) \\ & (\omega_1, \omega_2) & \mapsto & (u, v) \end{array}$$

where $B(\omega_1, \omega_2) = (u, v)$ if and only if for $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\begin{aligned} -\Delta u &= au + (b - m)v + \omega_2, \quad \text{in } \Omega \\ -\Delta v &= (c - n)u + av + \omega_1, \quad \text{in } \Omega\,. \end{aligned} \tag{2.2}$$

3. The Dual Variational Framework and the main Theorems

The use of the Dual Variational Principle allows us to find a solution for (1.1) as a minimum or as a critical point of a certain C^1 functional associated with the system.

Let the Hilbert space $W := L^2(\Omega) \times L^2(\Omega)$ be endowed with the norm.

$$\|(u, v)\|_W^2 := |u|_2^2 + |v|_2^2$$

where $|u|_2^2 = \int_\Omega |u|^2 dx$. We define the functional $\Psi : W \to \mathbb{R}$ by

$$\Psi(\omega_1, \omega_2) = \int_\Omega G(\omega_1)dx + \int_\Omega F(\omega_2)dx - \frac{1}{2} \int_\Omega \langle B\omega, \omega\rangle dx$$

where $\omega = (\omega_1, \omega_2)$, $G(t) = \int_0^t g^{-1}(\sigma)d\sigma$, $F(t) = \int_0^t f^{-1}(\sigma)d\sigma$, and

$$\langle B\omega, \eta\rangle = \int_\Omega (u_1\eta_1 + u_2\eta_2)dx,$$

if $B\omega = (u_1, u_2)$ and $\eta = (\eta_1, \eta_2)$. Since A is symmetric, we have $\langle B\omega, \eta\rangle = \langle \omega, B\eta\rangle$.

With additional hypotheses on the non-linearities we assure that $\Psi \in C^1(W, \mathbb{R})$. Hence, a straightforward calculation leads to

$$\langle \Psi'(\omega_1, \omega_2), (\eta_1, \eta_2)\rangle = \int_\Omega (g^{-1}(\omega_1)\eta_1 + f^{-1}(\omega_2)\eta_2)dx - \int_\Omega \langle B\omega, \eta\rangle dx. \tag{3.1}$$

We see that a pair $(\omega_1, \omega_2) \in W$ is a critical point of Ψ if and only if

$$B(\omega_1, \omega_2) = (g^{-1}(\omega_1), f^{-1}(\omega_2)).$$

Therefore, if $(u, v) = (g^{-1}(\omega_1), f^{-1}(\omega_2))$, then

$$(-\Delta u - au + (m - b)v, -\Delta v + (n - a)u - cv) \in (\hat{f}_m(v), \hat{g}_n(u)). \tag{3.2}$$

In our context, a pair (u, v) satisfying the above inclusion is said to be a *solution* to (1.1).

The following are our main results.

Theorem 3.1 (The nonresonant case). *Suppose that f and g are real functions with jumping nonlinearities at θ and ξ, respectively. If $(\omega_1, \omega_2) \in W$ is a minimum for Ψ,*

$$p(\lambda_1, A) > 0 \tag{3.3}$$

holds and

$$(u, v) := (g^{-1}(\omega_1), f^{-1}(\omega_2))$$

then

$$\begin{aligned} -\Delta u(x) &= au(x) + bv(x) + f(v(x)) \\ -\Delta v(x) &= cu(x) + av(x) + g(u(x)) \end{aligned}$$

a.e. for $x \in \Omega$, i. e. (u, v) is a strong solution for system (1.1).

