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SOME METRIC-SINGULAR PROPERTIES OF THE GRAPH OF SOLUTIONS OF THE ONE-DIMENSIONAL P-LAPLACIAN

MERVAN PAŠIĆ & VESNA ŽUPANOVIĆ

Abstract. We study the asymptotic behaviour of ε-neighbourhood of the graph of a type of rapidly oscillating continuous functions. Next, we estate necessary and sufficient conditions for rapid oscillations of solutions of the main equation. This enables us to verify some new singular properties of bounded continuous solutions of a class of nonlinear p-Laplacian by calculating lower and upper bounds for the Minkowski content and the s-dimensional density of the graph of each solution and its derivative.

1. Introduction

Let $-\infty < a < b < \infty$. Let y be a real function defined on $[a, b]$ and let y' denote the derivative of y in the classical sense. The main subjects of the paper are the graph $G(y)$ of a function y and its $\varepsilon-$ neighbourhood $G_\varepsilon(y)$, that is

$$G(y) = \{(t, y(t)) : a \le t \le b\}$$
$$G_\varepsilon(y) = \{(t_1, t_2) \in \mathbb{R}^2 : d((t_1, t_2), G(y)) \le \varepsilon\}.$$

Here $\varepsilon > 0$ and $d((t_1, t_2), G(y))$ denotes the distance between (t_1, t_2) and $G(y)$.

In the author's paper [9] for arbitrarily given $s \in (1, 2)$ it is constructed a class of Caratheodory functions $f(t, \eta, \xi)$ such that the graph $G(y)$ of each smooth enough solution y of the main equation

$$\begin{gathered} -(|y'|^{p-2}y')' = f(t, y, y') \quad \text{in } (a, b), \\ y(a) = y(b) = 0, \\ y \in W^{1,p}_{\rm loc}((a, b]) \cap C([a, b]), \end{gathered} \tag{1.1}$$

satisfies

$$\begin{gathered} \dim_M G(y) = s \quad \text{and} \quad \dim_M G(y') > 1, \\ \dim_{\rm Mloc}(G(y); a) = s \qquad \text{and} \quad \dim_{\rm Mloc}(G(y); t) = 1 \text{ for each } t \in (a, b]. \end{gathered} \tag{1.2}$$

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Here $\dim_M G(y)$ denotes the upper Minkowski-Bouligand (box-counting) dimension of the graph $G(y)$ and $\dim_{\mathrm{Mloc}}(G(y);t)$ denotes the locally upper Minkowski-Bouligand dimension of $G(y)$ at a point $t \in [a,b]$, defined by

$$\dim_M G(y) = \limsup_{\varepsilon\to 0} \big(2 - \frac{\log|G_\varepsilon(y)|}{\log\varepsilon}\big),$$
$$\dim_{\mathrm{Mloc}}(G(y);t) = \limsup_{\varepsilon\to 0} \dim_M(G(y)\cap B_\varepsilon(t,y(t))).$$

Here $|G_\varepsilon(y)|$ denotes the Lebesgue measure of $G_\varepsilon(y)$ and $B_\varepsilon(t_1,t_2)$ denotes a ball in $\mathbb{R}^2$ centered at the point (t_1,t_2) with radius $\varepsilon > 0$.

The orders of growth for the asymptotic behaviour of $|G_\varepsilon(y)|$ and $|G_\varepsilon(y')|$ are given in (1.2). That is, when $\varepsilon \approx 0$ we have

$$|G_\varepsilon(y)| \approx \varepsilon^{2-s} \quad \text{and} \quad |G_\varepsilon(y')| \approx \varepsilon^{2-q} \quad \text{for some } q > 1, \tag{1.3}$$

where y is any smooth enough solution of (1.1). In particular from (1.2) we also have

$$\begin{gathered} y \notin W^{1,p}(a,b) \quad \text{and} \quad \mathrm{length}(G(y)) = \mathrm{length}(G(y')) = \infty, \\ y \in W^{1,p}(a+\varepsilon,b) \quad \text{and} \quad \mathrm{length}(G(y|_{[a+\varepsilon,b]})) < \infty \text{ for any } \varepsilon > 0. \end{gathered} \tag{1.4}$$

In the present paper, we derive some new singular properties of the graphs $G(y)$ and $G(y')$, improving (1.2), (1.3) and (1.4). In this purpose, we need an equivalent way to define box-counting dimension

$$\dim_M G(y) = \inf\{\tau \ge 0 : M^\tau(G(y)) = 0\} = \sup\{\tau \ge 0 : M^\tau(G(y)) = \infty\}, \tag{1.5}$$

where $M^\tau(G(y))$ denotes the τ-dimensional upper Minkowski content of the graph $G(y)$ defined by

$$M^\tau(G(y)) = \limsup_{\varepsilon\to 0} (2\varepsilon)^{\tau-2}|G_\varepsilon(y)|.$$

According to (1.2) and (1.5) we may conclude that

$$M^\tau(G(y)) = 0 \text{ for all } \tau > s \quad \text{and} \quad M^\tau(G(y)) = \infty \text{ for all } \tau < s.$$

Thus, the following three cases are possible

(i) $M^s(G(y)) = 0$
(ii) $M^s(G(y)) = \infty$
(iii) $0 < M^s(G(y)) < \infty$.

