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LOCAL INVARIANCE VIA COMPARISON FUNCTIONS

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Abstract. We consider the ordinary differential equation $u'(t) = f(t, u(t))$, where $f : [a, b] \times D \to \mathbb{R}^n$ is a given function, while D is an open subset in $\mathbb{R}^n$. We prove that, if $K \subset D$ is locally closed and there exists a comparison function $\omega : [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\liminf_{h \downarrow 0} \frac{1}{h}\big[d(\xi + hf(t, \xi); K) - d(\xi; K)\big] \le \omega(t, d(\xi; K))$$

for each $(t, \xi) \in [a, b] \times D$, then K is locally invariant with respect to f. We show further that, under some natural extra condition, the converse statement is also true.

1. Introduction

In this paper we prove a local invariance result for a locally closed subset K in $\mathbb{R}^n$ with respect to an ordinary differential equation

$$u'(t) = f(t, u(t)), \tag{1.1}$$

where $f : [a, b] \times D \to \mathbb{R}^n$ is a given function, while D is an open subset in $\mathbb{R}^n$ including K. By a *solution* of (1.1) we mean a differentiable function $u : [\tau, c] \to D$ satisfying (1.1) for each $t \in [\tau, c]$. Clearly, whenever f is continuous, each solution of (1.1) is of class C^1. We recall that the subset K is *viable* with respect to f if for each $(\tau, \xi) \in [a, b) \times K$ there exists at least one solution $u : [\tau, c] \to K$, $c \in (\tau, b]$, of (1.1) satisfying the initial condition

$$u(\tau) = \xi. \tag{1.2}$$

Nagumo [16] was the first who showed that a necessary and sufficient condition, in order that a closed subset K in $\mathbb{R}^n$ be viable (*"rechts zulässig", i.e. right admissible* in his terminology) with respect to a continuous function f, is the tangency condition

$$\liminf_{h \downarrow 0} \frac{1}{h} d(\xi + hf(t, \xi); K) = 0 \tag{1.3}$$

for each $(t, \xi) \in [a, b] \times K$. Here and thereafter $d(x; C)$ denotes the distance from the point $x \in \mathbb{R}^n$ to the subset C in $\mathbb{R}^n$. An one-dimensional problem of this kind was considered earlier by Perron [18] who proved that a sufficient condition in order

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that, for each $\tau \in [a,b]$ and each $\xi \in [\omega_1(\tau), \omega_2(\tau)]$, (1.1) has at least one solution $u : [\tau, T] \to \mathbb{R}$ satisfying $\omega_1(t) \le u(t) \le \omega_2(t)$ for each $t \in [\tau, T]$ is

$$D_{\pm}\omega_1(t) \le f(t, \omega_1(t)) \quad \text{and} \quad D_{\pm}\omega_2(t) \ge f(t, \omega_2(t))$$

for each $t \in [a,b]$. Here $D_{\pm}g(t)$ denotes the right/left lower Dini derivative of the function g calculated at t. It is interesting to notice that Nagumo's result (or variants of it) has been rediscovered independently in the sixties and seventies by Yorke [24], [25], Brezis [5], Crandall [10], Hartman [12] and Martin [15] among others. More precisely, Yorke [24] uses viability (*weak positive invariance* in his terminology) in order to get sufficient conditions for stability, as well as to give very simple and elegant proofs for both Hukuhara and Kneser's Theorems. Brezis [5] analyzes the case when D is open, $K \subset D$ is relatively closed and f is locally Lipschitz, and proves that (1.3) with "lim" instead of "lim inf" is necessary and sufficient for K to be "flow invariant" for (1.1). Crandall [10] considers the case when D is arbitrary, $K \subset D$ is locally closed and $f : D \to \mathbb{R}^n$ is continuous and shows that a sufficient condition for K to be viable (*forward invariant* in his terminology) is (1.3). Hartman [12] proves essentially the same result for D open, $K \subset D$ relatively closed and $f : D \to \mathbb{R}^n$ continuous, and shows in addition that (1.3) is necessary for the viability of K with respect to f. Finally, Martin analyzes the special case when f is continuous and dissipative. It should be also noticed that related results expressed in terms of a generalized normal vector to the points of the set K had been obtained earlier by Bony [4]. The results of Brezis [5] and Bony [4] were refined by Redheffer [19], whose approach is more closely related to ours. An extension of Nagumo's Viability Theorem to Carathéodory functions f was proved by Ursescu [20]. At this point we can easily see that viability is independent of the values of f on $D \setminus K$, and therefore, in the study of viability problems there is no need for f to be defined "outside" K. This is no longer true if we consider the case of *local invariance* to be defined below. Namely, the subset K is *locally invariant* with respect to f if for each $(\tau, \xi) \in [a, b) \times K$ and each solution $u : [\tau, c] \to D$, $c \in (\tau, b]$, of (1.1), satisfying the initial condition (1.2), there exists $c_1 \in (a, c]$ such that we have $u(t) \in K$ for each $t \in [\tau, c_1]$. It is *invariant* if it is locally invariant and $c_1 = c$.