Theorem 3.2 (A resonant case). *Let $f(v) = \alpha v + a(v)$ and $g(u) = \beta u + b(u)$, be real functions with jumping nonlinearities at θ and ξ, respectively, such that*

$$0 < \alpha, \beta\,; \quad \alpha, \beta = \lambda_1^2 \tag{3.4}$$

$$\begin{aligned} \lim_{v \to \pm\infty} a(v) = a^{\pm}\,, &\quad a^- < 0 < a^+ \\ \lim_{u \to \pm\infty} b(u) = b^{\pm}\,, &\quad b^- < 0 < b^+. \end{aligned} \tag{3.5}$$

If

$$(0, 0) \notin (T_{f,\theta}, T_{g,\xi}) \tag{3.6}$$

then the system

$$\begin{gathered} -\Delta u = \alpha v + a(v), \quad \text{in } \Omega \\ -\Delta v = \beta u + b(u), \quad \text{in } \Omega \\ u = v = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{3.7}$$

has a strong solution.

Regarding the proofs of the previous theorems, following our approach, we will see that the variational techniques lead to a pair $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that (3.2) holds. Hence, to assure that (u, v) is in fact a solution for (1.1) it suffices to prove that the sets

$$\Omega_\xi = \{x \in \Omega : u(x) = \xi\} \quad \text{and} \quad \Omega_\theta = \{x \in \Omega : v(x) = \theta\}$$

have zero Lebesgue measure. These sets play an important role when inverting the functions $\hat{f}_m, \hat{g}_n$.

For the well known example in the scalar case

$$\begin{gathered} -\Delta u = h(\theta - v) f(v), \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{3.8}$$

by a result of Stampacchia [5], $\Delta u = 0$ a.e. on Ω_θ. Therefore, the set Ω_θ does not play any important role in this case, and follows immediately that the found wanted critical point u is a strong solution for (3.8), since $h(0) = 0$. However due to the coupling of similar equations in the system case, this fact does not occur and others procedures are demanded. Theorem 3.1 may be applied to study these cases.

4. Proof of Theorem 3.1

As mentioned before, the proof consists in showing that the sets Ω_θ and Ω_ξ have zero Lebesgue measure. Firstly, let us prove that $|\Omega_\xi| = 0$, where $|\bullet|$ is the Lebesgue measure. Let

$$T = T_{g,\xi} + n\xi = [c_1, c_2], \quad T^+ = [c_1, 1/2\,(c_1 + c_2)], \quad T^- = T - T^+,$$

where $c_1 = g(\xi^-) + n\xi$, $c_2 = g(\xi^+) + n\xi$ and $\Omega^\pm = \{x \in \Omega_\xi : \omega_1(x) \in T^\pm\}$. Define the function $\chi \in L^2(\Omega)$ by

$$\chi(x) = \begin{cases} 1, & x \in \Omega^+ \\ -1, & x \in \Omega^- \\ 0, & x \in \Omega - \Omega_\xi. \end{cases} \tag{4.1}$$

For $t > 0$, small enough, we have that

$$\omega_1(x) + t\chi(x) \in T, \quad \text{a.e. for } x \in \Omega_\xi.$$

Since $(\omega_1, \omega_2) \in E$ is a minimum for Ψ, we have that

$$\frac{d}{dt}\Psi(\omega_1 + \theta t\chi, \omega_2) \geq 0$$

for some $0 < \theta < 1$. Hence, by (3.1)

$$\begin{aligned} &\langle \Psi'(\omega_1 + \theta t\chi, \omega_2), (\chi, 0) \\ &= \int_\Omega g^{-1}(\omega_1 + \theta t\chi)\chi\, dx - \int_\Omega u\chi dx - \theta t \int_\Omega \langle B(\chi, 0), (\chi, 0)\rangle dx \geq 0. \end{aligned} \tag{4.2}$$

On the other hand

$$\int_\Omega g^{-1}(\omega_1 + \theta t\chi)\chi\, dx = \int_{\Omega_\xi} g^{-1}(\omega_1 + \theta t\chi)\chi\, dx = \xi \int_{\Omega_\xi} \chi\, dx$$

and

$$\int_\Omega u\chi\, dx = \int_{\Omega_\xi} u\chi\, dx = \xi \int_{\Omega_\xi} \chi\, dx.$$