In Section 2 and Section 5, for arbitrarily given $s \in (1,2)$ and under related assumptions on the nonlinearity $f(t,\eta,\xi)$ as in [9], we will prove that each solution y of (1.1) satisfies $0 < M^s(G(y)) < \infty$. That is, the graph $G(y)$ can be called as an s-set in respect to Minkowski content. Moreover, we find lower and upper bounds for $M^s(G(y))$ such that

$$0 < \frac{1}{2^7}(b-a)^s \le M^s(G(y)) \le m_s(b-a)^s < \infty, \tag{1.6}$$

where the positive constant m_s only depends on s. It is interesting in (1.6) that the s-dimensional "length" of $G(y)$ depends on the $s-$ power of the length of interval $[a,b]$, which improves $\mathrm{length}(G(y)) = \infty$ appearing in (1.4). Furthermore, we will give the existence of an $\varepsilon_0 > 0$ such that

$$0 < \frac{1}{2^6}(b-a)^s\varepsilon^{2-s} \le |G_\varepsilon(y)| \quad \text{for each } \varepsilon \in (0,\varepsilon_0), \tag{1.7}$$

where y is any solution of (1.1). This statement gives us a lower bound for asymptotic behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$. It is more precise than corresponding one in (1.3).

As the second, for arbitrarily given $s \in (1, 2)$ and under the same assumptions on the nonlinearity $f(t, \eta, \xi)$ as in getting of (1.6)–(1.7), we will prove in Section 3 that each smooth enough solution y of (1.1) satisfies

$$0 < \frac{1}{2^4}(b-a)^{s/2} \le M^{1+s/2}(G(y')). \tag{1.8}$$

Because of (1.5), the inequality (1.8) improves corresponding result from (1.2), where $\dim_M G(y') > 1$. Here, we have shown that $\dim_M G(y') \ge 1 + s/2$. Furthermore, we will show the existence of an $\varepsilon_0 > 0$ such that

$$0 < \frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2} \le |G_\varepsilon(y')| \quad \text{for each } \varepsilon \in (0, \varepsilon_0). \tag{1.9}$$

It gives us a lower bound for the asymptotic behaviour of $|G_\varepsilon(y')|$ as $\varepsilon \approx 0$.

Next, by means of (1.4) we have that the graph $G(y)$ of each solution y of (1.1) is high concentrated (in some sense) at the boundary point $t = a$. How much of the graph $G(y)$ is concentrated near $t = a$ in the sense of Minkowski content, we will consider in Section 6. In this purpose, we define the s-dimensional upper (Minkowski) density of $G(y)$ at a point $t \in [a, b]$ as follows

$$D^s(G(y); t) = \limsup_{r\to 0} \frac{M^s(G(y) \cap B_r(t, y(t)))}{(2r)^s}.$$

Let us remark that by means of $y \in W^{1,p}(a + \varepsilon, b)$, where $\varepsilon > 0$, it is clear that

$$D^s(G(y); t) = 0 \quad \text{for any } t \in (a, b] \quad \text{and} \quad s \in (1, 2).$$

In Section 6, for arbitrarily given $s \in (1, 2)$ and under the same assumptions on the nonlinearity $f(t, \eta, \xi)$ as in getting of (1.6)–(1.7), we will find a constant d_s such that each solution y of (1.1) satisfies

$$0 < d_s \le D^s(G(y); t = a). \tag{1.10}$$

Moreover, we will prove that

$$d_s(2r)^s \le M^s(G(y) \cap B_r(a, y(a))) \quad \text{for each } r \in (0, b-a), \tag{1.11}$$

where y is any smooth enough solution of (1.1). This inequality complete the statement (1.6).

To derive the statements (1.6)–(1.9), in Section 2, Section 3 and Section 5, we consider some metric properties of two types of rapid oscillations of real continuous functions: the first one is a kind of oscillations where the function is rapidly jumping over given obstacles, and the second one is a kind of oscillations where the convex and concave properties of the function are rapidly changing. In Section 4, some necessary conditions on the nonlinearity $f(t, \eta, \xi)$ will be given such that each solution of (1.1) is rapidly oscillating. These conditions on $f(t, \eta, \xi)$ will be very close to corresponding sufficient conditions used in Section 2. In appendix of the paper, we will give some technical results which play an important role in the proofs of the main results.

Further Remarks. A. It seems there is no known article dealing with this kind of problems. But, in the proofs of the main results we use some methods recently introduced in the author's paper [9].
B. The existence result for the equation (1.1) where the Caratheodory function $f(t,\eta,\xi)$ satisfies the assumptions as in the paper, was considered in the appendix of [9]. It was based on some known results about the existence of continuous solutions for the equations with singular nonlinearity, explored via the sub- and super-solutions technique. See O'Regan's book [8, Chapter 14].
C. Even the Minkowski content M^s is not a measure in the axiomatic sense, see for instance [3] and [7], the main results of the paper give some additional informations about the singular-metric boundary behaviour of the graph of any sufficiently smooth bounded solutions of the equation (1.1).
D. To make a comparison between singularity and regularity of bounded solutions, we refer to [12] and references therein, where regularity of solutions of quasilinear elliptic equations associated with p-Laplacian are studied.
E. About the properties and calculations of fractal dimensions and Minkowski content of several types of sets in $\mathbb{R}^n$, we refer to [1, 2, 3, 5, 7, 11, 14, 15].
F. Our main results can be generalized to the case of nonlinear variational inequalities and quasilinear elliptic systems associated with the one-dimensional p-Laplacian. See [10].