A first question which one may raise is whether, as in the case of viability, the local invariance is merely a consequence of a "good behavior" of $f : [a,b] \times D \to \mathbb{R}^n$ at the points of the boundary of K. In order to answer this question, let us recall first that there exists a closed set K, and a function $g : K \to \mathbb{R}^n$, such that K is viable with respect to g, but nevertheless there is no continuous extension $\widetilde{g} : D \to \mathbb{R}^n$ of g, with D an open neighborhood of K, such that K be locally invariant with respect to $\widetilde{g}$. See Aubin-Cellina [2], Example on page 203. To see that the situation is far from being simple, we notice that there exists a closed subset K which is viable with respect to a given function g which has two continuous extensions: one $\widetilde{g} : D \to \mathbb{R}^n$ with respect to which K is locally invariant, and another one $\overline{g} : D \to \mathbb{R}^n$ with respect to which K is not locally invariant. Indeed, let us consider $K = \{(x,y) \in \mathbb{R}^2;\ y \le 0\}$ and let $g : K \to \mathbb{R}^2$ be defined by $g((x,y)) = (1,0)$ for each $(x,y) \in K$. Obviously K is viable with respect to g. Next, let us define $\widetilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\widetilde{g}((x,y)) = (1,0)$ for each $(x,y) \in \mathbb{R}^2$, and $\overline{g} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\overline{g}((x,y)) = \begin{cases} (1,0) & \text{if } y \le 0 \\ (1, 3\sqrt[3]{y^2}) & \text{if } y > 0. \end{cases}$$

Clearly K is locally invariant (in fact invariant) with respect to $\widetilde{g}$, while K is not locally invariant with respect to $\overline{g}$. The latter assertion follows from the remark that, from each initial point, $\xi = (x, 0)$ (on the boundary of K), we have at least two solutions of (1.1) with f replaced by $\overline{g}$, the first one $u(t) = (t + x, 0)$ which lies in K, and the second one $v(t) = (t + x, t^3)$ which leaves K instantaneously. These examples show that the behavior of $f : [a, b] \times D \to \mathbb{R}^n$ on the boundary of K plays a crucial role, but is far from being sufficient for the local invariance of K with respect to f. Namely, to get local invariance, we need a "good behavior" of f at the points which are "very close" to the "exterior" boundary of K, in order to avoid the possible occurrence of "turning external flows".

Sufficient conditions for local invariance of a given set with respect to a given function, or even to a l.s.c. multifunction, were obtained by imposing some usual uniqueness hypotheses along with viability assumptions. See for instance [2], [5], [6], [7], [8], [10], [14], [15], [24]. Necessary conditions, which are expressed in the terms of a tangency condition of the type (1.3), besides the continuity of f, require also some uniqueness hypotheses. In a different spirit, more closely related to dynamical systems than to differential equations, the local invariance problem was studied by Ursescu [21], [22]. The main idea in [22] was to consider from the very beginning that a given abstract evolution operator which stands for the set of "all solutions" satisfies a certain tangency condition coupled with a uniqueness hypothesis. It should be mentioned that, in this general context, there is no need of a "right-hand side" f of the associated differential equation - if any.

In contrast with the above mentioned approaches, here, we are considering the classical differential equation (1.1) and we are looking for general sufficient and even necessary conditions for invariance expressed only in terms of f, K and D, but not in the terms of the panel of solutions of (1.1). Moreover, we are interested in those conditions allowing (1.1) to have multiple solutions in K. It should be noticed that, there are situations when (1.1) fails to have the local uniqueness property on K, cases in which the results just mentioned are not applicable. An extremely simple example of this sort is suggested by the preceding one. Namely let us consider the function $\overline{g} : \mathbb{R}^2 \to \mathbb{R}^2$, defined as above. Then, one can easily see that (1.1), with f replaced with $\overline{g}$, has multiple solutions in $P = \{(x, y) \in \mathbb{R}^2 ;\ y \geq 0\}$, and the latter is locally invariant with respect to $\overline{g}$.

The main goal of this paper is to show that, whenever f — possibly discontinuous — satisfies the surprisingly simple "exterior tangency" condition: *there exists an open neighborhood V of K, with $V \subset D$, such that*

$$\liminf_{h \downarrow 0} \frac{1}{h} \left[d(\xi + hf(t, \xi); K) - d(\xi; K) \right] \leq \omega(t, d(\xi; K)) \tag{1.4}$$

for each $(t, \xi) \in [a, b] \times V$, where ω is a certain comparison function, K is locally invariant with respect to f. A specific form of this condition, i.e. with $\omega \equiv 0$, was considered in Aubin [1], Theorem 5.2.1, p. 168, in order to get the local invariance of a subset K with respect to a locally bounded l.s.c. multi-valued function F. The condition (1.4) reduces to the classical Nagumo's tangency condition (1.3) when applied to $\xi \in K$, and this simply because, at each such point $\xi \in K$, $d(\xi; K) = 0$. So, we can easily see that, whenever K is open, (1.4) is automatically satisfied for the choice $V = K$. More than this, we will show that in many situations, the condition above is even necessary for local invariance. The truth of this result rests on the simple observation that local invariance is equivalent to the "(D, K)-separating uniqueness" property defined below, while (1.4), along with the continuity of f, implies both local viability and (D,K)-separating uniqueness. More precisely, we say that (1.1) has the *(D,K)-separating uniqueness property* if, for each $(\tau,\xi)\in[a,b)\times D$ and every pair of solutions $u,v:[\tau,T]\to D$ of (1.1), satisfying $u(\tau)=v(\tau)=\xi$, there exists $c\in(\tau,T]$ such that both $u((\tau,c])$ and $v((\tau,c])$ are included either in $D\setminus K$, or in K. In fact, the condition of viability and (D,K)-separating uniqueness is nothing else than a simple rephrasing of the local invariance property. Indeed, *if $K\subset D\subset\mathbb{R}^n$, with K closed and D open, and $f:[a,b]\times D\to\mathbb{R}^n$ is continuous, then K is locally invariant with respect to f if and only if* (1.3) *is satisfied and* (1.1) *has the (D,K)-separating uniqueness property.*

Although our main results can be extended to some infinite-dimensional differential inclusions, in order to avoid distracting technicalities, we preferred this simple but classical, both single-valued and finite-dimensional, framework . For details on a more general setting the interested reader is referred to Necula-Vrabie [17]. We also notice that one can treat the Carathéodory case as well with natural "a.e." simple modifications in both statements and proofs.