Accordingly, the difference between the first two members of the right hand side of (4.2) is zero and thus

$$\int_\Omega \langle B(\chi, 0), (\chi, 0)\rangle dx \leq 0. \tag{4.3}$$

For $t < 0$, replacing χ by $-\chi$ in the afore steps we get that

$$\int_\Omega \langle B(\chi, 0), (\chi, 0)\rangle dx \geq 0. \tag{4.4}$$

Consequently, by (4.3) and (4.4)

$$\langle B(\chi, 0), (\chi, 0)\rangle dx = 0. \tag{4.5}$$

Let $B(\chi, 0) = (u_1, v_1)$. Then by (2.2) and (4.5)

$$\begin{gathered} -\Delta u_1 = au_1 + (b - m)\, v_1, \quad \text{in } \Omega \\ -\Delta v_1 = (c - n)\, u_1 + av_1 + \chi, \quad \text{in } \Omega \\ u_1 = v_1 = 0, \quad \text{on } \partial\Omega \\ \int_\Omega \chi u_1 dx = 0. \end{gathered} \tag{4.6}$$

Claim. Let $u, v \in H_0^1(\Omega)$ and $\chi \in L^2(\Omega)$ satisfy (4.6). If (3.3) holds, then $u = v = 0$.

Proof. Multiplying the first equation in (4.6) by v_1, the second by u_1 respectively, and integrating by parts we achieve

$$\int_\Omega v_1^2 dx = \frac{c-n}{b-m} \int u_1^2, \quad \text{for small } m, n > 0. \tag{4.7}$$

Hence both $c - n$ and $b - m$ have the same signs. By the first equation in (4.6)

$$\lambda_1 \int_\Omega u_1^2 dx \ \le \int_\Omega |\nabla u_1|^2 dx = a \int_\Omega u_1^2 dx + (b-m) \int u_1 v_1.$$

Therefore, if $b - m > 0$

$$\lambda_1 \int_\Omega u_1^2 dx \le a \int_\Omega u_1^2 dx + (b-m)\ |u_1|_2\ \ |v_1|_2. \tag{4.8}$$

By (4.7) and (4.8),

$$(\lambda_1 - a) \int_\Omega u_1^2 dx \le (b-m) \sqrt{\frac{(c-n)}{(b-m)}} \int_\Omega u_1^2 dx$$

and thus

$$\Big\{(\lambda_1 - a) - \sqrt{(b-m)\ (c-n)}\Big\} \int_\Omega u_1^2 dx \le 0.$$

But $(\lambda_1 - a) > \sqrt{(b-m)\ (c-n)}$, for small m and n, since $p(\lambda_1, A) > 0$. This yields that $u_1 = 0$. Then $v_1 = 0$. When $b - m < 0$ the proof follows analogous steps. $\square$

By the above claim, we infer that $u_1 \equiv v_1 \equiv 0$ and hence $\chi \equiv 0$.

By the definition of the function (4.1) we conclude that $|\Omega_\xi| = 0$. A similar reasoning applies to show that $|\Omega_\theta| = 0$.

5. An application of Theorem 3.1

Our application of Theorem 3.1 consists of coercive functional bounded from below and associated to system (1.1). It is well known that this kind of functional has a minimum.

Theorem 5.1. *Let f and g satisfy*

$$|f(s)| \le c_1 + k_1|s|, \quad |g(s)| \le c_2 + k_2|s| \tag{5.1}$$

and define

$$0 < K := \max\{k_1, k_2\} < \widetilde{\lambda}_1(A) : \frac{(\lambda_1 - a) + \max\{|b|, |c|\}}{p(\lambda_1, A)}.$$

Suppose that

$$a < \lambda_1, \quad bc > 0 \tag{5.2}$$

and that (3.3) holds. Then, system (1.1) possesses a strong solution $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Proof. By (5.1)

$$\int_\Omega G(w_1)dx \geq \frac{1}{2K}\ |w_1|_2^2 - C|w_1|_2, \tag{5.3}$$

$$\int F(w_2)dx \geq \frac{1}{2K}\ |w_2|_2^2 - C|w_2|_2. \tag{5.4}$$

Let us estimate $\int_\Omega \langle Bw, w\rangle dx$. Let us first suppose that $b, c > 0$. By (2.2), Poincaré inequality and Hölder inequality we have

$$(\lambda_1 - a)|u|_2 \leq (b-m)|v|_2 + |w_2|_2$$
$$(\lambda_1 - a)|v|_2 \leq (c-n)|u|_2 + |w_1|_2.$$