2. Rapid oscillations and lower bounds for $|G_\varepsilon(y)|$ and $M^s(G(y))$

For a function $y : [a,b] \to \mathbb{R}$ let us introduce a type of very rapid oscillations of y near the boundary point $t = a$. A classical example for such type of oscillations is the function $y(t) = t^\alpha \cos 1/t^\beta$ near $t = 0$, where $0 < \alpha < \beta$.

Definition 2.1. Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying

$$\begin{gathered} a_k \searrow a \text{ and there is an } \varepsilon_0 > 0 \text{ such that for each } \varepsilon \in (0,\varepsilon_0) \\ \text{there is a } k(\varepsilon) \in \mathbb{N} \text{ such that } a_{j-1} - a_j \le \varepsilon/2 \text{ for each } j \ge k(\varepsilon). \end{gathered} \tag{2.1}$$

Let θ and ω be two measurable and bounded functions, both defined on $[a,b]$, such that $\theta(t) \le \omega(t)$ for each $t \in [a,b]$. We say that a function y defined on $[a,b]$ is (θ,ω,a_k)-*rapidly oscillating* if there is a sequence $\sigma_k \in (a_k, a_{k-1})$, $k > 1$ such that

$$y(\sigma_{2k}) \ge \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega \quad \text{and} \quad y(\sigma_{2k+1}) \le \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta, \ k \ge 1. \tag{2.2}$$

For the record, the number $k(\varepsilon)$ could be called as the index of $\varepsilon-$ density of a sequence a_k. It is interesting to show how to calculate the number $k(\varepsilon)$ for a given sequence a_k which satisfies (2.1).

Example 2.2. Let a_k be a sequence of real numbers from interval (a,b) given by

$$a_k = a + \frac{b-a}{2}(\frac{1}{k})^{1/\beta}, \quad k \ge 1 \text{ and } 0 < \beta < \infty. \tag{2.3}$$

Such a type of the sequence a_k is appearing in oscillations of the function $y(t) = t^\alpha \sin 1/t^\beta$ near $t = 0$, where $a = 0$, $b = 1$ and $0 < \alpha < \beta$. One can take for $k(\varepsilon)$ to be any natural number which satisfies

$$2(\frac{\beta\varepsilon}{b-a})^{-\frac{\beta}{\beta+1}} \le k(\varepsilon) \quad \text{for each } \varepsilon \in (0,\varepsilon_0) \text{ and for any } \varepsilon_0 > 0. \tag{2.4}$$

Indeed, using an elementary inequality

$$\Big(\frac{1}{j-1}\Big)^{1/\beta}-\Big(\frac{1}{j}\Big)^{1/\beta}\le\frac{2^{1+1/\beta}}{\beta}\Big(\frac{1}{j}\Big)^{1+1/\beta},$$

where $\beta>0$ and $j\ge 2$, it is easy to show that the sequence a_k defined in (2.3) satisfies the statement (2.1) in respect to $k(\varepsilon)$ determined in (2.4).

Now, we consider the following lemma which is a modification of [9, Lemma 2.1, p. 271]. It gives us an elementary and useful metric property of the (θ,ω,a_k)-rapidly oscillating functions.

Lemma 2.3. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1). *Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t)\le\omega(t)$, $t\in[a,b]$, such that*

$$\operatorname*{ess\,inf}_{(a_{2k+2},a_{2k+1})}\theta\ge\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})}\theta,\quad \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})}\omega\le\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})}\omega,\quad k\ge 1. \tag{2.5}$$

Let y be (θ,ω,a_k)-rapidly oscillating function on $[a,b]$ and let $y\in C((a,b])$. Then we have

$$|G_\varepsilon(y)|\ge\int_a^{a_{k(\varepsilon)}}(\omega(t)-\theta(t))dt\quad \text{for each } \varepsilon\in(0,\varepsilon_0), \tag{2.6}$$

where $k(\varepsilon)$ and ε_0 are appearing in (2.1).

Let us remark that the condition (2.5) can be easy verified if, for instance, θ is decreasing and ω is increasing on $[a,b]$.

Proof of Lemma 2.3. Let ε be a fixed real number such that $\varepsilon\in(0,\varepsilon_0)$, where ε_0 is from (2.1). We use the notation

$$\begin{gathered}A(\varepsilon,\theta,\omega)=\{(t,y)\in\mathbb{R}^2: t\in(a,a_{k(\varepsilon)}],\ y\in[\theta(t),\omega(t)]\},\\ B_{2k}=[a_{2k},a_{2k-1}]\times[\operatorname*{ess\,inf}_{(a_{2k-1},a_{2k-2})}\theta,\ \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})}\omega],\\ B_{2k+1}=[a_{2k+1},a_{2k}]\times[\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})}\theta,\ \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})}\omega].\end{gathered}$$

Let y be a (θ,ω,a_k)-rapidly oscillating function on $[a,b]$ such that $y\in C((a,b])$. By Definition 2.1, it implies the existence of a sequence $\sigma_k\in(a_k,a_{k-1})$ such that

$$y(\sigma_{2k})\ge\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})}\omega\quad\text{and}\quad y(\sigma_{2k+1})\le\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})}\theta,\quad k\ge 1. \tag{2.7}$$