The paper is divided into six sections the second one being merely devoted to the statement of our main results. The complete proofs are included in Section 3, while, in Section 4, we prove some sufficient conditions for local invariance which are variants of Theorem 2.1. Section 5 concerns a case in which viability comes from invariance, while in Section 6 we discuss the relationship between the viability of the epigraph of a certain function v with respect to a comparison function ω, and the differential inequality $D_+v\le\omega(t,v)$.

2. Statement of the main result

Throughout, we consider $\mathbb{R}^n$ endowed with one of its norms, $\|\cdot\|$. As usual, if $\xi\in\mathbb{R}^n$ and $\rho>0$, $B(\xi,\rho)$ denotes the closed ball centered at ξ and of radius ρ.

Definition 2.1. A subset K in $\mathbb{R}^n$ is *locally closed* if for each $\xi\in K$, there exists $\rho>0$ such that $K\cap B(\xi,\rho)$ is closed.

Clearly, each open subset, as well as each closed subset in $\mathbb{R}^n$ is locally closed, but there exist locally closed subsets which are neither open nor closed, as for instance the "interior" of a two-dimensional disk in $\mathbb{R}^3$.

Throughout in what follows we denote by $[D_+x](t)$ the right lower Dini derivative of the function $x:[a,b]\to\mathbb{R}$ at $t\in[a,b)$, i.e.

$$[D_+x](t)=\liminf_{h\downarrow 0}\frac{x(t+h)-x(t)}{h}.$$

If $\xi,\eta\in\mathbb{R}^n$, we denote by $[\xi,\eta]_+$ *the right directional derivative* of the norm $\|\cdot\|$ calculated at ξ in the direction η. Similarly, $(\xi,\eta)_+$ denotes *the right directional derivative* of $\frac{1}{2}\|\cdot\|^2$ calculated at ξ in the direction η. One may easily see that

$$(\xi,\eta)_+=\|\xi\|[\xi,\eta]_+$$

for each $\xi,\eta\in\mathbb{R}^n$, and, if $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$, where $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^n$,

$$(\xi,\eta)_+=\langle\xi,\eta\rangle.$$

Definition 2.2. A function $\omega:[a,b]\times[0,\rho)\to\mathbb{R}$ is a *comparison* function if $\omega(t,0)=0$ for each $t\in[a,b]$, and, for each $[\tau,T]\subset[a,b]$, the only continuous function $x : [\tau, T] \to [0, \rho)$, satisfying

$$[D_+x](t) \le \omega(t, x(t)) \quad \text{for all } t \in [\tau, T]$$
$$x(\tau) = 0,$$

is the null function.

A point $\xi \in \mathbb{R}^n$ *has projection on* K if there exists $\eta \in K$ with $\|\xi - \eta\| = d(\xi; K)$. Any $\eta \in K$ enjoying the above property is called *a projection* of ξ on K, and the set of all projections of ξ on K is denoted by $\Pi_K(\xi)$. We recall that if K is locally closed, then the set of all points $\xi \in \mathbb{R}^n$ for which $\Pi_K(\xi) \neq \emptyset$ is a neighborhood of K. See Lemma 18 in Cârjă-Ursescu [6].

Definition 2.3. An open neighborhood V of K is called a *proximal neighborhood* of K if, for each $\xi \in V$, $\Pi_K(\xi) \neq \emptyset$. If V is a proximal neighborhood of K, then every single-valued selection, $\pi_K : V \to K$, of Π_K, i.e. $\pi_K(\xi) \in \Pi_K(\xi)$ for each $\xi \in V$, is a *projection subordinated to* V.

Definition 2.4. Let $K \subset \mathbb{R}^n$ be locally closed and let D be an open neighborhood of K. We say that a function $f : [a, b] \times D \to \mathbb{R}^n$ has *the comparison property with respect to* (D, K) if there exists a proximal neighborhood $V \subset D$ of K, such that, for each $\xi_0 \in K$ there exist $r > 0$, one projection $\pi_K : V \to K$ subordinated to V, and one comparison function $\omega : [a, b] \times [0, \rho) \to \mathbb{R}$, with $\rho = \sup_{\xi \in V} d(\xi; K)$, such that

$$[\xi - \pi_K(\xi), f(t, \xi) - f(t, \pi_K(\xi))]_+ \le \omega(t, \|\xi - \pi_K(\xi)\|) \tag{2.1}$$

for each $(t, \xi) \in [a, b] \times [B(\xi_0, r) \cap V]$.

Clearly (2.1) is always satisfied for each $\xi \in K$, and therefore, in Definition 2.4, we have merely to assume that (2.1) holds for each $\xi \in B(\xi_0, r) \cap [V \setminus K]$.

Definition 2.5. The function $f : [a, b] \times D \to \mathbb{R}^n$ is called:

(i) (D, K)-*Lipschitz* if there exist a proximal neighborhood $V \subset D$ of K, a subordinated projection $\pi_K : V \to K$, and $L > 0$, such that

$$\|f(t, \xi) - f(t, \pi_K(\xi))\| \le L\|\xi - \pi_K(\xi)\|$$

for each $(t, \xi) \in [a, b] \times [V \setminus K]$;

(ii) (D, K)-*dissipative* if there exist a proximal neighborhood $V \subset D$ of K, and a projection, $\pi_K : V \to K$, subordinated to V, such that

$$[\xi - \pi_K(\xi), f(t, \xi) - f(t, \pi_K(\xi))]_+ \le 0$$

for each $(t, \xi) \in [a, b] \times [V \setminus K]$.