Since $\lambda_1 > a$, by the above inequalities

$$\begin{aligned} \frac{p(\lambda_1, A_{m,n})}{\lambda_1 - a}\ |u|_2\ |w_1|_2 &\leq \frac{|b-m|}{\lambda_1 - a}|w_1|_2^2\ + |w_1|_2\ |w_2|_2 \\ \frac{p(\lambda_1, A_{m,n})}{\lambda_1 - a}\ |v|_2\ |w_2|_2 &\leq \frac{|c-n|}{\lambda_1 - a}\ |w_2|_2^2 + |w_1|_2\ |w_2|_2. \end{aligned} \tag{5.5}$$

If m, n are such that $p(\lambda_1, A_{m,n}) > 0$, inequalities (5.5) yields

$$|u|_2|w_1|_2 + |v|_2\ |w_2|_2 \leq \frac{(\lambda_1 - a) + max\{|b-m|, |c-n|\}}{p(\lambda_1, A_{m,n})} \left(|w_1|_2^2 + |w_2|_2^2\right).$$

Denoting

$$a(m,n) = \frac{(\lambda_1 - a) + max\{|b-m|, |c-n|\}}{p(\lambda_1, A_{m,n})}$$

we see that

$$\int \langle Bw, w\rangle\, dx \leq a(m,n)\left(|w_1|_2^2 + |w_2|_2^2\right). \tag{5.6}$$

If $b, c < 0$ a similar reasoning leads to the last inequality.

Observe that $\lim_{m,n\to 0}\ a(m,n) = 1/\widetilde{\lambda}_1$. Choosing m, n such that $\frac{1}{K} < a(m,n) < \frac{1}{\lambda_1}$, by (5.3), (5.4) and (5.6) we get

$$\Psi(w_1, w_2) \geq \frac{1}{2K}\left(|w_1|_2^2 + |w_2|_2^2\right) - \frac{a(m,n)}{2}\left(|w_1|_2^2 + |w_2|_2^2\right) - C\left(|w_1|_2 + |w_2|\right).$$

Thus, $\Psi(w)$ is bounded from below and coercive and hence it has a minimum (w_1, w_2). □

6. Proof of Theorem 3.2

In this resonant case, the operator B is compact and defined as

$$B(w_1, w_2) = (u, v) \Longleftrightarrow \begin{cases} -\Delta u = w_2 \\ -\Delta v = w_1 \end{cases} \tag{6.1}$$

The proof follows from the next two theorems. Firstly let us prove that the functional Ψ associated to the system in this case has a linking at the origin.

Theorem 6.1. *Suppose that* (3.4) *and* (3.5) *hold. Then* Ψ *has a linking at the origin, i.e., there exist two subspaces* $Z, V \subset W(:= L^2(\Omega) \times L^2(\Omega)\ )$ *such that* $W = Z \oplus V$, $\dim Z < \infty$ *and there exists* $(z_1, z_2) \in Z$ *such that*

$$\lim_{|t| \to +\infty} \Psi(tz_1, tz_2) = -\infty \tag{6.2}$$

$$\inf_V \ \Psi \ > -\infty \tag{6.3}$$

Proof. For short notation we shall consider $m = n = 0$. By (3.4) and (3.5) it follows that

$$\begin{aligned} G(w_1) &= \frac{w_1^2}{2\beta} - p(w_1) \\ F(w_2) &= \frac{w_2^2}{2\alpha} - q(w_2)\,. \end{aligned} \tag{6.4}$$

Where $(w_1, w_2) = w \in W, p$ and q are functions with linear growth such that

$$\lim_{s \to \pm\infty} p(s) = \lim_{s \to \pm\infty} q(s) = \pm\infty. \tag{6.5}$$