Let k be a fixed natural number such that $k\ge k(\varepsilon)+1$. Let (t_0,y_0) be an arbitrarily given point of B_k. Let us remark that from (2.7) we get:

$$\begin{gathered}\text{if } (t_0,y_0)\in B_{2j} \text{ then } y(\sigma_{2j-1})\le y_0\le y(\sigma_{2j}),\\ \text{if } (t_0,y_0)\in B_{2j+1} \text{ then } y(\sigma_{2j+1})\le y_0\le y(\sigma_{2j}),\ j\ge 1.\end{gathered}$$

In particular, it implies that

$$\big\{(t,y_0): t\in(\sigma_k,\sigma_{k-1})\big\}\cap G(y|_{[\sigma_k,\sigma_{k-1}]})\ne\emptyset,$$

where $G(y|_{[\sigma_k,\sigma_{k-1}]})$ denotes the graph of the function $y|_{[\sigma_k,\sigma_{k-1}]}$ (here $y|_I$ denotes the function-restriction of y on interval I). Hence, there is a point $s\in(\sigma_k,\sigma_{k-1})$

such that $(s,y_0)\in G(y|_{[\sigma_k,\sigma_{k-1}]})$. Now, by the help of (2.1) we get

$$\begin{aligned} d((t_0,y_0),G(y|_{[\sigma_k,\sigma_{k-1}]})) &\le d((t_0,y_0),(s,y_0)) \le d((a_k,y_0),(\sigma_{k-1},y_0))\\ &= \sigma_{k-1}-a_k \le a_{k-2}-a_k\\ &= a_{k-1}-a_k+a_{k-2}-a_{k-1} \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Thus, we have proved that

$$B_k \subseteq G_\varepsilon(y|_{[\sigma_k,\sigma_{k-1}]}) \quad \text{for any } k\ge k(\varepsilon)+1 \quad\text{and}\quad \varepsilon\in(0,\varepsilon_0), \tag{2.8}$$

where $G_\varepsilon(y|_{[\sigma_k,\sigma_{k-1}]})$ denotes the ε-neighbourhood of the graph $G(y|_{[\sigma_k,\sigma_{k-1}]})$. Let us remark that from (2.5) we may easy conclude that

$$A(\varepsilon,\theta,\omega)\subseteq \cup_{k=k(\varepsilon)+1}^{\infty} B_k, \tag{2.9}$$

where $\varepsilon\in(0,\varepsilon_0)$. According to (2.8) and (2.9) we obtain:

$$A(\varepsilon,\theta,\omega)\subseteq \cup_{k=k(\varepsilon)+1}^{\infty} G_\varepsilon(y|_{[\sigma_k,\sigma_{k-1}]}) \subseteq G_\varepsilon\big(\cup_{k=k(\varepsilon)+1}^{\infty} y|_{[\sigma_k,\sigma_{k-1}]}\big)\subseteq G_\varepsilon(y),$$

where $\varepsilon\in(0,\varepsilon_0)$. Taking the Lebesgue measure on the both sides in the previous statement, it proves the inequality (2.6). □

How to calculate the right hand side in (2.6) it is shown in the following example.

Example 2.4. Let y be a real continuous function on $[a,b]$ and let y be (θ,ω,a_k)-rapidly oscillating, where θ, ω and a_k are given by:

$$\begin{gathered} a_k = a+\frac{b-a}{2}\big(\frac{1}{k}\big)^{1/\beta}, \quad k\ge 1,\\ \theta(t)=-(t-a)^\alpha \quad\text{and}\quad \omega(t)=(t-a)^\alpha, \quad t\in[a,b],\\ \text{where } \alpha \text{ and } \beta \text{ satisfy } 0<\alpha<\beta<\infty. \end{gathered} \tag{2.10}$$

We can take for $k(\varepsilon)$ to be any number that satisfies

$$\begin{gathered} c_0\big(\frac{1}{\varepsilon}\big)^{\frac{\beta}{\beta+1}} \le k(\varepsilon) \le 2c_0\big(\frac{1}{\varepsilon}\big)^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon\in(0,\varepsilon_0),\\ \text{where } c_0 = 2\big(\frac{b-a}{\beta}\big)^{\frac{\beta}{\beta+1}} \text{ and } \varepsilon_0=\frac{b-a}{\beta}. \end{gathered} \tag{2.11}$$

It is clear that (2.11) implies (2.4) and so the main conclusion of Example 2.2 is still valid. That is, the sequence a_k given in (2.10) satisfies the condition (2.1), where $k(\varepsilon)$ and ε_0 are taken to be as in (2.11).

In contrast to (2.4), where ε_0 is an arbitrary positive number, we need in (2.11) an $\varepsilon_0=(b-a)/\beta$. This condition on ε_0 is essentialy to ensure that $k(\varepsilon)\in\mathbb{N}$ for all $\varepsilon\in(0,\varepsilon_0)$.

Next, since $\theta(t)=-(t-a)^\alpha$ and $\omega(t)=(t-a)^\alpha$ are decreasing and increasing on $[a,b]$, respectively, it is clear that θ and ω satisfy the condition (2.5).