Let V be a proximal neighborhood of K, and let $\pi_K : V \to K$ be a projection subordinated to V. If $f : [a, b] \times V \to K$ is a given function with the property that, for each $\eta \in K$, its restriction to the "rectangle"

$$V_\eta = \{(t, \xi) \in [a, b] \times [V \setminus K]\,;\ \pi_K(\xi) = \eta\}$$

is dissipative, then f is (D, K)-dissipative.

It is easy to see that if f is either (D, K)-Lipschitz, or (D, K)-dissipative, then it has the comparison property with respect to (D, K). Simple examples show that there are functions f which, although neither (D, K)-Lipschitz, nor (D, K)-dissipative, do have the comparison property. Moreover, there exist functions which, although (D, K)-Lipschitz, are not Lipschitz on D, as well as, functions which although (D, K)-dissipative, are not dissipative on D. In fact, these two properties describe merely the local "exterior" behavior of f at the interface between K and $D \setminus K$. We include below two "autonomous" examples: the first one of an (D, K)-Lipschitz function which is not locally Lipschitz, and the second one of a function which, although non-dissipative, is (D, K)-dissipative.

Example 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function whose graph is as in Figure 1 below, and let $K = [a, b]$.

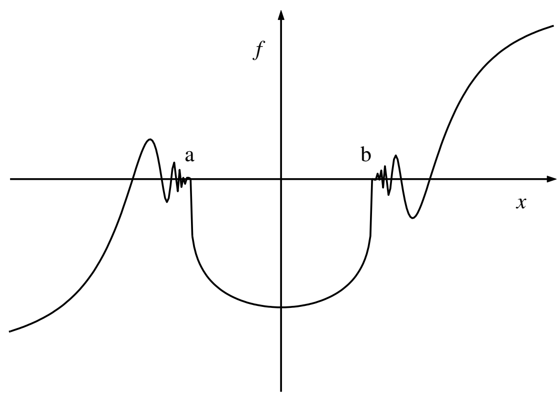

FIGURE 1.

Then, f is $(\mathbb{R}, K)$-Lipschitz with $V = \mathbb{R}$. As both the right derivative at a and the left derivative at b are not finite, f is not locally Lipschitz on $\mathbb{R}$.

Example 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function whose graph is illustrated in Figure 2.

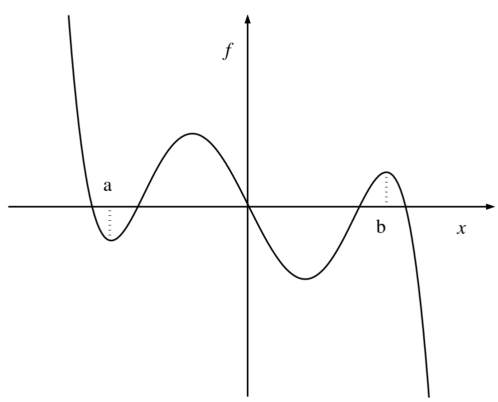

FIGURE 2.

Let K be any set including $[a, b]$ and let D be any open subset in $\mathbb{R}$ with $K \subset D$. One may easily verify that f is (D, K)-dissipative with $V = D$, but it is not dissipative.

Our main result is:

Theorem 2.1. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$ be such that* (1.4) *is satisfied. Then K is locally invariant with respect to f.*

Remark 2.1. Clearly (1.4) is satisfied with $\omega = \omega_f$, where $\omega_f : [a,b] \times [0,\rho) \to \overline{\mathbb{R}}$, $\rho = \sup_{\xi \in V} d(\xi; K)$, is defined by

$$\omega_f(t,r) = \sup_{\substack{\xi \in V \\ d(\xi;K)=r}} \liminf_{h \downarrow 0} \frac{1}{h}[d(\xi + hf(t,\xi); K) - d(\xi; K)] \tag{2.2}$$

for each $(t,\xi) \in [a,b] \times [0,\rho)$. If f is locally bounded, then ω_f is finite valued.

So, Theorem 2.1 can be reformulated as:

Theorem 2.2. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If there exists an open neighborhood V of K with $V \subset D$ such that ω_f defined by* (2.2) *is a comparison function, then K is locally invariant with respect to f.*

If f has the comparison property, the converse statement in Theorem 2.1 holds also true.

Theorem 2.3. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If f has the comparison property with respect to (D,K), and* (1.3) *is satisfied, then* (1.4) *is also satisfied.*

By observing that the necessity part of Nagumo's Viability Theorem holds true without the continuity assumption on f, from Theorem 2.3, we deduce:

Theorem 2.4. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If f has the comparison property with respect to (D,K), and K is viable with respect to f, then* (1.4) *is also satisfied.*

An immediate consequence of Theorems 2.3 and 2.4 is:

Theorem 2.5. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If f has the comparison property with respect to (D,K), and K is viable with respect to f, then K is locally invariant with respect to f.*

3. Proofs of the Theorems

We begin with the proof of Theorem 2.1.