Accordingly, by (6.4), we have that

$$\Psi(w_1, w_2) = \frac{1}{2} \int (\frac{w_1^2}{\beta} + \frac{w_2^2}{\alpha}) dx - \int (p(w_1) + q(w_2)) dx - \frac{1}{2} \int \langle Bw, w \rangle dx \tag{6.6}$$

Let us choose $(w_1, w_2) = (a\phi_1, b\phi_1) := (z_1, z_2)$, where $a, b \in \mathbb{R}$ will be picked up later. If $B(z_1, z_2) = (u, v)$, then

$$\int \langle Bz, z \rangle dx = \frac{2ab}{\lambda_1} \int \phi_1^2 dx, \quad z = (z_1, z_2). \tag{6.7}$$

Hence, if $a_0, b_0 > 0$ is chosen such that

$$(\frac{a_0}{b_0})^2 = \frac{\beta}{\alpha} \tag{6.8}$$

using (3.4) we have

$$\begin{aligned} \frac{1}{2}\Big[\int (\frac{z_1^2}{\beta} + \frac{z_2^2}{\alpha}) dx - \int \langle Bz, z \rangle dx\Big] &= \frac{1}{2}\big[\frac{a_0^2}{\beta} + \frac{b_0^2}{\alpha} - \frac{2a_0 b_0}{\lambda_1}\big] \int \phi_1^2 \\ &= \frac{1}{2}(\frac{a_0}{\sqrt{\beta}} - \frac{b_0}{\sqrt{\alpha}})^2 \int \phi_1^2 = 0 \end{aligned}$$

Therefore, by (6.5) and the above result, we obtain

$$\Psi(tz_1, tz_2) = -\int (p(tz_1) + q(tz_2)) dx \to -\infty, \text{ as } |t| \to +\infty. \tag{6.9}$$

□

Before proving (6.3) we need some remarks. Observe that we may decompose $W = Z \oplus V$. Where $Z = \langle (z_1, z_2) \rangle$ (the space generated by $(z_1, z_2) \in W$) and

$$V = \{(r\phi_1 + v_1, s\phi_1 + v_2) : v_1, v_2 \in \langle \phi_1 \rangle^\perp \text{ and } r, s \in \mathbb{R},\ a_0 r + b_0 s = 0\}.$$

Claim. If $\mathrm{v} = (v_1, v_2) \in E$, and $v_1, v_2 \in \langle \phi_1 \rangle^{\perp}$, then
$$\int \langle B\mathrm{v}, \mathrm{v} \rangle dx \leq \frac{2}{\lambda_2} |v_1|_2 |v_2|_2. \tag{6.10}$$
Indeed, it follows from (6.1) that if $B\mathrm{v} = (u, v)$, then
$$\int \langle B\mathrm{v}, \mathrm{v} \rangle dx = \int uv_1 + vv_2 = 2 \int \nabla u \nabla v.$$
If $\{\phi_j\}$ is the orthonormal base of $L^2(\Omega)$ composed of eigenvalues of the Laplacean we may write
$$v_1 = \sum\nolimits_{j=2}^{\infty} \alpha_j^1 \phi_j$$
$$v_2 = \sum\nolimits_{k=2}^{\infty} \alpha_k^2 \phi_j$$
and hence
$$u = \sum\nolimits_{j=2}^{\infty} \frac{\alpha_j^1 \phi_j}{\lambda_j}$$
$$v = \sum\nolimits_{k=2}^{\infty} \frac{\alpha_k^2 \phi_k}{\lambda_k}$$
Using that $\lambda_1 < \lambda_2 \leq \lambda_3 \dots .$ and the liner product properties it follows that
$$\int \nabla u \nabla v \leq \frac{1}{\lambda_2} |v_1|_2 \ |v_2|_2\,. \tag{6.11}$$
Thus, (6.10) is proved.