Thus, the (θ,ω,a_k) defined by (2.10) satisfies the hypotheses of Lemma 2.3. Hence, the inequality (2.6) can be applied to our situation here. It gives us

$$\begin{aligned} |G_\varepsilon(y)| &\ge \int_a^{a_{k(\varepsilon)}}(\omega(t)-\theta(t))dt = 2\int_a^{a_{k(\varepsilon)}}(t-a)^\alpha dt\\ &= \frac{2}{\alpha+1}(a_{k(\varepsilon)}-a)^{\alpha+1} = \frac{2^{-\alpha}(b-a)^{\alpha+1}}{\alpha+1}\big(\frac{1}{k(\varepsilon)}\big)^{\frac{\alpha+1}{\beta}} \quad \text{for each } \varepsilon\in(0,\varepsilon_0). \end{aligned}$$

From the right inequality in (2.11) we have in particular

$$\frac{1}{k(\varepsilon)} \geq \frac{1}{4}(\frac{\beta}{b-a})^{\frac{\beta}{\beta+1}}\varepsilon^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}).$$

These inequalities give us

$$|G_\varepsilon(y)| \geq c\varepsilon^{\frac{\alpha+1}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_0),$$

where

$$\varepsilon_0 = \frac{b-a}{\beta} \quad \text{and} \quad c = \frac{2^{-\alpha}}{\alpha+1}(\frac{1}{4})^{\frac{\alpha+1}{\beta}}\beta^{\frac{\alpha+1}{\beta+1}}(b-a)^{\frac{(\alpha+1)\beta}{\beta+1}}.$$

To derive a lower bound for $M^s(G(y))$ where y is any solution of (1.1) we need to impose on the function $f(t,\eta,\xi)$ some sufficient conditions such that each solution of (1.1) is (θ,ω,a_k)-rapidly oscillating. It is the subject of the following lemma which is a modification of [9, Lemma 4.1, p. 280] and [9, Lemma 4.2, p. 281].

Lemma 2.5. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying (2.1). Let $\tilde{\theta}_0$ and $\tilde{\omega}_0$ be two arbitrarily given real numbers, and let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t) \leq \omega(t)$, $t \in [a,b]$, which satisfy (2.5) and*

$$\tilde{\theta}_0 \leq \operatorname*{ess\,inf}_{(a,b)} \theta < \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta\,, \quad \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega < \operatorname*{ess\,sup}_{(a,b)} \omega \leq \tilde{\omega}_0, \quad k \geq 1. \tag{2.12}$$

Let the sets J_k be defined by

$$J_{2k} = (\tilde{\theta}_0, \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega) \quad \text{and} \quad J_{2k+1} = (\operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta, \tilde{\omega}_0), \quad k \geq 1.$$

Next, let for each $k \geq 1$ the Caratheodory function $f(t,\eta,\xi)$ satisfy

$$f(t,\eta,\xi) \geq 0, \quad t \in (a_{2k},a_{2k-1}),\ \eta \in J_{2k},\ \xi \in \mathbb{R}, \tag{2.13}$$

$$\int_{A_{2k}} \operatorname*{ess\,inf}_{(\eta,\xi)\in J_{2k}\times R} f(t,\eta,\xi)dt > \frac{c(p)}{(a_{2k-1}-a_{2k})^{p-1}} \frac{(\tilde{\omega}_0-\tilde{\theta}_0)^p}{(\tilde{\omega}_0 - \operatorname{ess\,sup}_{(a_{2k},a_{2k-1})}\omega)}, \tag{2.14}$$

$$f(t,\eta,\xi) \leq 0, \quad t \in (a_{2k+1},a_{2k}),\ \eta \in J_{2k+1},\ \xi \in \mathbb{R}, \tag{2.15}$$

$$\int_{A_{2k+1}} \operatorname*{ess\,sup}_{(\eta,\xi)\in J_{2k+1}\times R} f(t,\eta,\xi)dt < -\frac{c(p)}{(a_{2k}-a_{2k+1})^{p-1}} \frac{(\tilde{\omega}_0-\tilde{\theta}_0)^p}{(\operatorname{ess\,inf}_{(a_{2k+1},a_{2k})}\theta - \tilde{\theta}_0)}, \tag{2.16}$$

where $c(p) = 2[4(p-1)]^{p-1}$ and A_k is a family of sets

$$A_k = [a_k + \frac{1}{4}(a_{k-1}-a_k), a_{k-1} - \frac{1}{4}(a_{k-1}-a_k)], \quad k \geq 1.$$

Then each solution y of the equation (1.1) such that

$$\tilde{\theta}_0 \leq y(t) \leq \tilde{\omega}_0, \quad t \in (a,b) \tag{2.17}$$

is (θ,ω,a_k)-rapidly oscillating on $[a,b]$.

The proof of this lemma will be sketched in Appendix of this paper. It is worth to mention that the condition (2.17) will be easy achieved in Theorem 2.7 below. An example for such a class of Caratheodory functions $f(t,\eta,\xi)$ which satisfies the hypotheses (2.13)–(2.16) is given as follows.