Proof. Let $V \subset D$ be the open neighborhood of K whose existence is ensured by (1.4), let $\xi \in K$ and let $u : [\tau, T] \to V$ be any local solution of (1.1) and (1.2). Diminishing c if necessary, we may assume that there exists $\rho > 0$ such that $B(\xi,\rho) \cap K$ is closed and $u(t) \in B(\xi, \rho/2)$ for each $t \in [\tau, T]$. Let $g : [\tau, T] \to \mathbb{R}_+$ be defined by $g(t) = d(u(t); K)$ for each $t \in [\tau, T]$. Let $t \in [\tau, T)$ and $h > 0$ with $t + h \in [\tau, T]$. We have

$$\begin{aligned} g(t+h) &= d(u(t+h); K) \\ &\leq h \left\| \frac{u(t+h) - u(t)}{h} - u'(t)) \right\| + d(u(t) + hu'(t)); K). \end{aligned}$$

Hence
$$\frac{g(t+h)-g(t)}{h} \le \alpha(h) + \frac{d(u(t)+hu'(t);K)-d(u(t);K)}{h},$$
where
$$\alpha(h) = \Big\| \frac{u(t+h)-u(t)}{h} - u'(t) \Big\|.$$
Since $u'(t) = f(t,u(t))$ and $\lim_{h\downarrow 0} \alpha(h) = 0$, passing to the inf-limit in the inequality above for $h \downarrow 0$, and taking into account that V, K, and f satisfy (1.4), we get
$$[D_+ g](t) \le \omega(t, g(t))$$
for each $t \in [\tau, T)$. So, $g(t) \equiv 0$ which means that $u(t) \in \overline{K} \cap B(\xi, \rho/2)$. But $\overline{K} \cap B(\xi,\rho/2) \subset K \cap B(\xi,\rho)$, and the proof is complete . ☐

We can now proceed to the proof of Theorem 2.3.

Proof. Let $V \subset D$ be the open neighborhood of K as in Definition 2.4, let $\xi \in V$ and $t \in [a,b]$. Let $r > 0$ and let π_K be the selection of Π_K as in Definition 2.4. Let $h > 0$ with $t + h \in [t,T]$. Since $\|\xi - \pi_K(\xi)\| = d(\xi;K)$, we have
$$\begin{aligned} d(\xi + hf(t,\xi);K) - d(\xi;K) &\le \|\xi - \pi_K(\xi) + h[f(t,\xi) - f(t,\pi_K(\xi))]\| \\ &\quad - \|\xi - \pi_K(\xi)\| + d(\pi_K(\xi) + hf(t,\pi_K(\xi));K). \end{aligned}$$
Dividing by h, passing to the inf-limit for $h \downarrow 0$, and using (1.3), we obtain
$$\begin{aligned} \liminf_{h\downarrow 0} \frac{1}{h}[d(\xi + hf(t,\xi);K) - d(\xi;K)] &\le [\xi - \pi_K(\xi), f(t,\xi) - f(t,\pi_K(\xi))]_+ \\ &\le \omega(t, \|\xi - \pi_K(\xi)\|). \end{aligned}$$
This inequality shows that (1.4) holds, and this completes the proof. ☐

Remark 3.1. Let K be locally closed and let V an open neighborhood of K such that $\Pi_K(\xi) \neq \emptyset$ for each $\xi \in V$. For each single-valued selection π_K of Π_K, we define $\omega_{f,\pi_K} : [a,b] \times [0,\rho) \to \overline{\mathbb{R}}$, where $\rho = \sup_{\xi\in V} d(\xi;K)$, by $\omega_{f,\pi_K}(t,0) = 0$ for $t \in [a,b]$ and
$$\omega_{f,\pi_K}(t,r) = \sup_{\substack{\xi\in V \\ \|\xi - \pi_K(\xi)\| = r}} [\xi - \pi_K(\xi), f(t,\xi) - f(t,\pi_K(\xi))]_+$$
if $r \in (0,\rho)$ and $t \in [a,b]$. Obviously, each locally bounded function f satisfies (2.1) with ω replaced by ω_{f,π_K}, the latter being *the smaller function* ω satisfying (2.1) for a certain π_K. So, we have the following consequence of Theorem 2.3:

Theorem 3.1. *Let $K \subset D \subset \mathbb{R}^n$, with K locally closed and D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If there exist an open neighborhood V of K with $\Pi_K(\xi) \neq \emptyset$ for each $\xi \in V$, and a selection π_K of Π_K such that ω_{f,π_K}, defined as above, is a comparison function and* (1.3) *is satisfied, then* (1.4) *is also satisfied.*

4. Sufficient conditions for invariance revisited

Let $K \subset \mathbb{R}^n$ and let V be a open neighborhood of K.

Definition 4.1. A function $g : V \to \mathbb{R}_+$ is *a proximal generalized distance* if:
- (i) g is Lipschitz continuous on bounded subsets in V;
- (ii) $g(\xi) = 0$ if and only if $\xi \in K$.

A simple example of a proximal generalized distance is $g(\xi) = \alpha(d(\xi;K))$, where $\alpha : [0,+\infty) \to [0,+\infty)$ is Lipschitz on bounded subsets, with $\alpha(r) = 0$ if and only if $r = 0$, while $d(\xi;K)$ is the usual distance from ξ to K. We notice that, whenever there exists a proximal generalized distance $g : V \to [0,+\infty)$, K is locally closed. Indeed, since $K = \{\xi \in V\,;\ g(\xi) = 0\}$ and g is continuous, K is relatively closed in V. But V is open and thus K is locally closed, as claimed.

Theorem 4.1. *Let $K \subset D \subset \mathbb{R}^n$, with D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If there exist an open neighborhood V of K and a generalized distance $g : V \to \mathbb{R}_+$ such that*

$$\liminf_{h\downarrow 0} \frac{1}{h}[g(\xi + hf(t,\xi)) - g(\xi)] \le \omega(t, g(\xi)) \tag{4.1}$$

for each $(t,\xi) \in [a,b] \times V$, then K is locally invariant with respect to f.