Now, by (6.10), if $\mathrm{v} = (v_1, v_2) \in \langle \phi_1 \rangle^{\perp} \times \langle \phi_1 \rangle^{\perp}$ we obtain that
$$\frac{1}{2} \int (\frac{|v_1|_2^2}{\beta} + \frac{|v_2|_2^2}{\alpha}) dx - \frac{1}{2} \int \langle B\mathrm{v}, \mathrm{v} \rangle \geq \frac{1}{2} [(\frac{|v_1|_2}{\sqrt{\beta}})^2 - \frac{2|v_1|_2 |v_2|_2}{\lambda_2} + (\frac{|v_2|_2}{\sqrt{\alpha}})^2]$$
since the discriminant of this quadric for is negative, because
$$\frac{1}{\lambda_2^2} < \frac{1}{\sqrt{\alpha\beta}} = \frac{1}{\lambda_1^2}\,,$$
there exists $k > 0$ such that it is greater or equal to $k(|v_1|_2^2 + |v_2|_2^2)$. Thus
$$\frac{1}{2} \int (\frac{|v_1|_2^2}{\beta} + \frac{|v_2|_2^2}{\alpha}) - \frac{1}{2} \int \langle B\mathrm{v}, \mathrm{v} \rangle \geq k(|v_1|_2^2 + \ |v_2|_2^2). \tag{6.12}$$
Let $(r\phi_1 + v_1, s\phi_1 + v_2) \in V$. Therefore, by (3.4) and (6.12),
$$\begin{aligned}
&\Psi(r\phi_1 + v_1, s\phi_1 + v_2) \\
&= \frac{1}{2} \Big[(\frac{r^2}{\beta} + \frac{s^2}{\alpha}) \int \phi_1^2 dx - \int \langle B(r\phi_1, s\phi_1), (r\phi_1, s\phi_1) \rangle dx \Big] \\
&\quad + \frac{1}{2} \Big[(\frac{|v_1|^2}{\beta} + \frac{|v_2|^2}{\alpha}) \int \phi_1^2 dx - \int \langle B\mathrm{v}, \mathrm{v} \rangle dx \Big] - \int (p(r\phi_1 + v_1) + q(s\phi_1 + v_2) dx \\
&\geq \frac{1}{2} \Big[(\frac{r^2}{\beta} - \frac{2rs}{\lambda_1} + \frac{s^2}{\alpha}) \int \phi_1^2 dx + k \int \big(|v_1|^2 + |v_2|^2 \big) \Big] \\
&\quad - \int (p(r\phi_1 + v_1) + q(s\phi_1 + v_2) dx
\end{aligned}$$
and since $\frac{r}{\beta} - \frac{s}{\alpha} \neq 0$ and the functions p and q have linear growth we have that $\inf_V \Psi > -\infty$.

Theorem 6.2. *With the hypotheses of Theorem 3.2, the functional Ψ has a critical point $(w_1, w_2) \in W$ at the level c. We also have that if $(u, v) := B(w_1, w_2)$, then (u, v) is a strong solution for system* (3.7).

Proof. Since Ψ has a linking geometry, if we prove that it satisfies $(PS)_c$ condition, by the Saddle Point, [11, Theorem 1.2], it has a critical point.

Let $(w_n^1, w_n^2) \subset W$ be a sequence such that for some $c \in \mathbb{R}$,

$$\Psi(w_n^1, w_n^2) \to c \in \mathbb{R}, \tag{6.13}$$

$$\Psi'(w_n^1, w_n^2) \to 0 \quad \text{in } W' \tag{6.14}$$

We have that $w_n^1 = t_n^1\phi_1 + v_n^1$ and $w_n^2 = t_n^2\phi_1 + v_n^2$ for some real sequences t_n^1 and t_n^2. Substituting these sequences in (6.2) and (6.3) and using (3.5) we have the boundedness of (w_n^1, w_n^2). Therefore, up to subsequences, we may suppose that $w_n^1 \rightharpoonup w_1$ and $w_n^2 \rightharpoonup w_2$ in $L^2(\Omega)$ for some w_1 and $w_2 \in L^2(\Omega)$. Since B is a compact operator,

$$B(w_n^1, w_n^2) \to B(w_1, w_2) := (u, v). \tag{6.15}$$

By (6.14) we achieve that

$$g^{-1}(w_n^1) \to u \quad \text{and} \quad f^{-1}(w_n^2) \to v \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{6.16}$$