Example 2.6. To simplify the notation, let $\tilde{\theta}_0$, $\tilde{\omega}_0$, θ_{2k+1}, and ω_{2k} be defined as follows:

$$\tilde{\theta}_0 = \operatorname*{ess\,inf}_{(a,b)} \theta \quad \text{and} \quad \tilde{\omega}_0 = \operatorname*{ess\,sup}_{(a,b)} \omega,$$
$$\theta_{2k+1} = \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \quad \text{and} \quad \omega_{2k} = \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega.$$

Let $g = g(t,\eta,\xi)$ be a Caratheodory function defined by

$$g = \frac{\pi c(p)(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{\sin\frac{\pi}{4}} \sum_{k=1}^{\infty} \Big[\frac{(\eta - \tilde{\omega}_0)^-}{(\tilde{\omega}_0 - \omega_{2k})^2} \frac{\sin(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k}))}{(a_{2k-1}-a_{2k})^p} K_{[a_{2k},a_{2k-1}]}(t)$$
$$- \frac{(\eta - \tilde{\theta}_0)^+}{(\theta_{2k+1} - \tilde{\theta}_0)^2} \frac{\sin(\frac{\pi}{a_{2k}-a_{2k+1}}(t-a_{2k+1}))}{(a_{2k}-a_{2k+1})^p} K_{[a_{2k+1},a_{2k}]}(t) \Big],$$

where $c(p)$ is appearing in (2.14) and (2.16), and where $K_A(t)$ denotes, as usual, the characteristic function of a set A. Also, $\eta^- = \max\{0,-\eta\}$ and $\eta^+ = \max\{0,\eta\}$.

Even $K_A(t)$ is not continuous in t, it is not dificult to check that $g(t,\eta,\xi)$ is continuous in all its variables. Next, for any fixed $k \in \mathbf{N}$, it is clear that

$$(\eta - \tilde{\omega}_0)^- \sin(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k})) \geq 0 \quad \text{for any } t \in [a_{2k},a_{2k-1}],\ \eta \in J_{2k},$$
$$\frac{(\eta - \tilde{\omega}_0)^-}{\tilde{\omega}_0 - \omega_{2k}} \geq 1 \quad \text{for any } \eta \in J_{2k},\ \text{where } J_{2k} = (\tilde{\theta}_0, \omega_{2k}).$$

Hence, for any $k \in \mathbb{N}$, $t \in [a_{2k},a_{2k-1}]$, $\eta \in J_{2k}$, and $\xi \in \mathbb{R}$ we have

$$g(t,\eta,\xi) = \frac{\pi c(p)(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{\sin\frac{\pi}{4}} \frac{(\eta - \tilde{\omega}_0)^-}{(\tilde{\omega}_0 - \omega_{2k})^2} \frac{\sin(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k}))}{(a_{2k-1}-a_{2k})^p} \geq 0,$$
$$\operatorname*{ess\,inf}_{(\eta,\xi)\in J_{2k}\times R} g(t,\eta,\xi) \geq \frac{\pi c(p)}{(\sin\frac{\pi}{4})} \frac{(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{(\tilde{\omega}_0 - \omega_{2k})} \frac{\sin(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k}))}{(a_{2k-1}-a_{2k})^p}.$$

From the first equality, it follows that $g(t,\eta,\xi)$ satisfies (2.13). Also, integrating the second inequality over the set

$$[a_{2k} + \frac{1}{4}(a_{2k-1}-a_{2k}), a_{2k-1} - \frac{1}{4}(a_{2k-1}-a_{2k})],$$

it immediately shows that $g(t,\eta,\xi)$ satisfies (2.14) too. On the same way the dual hypotheses (2.15) and (2.16) can be verified. Thus, the function $f(t,\eta,\xi) = g(t,\eta,\xi)$ satisfies the assumptions of Lemma 2.5.

Now, we are able to formulate the first main result of the paper.

Theorem 2.7. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1). *Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t) \leq \omega(t)$, $t \in [a,b]$, which satisfy* (2.5), (2.12) *and*

$$\operatorname*{ess\,inf}_{(a,b)} \theta < 0 < \operatorname*{ess\,sup}_{(a,b)} \omega. \tag{2.18}$$

Next, let for each $k \geq 1$ the Caratheodory function $f(t,\eta,\xi)$ satisfy (2.13)–(2.16), *and*

$$\begin{aligned} f(t,\eta,\xi) < 0, &\quad t \in (a,b),\ \eta > \tilde{\omega}_0 \text{ and } \xi \in \mathbb{R}, \\ f(t,\eta,\xi) > 0, &\quad t \in (a,b),\ \eta < \tilde{\theta}_0 \text{ and } \xi \in \mathbb{R}, \end{aligned} \tag{2.19}$$

where $\tilde\theta_0$ and $\tilde\omega_0$ are two arbitrarily given real numbers satisfying (2.12)*. Then for any $s \in (1,2)$ and for each solution y of* (1.1) *there holds true*

$$
\begin{gathered}
|G_\varepsilon(y)| \geq \int_a^{a_{k(\varepsilon)}} (\omega(t) - \theta(t))dt \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \\
M^s(G(y)) \geq 2^{s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \int_a^{a_{k(\varepsilon)}} (\omega(t) - \theta(t))dt,
\end{gathered}
\tag{2.20}
$$

where ε_0 is appearing in (2.1)*.*

Proof. Let y be a solution of the equation (1.1). Because of [9, Lemma 3.2, p. 279], from (2.12), (2.18) and (2.19) we get $\tilde\theta_0 \leq y(t) \leq \tilde\omega_0$ for each $t \in [a,b]$. Therefore the assumptions of Lemma 2.5 are satisfied and so y is (θ, ω, a_k)-rapidly oscillating function on $[a,b]$. Now, by means of Lemma 2.3 and the definition of $M^s(G(y))$, the desired statement (2.20) easy follows. □

The following result is the main consequence of (2.20). We can derive also lower bounds appearing in (1.6) and (1.7).