Proof. The proof follows closely that one of Theorem 2.1, with the special mention that here one has to use the obvious inequality $g(\lambda) \le g(\eta) + L\|\lambda - \eta\|$ for each $\lambda, \eta \in B(\xi,\rho) \cap V$, where $L > 0$ is the Lipschitz constant of g on $B(\xi,\rho) \cap V$. □

In order to obtain a simple, but useful, extension of Theorem 2.1, some observations are needed. Namely, if $g : V \to [0,+\infty)$ is a generalized distance, we may consider the generalized tangency condition

$$\liminf_{h\downarrow 0} \frac{1}{h} g(\xi + hf(t,\xi)) = 0 \tag{4.2}$$

for each $(t,\xi) \in [a,b] \times K$, and one may ask whether this implies viability, whenever, of course f is continuous. The answer to this question is in the negative as the simple example below shows.

Example 4.1. Let K be locally closed, let V be any open neighborhood of K and let $g : V \to [0,+\infty)$ be defined as $g(\xi) = d^2(\xi;K)$ for each $\xi \in V$. Further, let $f : [a,b] \times K \to \mathbb{R}^n$ be a continuous function such that K is not viable with respect to f. We can always find such a function whenever K is not open. Now, since $g(\xi + hf(t,\xi)) \le \|\xi + hf(t,\xi) - \xi\|^2 \le h^2\|f(t,\xi)\|^2$ for each $(t,\xi) \in [a,b] \times K$, (4.2) is trivially satisfied. Thus the generalized tangency condition (4.2) does not imply the viability of K with respect to f.

This example shows that if g is a generalized distance and g^2 satisfies (4.2), it may happen that g does not satisfy (4.2). Therefore it justifies why, in the next result, we assume explicitly that (4.2) holds true, even though it is automatically satisfied by g^2.

Theorem 4.2. *Let $K \subset D \subset \mathbb{R}^n$, with D open, and let $f : [a,b] \times D \to \mathbb{R}^n$. If there exist an open neighborhood V of K and a generalized distance $g : V \to \mathbb{R}_+$ satisfying* (4.2) *and such that*

$$\liminf_{h\downarrow 0} \frac{1}{2h}[g^2(\xi + hf(t,\xi)) - g^2(\xi)] \le g(\xi)\omega(t, g(\xi)) \tag{4.3}$$

for each $(t,\xi) \in [a,b] \times V$, then K is locally invariant with respect to f.

Proof. We have only to observe that, in the presence of (4.2), (4.3) and (4.1) are equivalent. □

Using Theorem 4.2, we will prove some other sufficient condition for invariance expressed in the terms of a generalized Lipschitz projection. Namely, $K \subset \mathbb{R}^n$ is *Lipschitz retract* if there exist an open neighborhood V of K and a Lipschitz continuous map, $r : V \to K$, with $r(\xi) = \xi$ if and only if $\xi \in K$. The function r as above is called *generalized Lipschitz projection*. On each Lipschitz retract K one can define a proximal generalized distance $g : V \to \mathbb{R}_+$, by $g(\xi) = \|r(\xi) - \xi\|$ for all $\xi \in V$. Consequently, each Lipschitz retract subset K is locally closed. Moreover, each open subset K is Lipschitz retract (take $V = K$ and r the identity). Another simple example of Lipschitz retract is given by a closed subset K which has an open neighborhood V for which there exists a single-valued continuous projection $\pi_K : V \to K$, i.e. $d(\xi; K) = \|\pi_K(\xi) - \xi\|$ for each $\xi \in V$. In the latter case we say that K is *a proximate retract*. It should be noticed that the class of Lipschitz retract subsets is strictly larger than that of proximate retracts as the simple example below shows.

Example 4.2. Let us consider $\mathbb{R}^2$, equipped with its usual Hilbert structure and let us observe that the set

$$K = \left\{(x, y) \in \mathbb{R}^2\,; y \leq |x|\right\},$$

although a Lipschitz retract, is not a proximate retract. Indeed, let $V = \mathbb{R}^2$, and let $r((x, y))$ be defined, either as (x, y) if $(x, y) \in K$, or as $(x, |x|)$ if $(x, y) \in V \setminus K$. It is easy to see that r is a generalized Lipschitz projection with Lipschitz constant $\sqrt{2}$, and thus K is a Lipschitz retract. Nevertheless, K is not a proximate retract since any selection π_K of the the projection Π_K is discontinuous at each point $(0, y)$, with $y > 0$.

It should be emphasized that all the results which will follow can be reformulated to handle also locally Lipschitz retract subsets, i.e. those subsets K satisfying: *for each $\xi \in K$ there exists $\rho > 0$ such that $B(\xi; \rho) \cap K$ is Lipschitz retract*, but for the sake of simplicity we confined ourselves to the simpler case of Lipschitz retracts. First, let K be Lipschitz retract with the corresponding generalized Lipschitz projection $r : V \to K$. In the next two theorems we assume that $\|\cdot\|$ is defined by $\|x\|^2 = \langle x, x\rangle$, where $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^n$.

Theorem 4.3. *Let $D \subset \mathbb{R}^n$ be open and let $f : [a, b] \times D \to \mathbb{R}^n$. Let us assume that $K \subset D$ is Lipschitz retract with generalized Lipschitz projection $r : V \to K$ satisfying*

$$\liminf_{h\downarrow 0} \frac{1}{h}\|r(\xi + hf(t, \xi)) - \xi - hf(t, \xi)\| = 0 \tag{4.4}$$

for each $(t, \xi) \in [a, b] \times K$. Assume in addition that there exists a comparison function $\omega : [a, b] \times [0, \rho) \to \mathbb{R}$, with $\rho = \sup_{\xi\in V} \|r(\xi) - \xi\|$[1] *such that*

$$\liminf_{h\downarrow 0} \frac{1}{h}\left\langle r(\xi + hf(t, \xi)) - r(\xi) - hf(t, \xi), r(\xi) - \xi\right\rangle \leq \|r(\xi) - \xi\|\omega(t, \|r(\xi) - \xi\|) \tag{4.5}$$

for each $(t, \xi) \in [a, b] \times V$. Then K is locally invariant with respect to f.