Thus

$$w_n^1(x) \to g(u(x)) \quad \text{a.e. in } \Omega - \Omega_\xi \tag{6.17}$$

$$w_n^2(x) \to f(v(x)) \text{ a.e. in } \Omega - \Omega_\theta$$

$$g^{-1}(w_n^1(x)) \to u(x) = \xi \text{ in } \Omega_\xi f^{-1}(w_n^2(x)) \to v(x) = \theta \text{ in } \Omega_\theta. \tag{6.18}$$

Our next step is proving that

$$\begin{gathered} g^{-1}(w_n^1) \to g^{-1}(w_1) \quad \text{in } L^2(\Omega_\xi) \\ \int_{\Omega-\Omega_\xi} G(w_n^1)dx \to \int_{\Omega-\Omega_\xi} G(w_1)dx \end{gathered} \tag{6.19}$$

$$\begin{gathered} f^{-1}(w_n^2) \to f^{-1}(w_2) \quad \text{in } L^2(\Omega - \Omega_\theta) \\ \int_{\Omega-\Omega_\theta} F(w_n^2)dx \to \int_{\Omega-\Omega_\theta} F(w_2)dx. \end{gathered} \tag{6.20}$$

$$\begin{gathered} |\Omega_\xi| = 0 \\ |\Omega_\theta| = 0. \end{gathered} \tag{6.21}$$

Hence, using the above assertions in (6.13) and (6.14), we have that $\Psi'(w_1, w_2) = 0$ and $\Psi(w_1, w_2) = c$. $\square$

Proof of (6.19). Firstly, since $g(s) = \beta s + b(s)$ with b bounded, we have that $|w_n^1(x)| \le c_1|g^{-1}(w_n^1(x))| + c_2$, and by (6.16), $|w_n^1(x)| \le c_1 h(x)$ for some $h \in L^2(\Omega)$. By (6.17) and Lebesgue Theorem we have that $w_n^1 \to g(u)$ in $L^2(\Omega - \Omega_\xi)$. But $w_n^1 \rightharpoonup w_1$, which implies that $w_n^1 \to w_1$ in $L^2(\Omega - \Omega_\xi)$. Since g^{-1} is asymptotically linear, assertion (6.19) follows. $\square$

Proof of (6.21). With the hypothesis (3.6) we may suppose that $T_{g,\xi} = [c_1, c_2]$ with $c_1 > 0$. (The case $c_2 < 0$ follows in the same way). Let

$$h^\xi(x) = \begin{cases} 1, & x \in \Omega_\xi \\ 0, & x \in \Omega - \Omega_\xi\,. \end{cases}$$

By a result of Stamppachia [5] already mentioned in the Introduction, $w_1 = 0$ a.e. in Ω_θ and $w_2 = 0$ a.e. in Ω_ξ. But, since (6.17) holds we also have that $u(x) = g^{-1}(0)$ a.e. in Ω_θ and by the same reasoning, $v(x) = f^{-1}(0)$ a.e. in Ω_ξ. Hence, by (6.1) we also have that $w_1 = 0$ a.e. in Ω_ξ and $w_2 = 0$ a.e. in Ω_θ.

On the other hand

$$\int_{\Omega_\xi} w_n^1 dx = \langle w_n^1, h^\xi \rangle \to \langle w_n^1, h^\xi \rangle = \int_{\Omega_\xi} w_1 dx. \tag{6.22}$$

Since g^{-1} is strictly monotonic, we see that $\liminf w_n^1(x) \geq c_1$ a.e. in Ω_ξ. But as $|w_n^1| \leq h \in L^2(\Omega)$, Fatou's Lemma and (6.22) imply that

$$0 = \liminf \int_{\Omega_\xi} w_n^1(x) dx \geq c_1 \int_{\Omega_\xi} dx = c_1 |\Omega_\xi|$$

and consequently (6.21) holds. The remaining part of the proof follows by making the respective changes. □

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