Corollary 2.8. *For arbitrarily given real number $s \in (1,2)$, let a_k, θ and ω be given by*

$$
\begin{gathered}
a_k = a + \frac{b-a}{2} (\frac{1}{k})^{1/\beta}, \quad k \geq 1, \\
\theta(t) = -(t-a) \quad \text{and} \quad \omega(t) = t - a, \quad t \in (a,b), \\
\text{where } \beta \text{ satisfies } 1 < \beta < \infty \text{ and } \tfrac{2\beta}{\beta+1} = s.
\end{gathered}
\tag{2.21}
$$

Let the Caratheodory function $f(t, \eta, \xi)$ satisfy (2.13)–(2.16)*, and* (2.19) *in respect to such given (θ, ω, a_k), where $\tilde\theta_0$ and $\tilde\omega_0$ are two arbitrarily given real numbers satisfying* (2.12)*. Then each solution y of the equation* (1.1) *satisfies*

$$
\begin{gathered}
|G_\varepsilon(y)| \geq \frac{1}{2^6}(b-a)^s \varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}), \\
M^s(G(y)) \geq \frac{1}{2^7}(b-a)^s.
\end{gathered}
\tag{2.22}
$$

Proof. First, we know, by Example 2.4, that the sequence a_k defined in (2.21) satisfies the condition (2.1) in respect to $k(\varepsilon)$ determined by (2.11). Next, since $\theta(t) = -(t-a)$ is decreasing and $\omega(t) = t - a$ is increasing, it is clear that the functions θ and ω defined in (2.21) satisfy the conditions (2.5), (2.12) and (2.18).

Thus, the hypotheses of Theorem 2.7 are satisfied and so the statement (2.20) may be used here. Putting given data from (2.21) into (2.20) we can calculate

$$
|G_\varepsilon(y)| \geq \int_a^{a_{k(\varepsilon)}} 2(t-a)dt = (\frac{b-a}{2})^2 (\frac{1}{k(\varepsilon)})^{2/\beta}. \tag{2.23}
$$

Since $\beta > 1$ and $2\beta/(\beta+1) = s$, by the help of the right inequality in (2.11) we get

$$
(\frac{b-a}{2})^2 (\frac{1}{k(\varepsilon)})^{2/\beta} \geq \frac{(b-a)^2}{4} (\frac{1}{4})^{2/\beta} \frac{\beta^{\frac{2}{\beta+1}}}{(b-a)^{\frac{2}{\beta+1}}} \varepsilon^{\frac{2}{\beta+1}} \geq \frac{1}{2^6}(b-a)^s \varepsilon^{2-s}.
$$

Putting this inequality into (2.23) we have proved (2.22). □

At the end of this section we give an example for such a class of Caratheodory functions $f(t, \eta, \xi)$ that satisfies the assumptions of Theorem 2.7.

Example 2.9. Let $h_1(t)$ and $h_2(t)$ be two measurable functions on (a,b) such that $h_1(t) > 0$ and $h_2(t) > 0$. Let $f(t,\eta,\xi)$ be a function defined by

$$f(t,\eta,\xi) = -h_1(t)(\eta - \operatorname*{ess\,sup}_{(a,b)} \omega)^+ + h_2(t)(\eta - \operatorname*{ess\,inf}_{(a,b)} \theta)^- + g(t,\eta,\xi),$$

where the function $g(t,\eta,\xi)$ is constructed in Example 2.6 above. Since $g(t,\eta,\xi)$ satisfies the assumptions of Lemma 2.5, it is clear that such given $f(t,\eta,\xi)$ satisfies the assumptions of Theorem 2.7.

3. Lower bounds for $|G_\varepsilon(y')|$ and $M^s(G(y'))$

We proceed with some observations from the previous section but they are pointed to the derivative y' of any smooth enough real function y. It is started with a relation between the $(\theta,\omega,a_k)-$rapid oscillations of a function y and the asymptotic behaviour of $|G_\varepsilon(y')|$ as $\varepsilon \approx 0$.

Lemma 3.1. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1)*. Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t) \le \omega(t)$, $t \in [a,b]$, which satisfy* (2.5)*. Next, let y be (θ,ω,a_k)-rapidly oscillating on $[a,b]$ and let $y \in C((a,b]) \cap C^1(a,b)$. Then we have*

$$|G_\varepsilon(y')| \ge \sum_{k=k(\varepsilon/2)}^{\infty} \Big(\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \Big) \quad \text{for each } \varepsilon \in (0,\varepsilon_0), \tag{3.1}$$

where $k(\varepsilon)$ and ε_0 are defined in (2.1)*.*

Proof. Let y be a (θ,ω,a_k)-rapidly oscillating function on $[a,b]$. By means of Definition 2.1, there is a sequence $\sigma_k \in (a_k,a_{k-1})$, $k > 1$ such that

$$y(\sigma_{2k}) \ge \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega \quad \text{and} \quad y(\sigma_{2k+1}) \le \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta, \quad k \ge 1. \tag{3.2}$$

Let $k(\varepsilon)$ and ε_0 be taken from (2.1). So, we have that $a_{j-1} - a_j \le \varepsilon/4$ for each $j \ge k(\frac{\varepsilon}{2})$. Hence, the sequence σ_k also satisfies

$$\sigma_k \searrow a \text{ and } \sigma_{j-1} - \sigma_j \le \varepsilon/2 \text{ for each } j \ge k(\tfrac{\varepsilon}{2}) \text{ and } \varepsilon \in (0,\varepsilon_0). \tag{3.3}$$