[1]Diminishing V if necessary, we may always assume that $\sup_{\xi\in D} \|r(\xi) - \xi\|$ is finite.

Proof. Let us define $g(\xi) = \|r(\xi) - \xi\|$ for each $\xi \in V$ and let $L > 0$ be the Lipschitz constant of r. We have

$$\begin{aligned}
&g^2(\xi + h\eta) - g^2(\xi)\\
&= \langle r(\xi + h\eta) - (\xi + h\eta) - (r(\xi) - \xi)), r(\xi + h\eta) - (\xi + h\eta) + (r(\xi) - \xi))\rangle\\
&= \langle r(\xi + h\eta) - r(\xi), r(\xi + h\eta) + r(\xi) - 2\xi\rangle - h\langle \eta, 2r(\xi + h\eta) - 2\xi\rangle + h^2\|\eta\|^2\\
&= \|r(\xi + h\eta) - r(\xi)\|^2 + 2\langle r(\xi + h\eta) - r(\xi), r(\xi) - \xi\rangle - h\langle \eta, 2r(\xi + h\eta) - 2\xi\rangle\\
&\quad + h^2\|\eta\|^2\\
&\le (L^2 + 1)h^2\|\eta\|^2 + 2\langle r(\xi + h\eta) - r(\xi), r(\xi) - \xi\rangle - 2h\langle \eta, r(\xi + h\eta) - \xi\rangle.
\end{aligned}$$

Hence,

$$\liminf_{h\downarrow 0} \frac{1}{2h}[g^2(\xi + h\eta) - g^2(\xi)] \le \liminf_{h\downarrow 0} \frac{1}{h}\,\langle r(\xi + h\eta) - r(\xi), r(\xi) - \xi\rangle - \langle \eta, r(\xi) - \xi\rangle.$$

Since, by (4.4), g satisfies (4.2), taking $\eta = f(t, \xi)$ and using (4.5) and Theorem 4.2, we get the conclusion. □

¿From Theorem 4.3 we deduce:

Theorem 4.4. *Let $K \subset D$ with D open and let $f : [a, b] \times D \to \mathbb{R}^n$. Let us assume that K is a Lipschitz retract with the generalized Lipschitz projection $r : V \to K$ satisfying* (4.4)*. Let us assume in addition that, for each $t \in [a, b]$ and $\xi \in V$, there exists the directional derivative, $r'(\xi)[f(t, \xi)]$, of r, at ξ in the direction $f(t, \xi)$, and*

$$\langle r'(\xi)[f(t, \xi)] - f(t, \xi), r(\xi) - \xi\rangle \le \|r(\xi) - \xi\|\omega(t, \|r(\xi) - \xi\|), \tag{4.6}$$

where $\omega : [a, b] \times [0, \rho)$ is a comparison function, with $\rho = \sup_{\xi\in V} \|r(\xi) - \xi\|$. Then K is locally invariant with respect to f.

Proof. It is easy to see that, in this specific case, (4.6) is equivalent with (4.5) and this completes the proof. □

Remark 4.1. Let $K \subset \mathbb{R}^n$ be a Lipschitz retract and let $r : V \to K$ be the corresponding generalized Lipschitz projection. Let $f : [a, b] \times V \to \mathbb{R}^n$ be a continuous function, $\rho = \sup_{\xi\in V} \|r(\xi) - \xi\|$, and let us define $\omega : [a, b] \times [0, \rho) \to \overline{\mathbb{R}}_+$ by $\omega(t, 0) = 0$ and

$$\omega(t, x) = \sup_{\substack{\xi\in V\\ \|r(\xi)-\xi\|=x}} \frac{\langle r'(\xi)[f(t, \xi)] - f(t, \xi), r(\xi) - \xi\rangle}{\|r(\xi) - \xi\|} \tag{4.7}$$

for each $(t, x) \in [a, b] \times (0, \rho)$, where $\rho = \sup_{\xi\in V} \|r(\xi) - \xi\|$. From Theorem 4.4 it follows that K is locally invariant with respect to f if (4.4) is satisfied and ω, defined by (4.7), is a comparison function. Furthermore, if $K \subset \mathbb{R}^n$ is a closed linear subspace in $\mathbb{R}^n$, and r is the projection of $\mathbb{R}^n$ on K, then r is linear, and $r'(\xi)[\eta] = r(\eta)$ for each $\xi, \eta \in \mathbb{R}^n$. So, in this case, the tangency condition (4.4) is equivalent to $f(K) \subset K$. Take $D = \{\xi \in \mathbb{R}^n\,;\ d(\xi; K) < \rho\}$, for some fixed $\rho > 0$, and let us observe that the function ω, defined by (4.7), is given by $\omega(t, 0) = 0$ and

$$\omega(t, x) = \sup_{\substack{\xi\in V\\ \|r(\xi)-\xi\|=x}} \frac{\langle r(f(t, \xi)) - f(t, \xi), r(\xi) - \xi\rangle}{\|r(\xi) - \xi\|} \tag{4.8}$$

for each $(t,x) \in [a,b] \times (0,\rho)$. Hence, if $f(K) \subset K$, and ω defined by (4.8) is a comparison function, then K is locally invariant with respect to (1.1).

5. Viability implies invariance

It is well-known that whenever $f : [a,b] \times K \to \mathbb{R}^n$ is continuous, satisfies the classical Nagumo tangency condition (1.3), and K is a proximate retract, then f can be extended to a function $\widetilde{f} : [a,b] \times D \to \mathbb{R}^n$, with $D \subset \mathbb{R}^n$ open and $K \subset D$, such that K be invariant with respect to $\widetilde{f}$. We notice that we can take $\widetilde{f}(t,\xi) = f(t, \pi_K(\xi))$ for each $(t,\xi) \in [a,b] \times D$, where $\pi_K : D \to K$ is the corresponding projection. Our aim here is to show that this simple result can be extended to Lipschitz retract subsets whenever f satisfies a suitable tangency condition. We notice that there are examples of locally closed subsets K and of continuous functions $f : [a,b] \times K \to \mathbb{R}^n$, satisfying (1.3), which cannot be extended to continuous functions $\widetilde{f}$ defined on $[a,b] \times D$ with D a certain open neighborhood of K, and such that $\widetilde{f}$ satisfy (1.4). See for instance Aubin-Cellina [2], Example on page 203.

The main result in this section is:

Theorem 5.1. *Let $f : [a,b] \times K \to \mathbb{R}^n$, and let us assume that K is Lipschitz retract with generalized Lipschitz projection $r : V \to K$. If there exists a generalized distance $g : V \to \mathbb{R}_+$ and a comparison function $\omega : [a,b] \times [0,\rho) \to \mathbb{R}$, with $\rho = \sup_{\xi \in V} g(\xi)$, such that*

$$\liminf_{h \downarrow 0} \frac{1}{h}[g(\xi + hf(t, r(\xi))) - g(\xi)] \leq \omega(t, g(\xi)), \tag{5.1}$$

for each $(t,\xi) \in [a,b] \times V$, then the function $\widetilde{f} : [a,b] \times V \to \mathbb{R}^n$, defined by $\widetilde{f}(t,\xi) = f(t, r(\xi))$, for each $(t,\xi) \in [a,b] \times V$, satisfies (4.1), and consequently K is invariant with respect to $\widetilde{f}$.

Proof. The conclusion is an immediate consequence of Theorem 4.1. □

6. Comparison and viability

The next result was called to our attention by Ursescu [23]. Similar results can be found in Clarke–Ledyaev–Stern [9].

Theorem 6.1. *Let $\omega : [a,b] \times \mathbb{R}_+ \to \mathbb{R}$ and $v : [\tau, T] \to \mathbb{R}_+$ be continuous, with $[\tau, T] \subset [a,b]$. Then $\operatorname{Epi}(v) = \{(t,\eta);\ v(t) \leq \eta,\ t \in [\tau, T)\}$ is viable with respect to $(t,y) \mapsto (1, \omega(t,y))$ if and only if v satisfies*

$$[D_+ v](t) \leq \omega(t, v(t)) \tag{6.1}$$

for each $t \in [\tau, T)$.

Proof. Sufficiency. It suffices to show that the set $\{(t, v(t));\ t \in [\tau, T)\}$, which is included in the boundary $\partial\operatorname{Epi}(v)$ of $\operatorname{Epi}(v)$, satisfies the Nagumo's tangency condition (1.3). From (6.1) it follows that

$$\Big[D_+\Big(v(\cdot) - \int_\tau^{\cdot} \omega(s, v(s))\,ds\Big)\Big](t) \leq 0$$

for each $t \in [\tau, T)$. Thus, in view of a classical result in Hobson [13], p. 365, we necessarily have that $t \mapsto v(t) - \int_\tau^t \omega(s, v(s))\,ds$ is non-increasing on $[\tau, T]$. So

$$\Big(t + h, v(t) + \int_t^{t+h} \omega(s, v(s))\,ds\Big) \in \mathrm{Epi}(v)$$

and therefore,

$$\begin{aligned} &d((t, v(t)) + h(1, \omega(t, v(t))); \mathrm{Epi}(v)) \\ &\le \Big\|(t, v(t)) + h(1, \omega(t, v(t))) - \Big(t + h, v(t) + \int_t^{t+h} \omega(s, v(s))\,ds\Big)\Big\| \\ &= \Big|h\omega(t, v(t)) - \int_t^{t+h} \omega(s, v(s))\,ds\Big|. \end{aligned}$$

Dividing by $h > 0$ and passing to lim inf for $h \downarrow 0$ we get (1.3) and this completes the proof of the sufficiency.

Necessity. Let us assume that $\mathrm{Epi}(v)$ is viable with respect to the function $(t, y) \mapsto (1, \omega(t, y))$, let $t \in [a, b)$, and let $(s, x); [0, \delta) \to \mathrm{Epi}(v)$ be a solution of the system $s'(\theta) = 1$, $x'(\theta) = \omega(s(\theta), x(\theta))$, subjected to the initial conditions $s(0) = t$ and $x(0) = v(t)$. This means that $v(s(h)) \le x(h)$ for all $h \in [0, \delta)$. But $s(h) = t+h$ and so we have

$$\frac{v(t + h) - v(t)}{h} \le \frac{x(h) - x(0)}{h}.$$

Hence

$$[D_+v](t) \le \omega(t, x(0)) = \omega(t, v(t))$$

for each $t \in [\tau, T)$. The proof is complete. □

Corollary 6.1. *Let $\omega : [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ be continuous and such that, for each $\tau \in [a, b)$, the Cauchy problem $y'(t) = \omega(t, y(t))$, $y(\tau) = 0$ has only the null solution. Then ω is a comparison function.*

Proof. Let $v : [\tau, T] \to \mathbb{R}$ be any solution of (6.1). By Theorem 6.1, $\mathrm{Epi}(v)$ is viable with respect to $(t, y) \mapsto (1, \omega(t, y))$. So, the unique solution $y : [\tau, T) \to \mathbb{R}_+$ of the Cauchy problem $y'(t) = \omega(t, y(t))$, $y(\tau) = 0$ satisfies $0 \le v(t) \le y(t) = 0$. □

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