Applying Lagrange's mean value theorem on (σ_k,σ_{k-1}) we get a sequence $s_k \in (\sigma_k,\sigma_{k-1})$, $k > 1$ such that

$$y'(s_{2k+1}) = \frac{y(\sigma_{2k}) - y(\sigma_{2k+1})}{\sigma_{2k} - \sigma_{2k+1}} \quad \text{and} \quad y'(s_{2k}) = \frac{y(\sigma_{2k-1}) - y(\sigma_{2k})}{\sigma_{2k-1} - \sigma_{2k}}. \tag{3.4}$$

Let us use the following notation

$$\begin{gathered} z(t) = y'(t),\ t \in (a,b), \\ \delta_{2k+1} = \frac{\operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname{ess\,inf}_{(a_{2k+1},a_{2k})} \theta}{\sigma_{2k} - \sigma_{2k+1}}, \\ \delta_{2k} = \frac{\operatorname{ess\,inf}_{(a_{2k-1},a_{2k-2})} \theta - \operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \omega}{\sigma_{2k-1} - \sigma_{2k}}. \end{gathered}$$

Hence, from (2.5), (3.2), and (3.4) follows

$$z(s_{2k+1}) \ge \delta_{2k+1} \ge 0 \quad \text{and} \quad z(s_{2k}) \le \delta_{2k} \le 0, \quad k \ge 1, \tag{3.5}$$

where $s_k \in (\sigma_k,\sigma_{k-1})$. To prove (3.1) we need a modification of Lemma 2.3. □

Lemma 3.2 (A version of Lemma 2.3)**.** *Let σ_k be a decreasing sequence of real numbers from interval (a,b) satisfying (3.3), where $k(\varepsilon)$ and ε_0 are taken from (2.1). Let δ_k be a sequence of real numbers such that $\delta_{2k+1} \ge 0$ and $\delta_{2k} \le 0$, $k \ge 1$. Let $z(t)$ be a continuous function in $(a,b]$ such that there is a sequence $s_k \in (\sigma_k, \sigma_{k-1})$ satisfying (3.5). Then there holds*

$$|G_\varepsilon(z)| \ge \sum_{k=k(\varepsilon/2)}^{\infty} \delta_{2k+1}(\sigma_{2k} - \sigma_{2k+1}) \quad \text{for each } \varepsilon \in (0,\varepsilon_0). \tag{3.6}$$

To prove (3.6), we suggest reader to use the same argumentation from the proof of Lemma 2.3 to prove that $B_{2k+1} \subseteq G_\varepsilon(z|_{[\sigma_{2k+1},\sigma_{2k}]})$ for each $k \ge k(\varepsilon)+1$ and $\varepsilon \in (0,\varepsilon_0)$, where $B_{2k+1} = [\sigma_{2k+1}, \sigma_{2k}] \times [0, \delta_{2k+1}]$.

Using the above notation for δ_k and $z(t)$, from (3.6) follows

$$\begin{aligned} |G_\varepsilon(y')| &\ge \sum_{k=k(\varepsilon/2)}^{\infty} \frac{(\operatorname{ess\,sup}_{(a_{2k},a_{2k-1})}\omega - \operatorname{ess\,inf}_{(a_{2k+1},a_{2k})}\theta)}{\sigma_{2k} - \sigma_{2k+1}} (\sigma_{2k} - \sigma_{2k+1}) \\ &= \sum_{k=k(\varepsilon/2)}^{\infty} (\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta). \end{aligned}$$

Thus, Lemma 3.1 is verified.

Next, we consider an easy example for the calculation of the right hand side in (3.1).

Example 3.3. Let y be a real function smooth enough in (a,b) and let y be (θ, ω, a_k)-rapidly oscillating, where the data θ, ω and a_k are given in (2.10) above. Regarding to Example 2.4, we know such given θ, ω and a_k satisfies the condition (2.1) and (2.5), where $k(\varepsilon)$ and ε_0 are given in (2.11). Hence, we may calculate the right hand side in (3.1) for such function y. First, we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta = \omega(a_{2k-1}) - \theta(a_{2k}) \\ &= (\frac{b-a}{2})^\alpha \big[(\frac{1}{2k-1})^{\alpha/\beta} + (\frac{1}{2k})^{a/\beta} \big] \ge \frac{(b-a)^\alpha}{2^{\alpha-1}} \Big(\frac{1}{2k}\Big)^{\alpha/\beta}. \end{aligned} \tag{3.7}$$

In particular, from (2.11) we have

$$\frac{1}{k(\varepsilon/2)} \ge \frac{1}{4}(\frac{\beta}{2(b-a)})^{\frac{\beta}{\beta+1}} \varepsilon^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}). \tag{3.8}$$

According to (3.7) and (3.8), from (3.1) follows

$$\begin{aligned} |G_\varepsilon(y')| &\ge \sum_{k=k(\varepsilon/2)}^{\infty} \frac{(b-a)^\alpha}{2^{\alpha-1}} \Big(\frac{1}{2k}\Big)^{\alpha/\beta} \ge \frac{(b-a)^\alpha}{2^{\alpha-1}} (\frac{1}{2k(\varepsilon/2)})^{\alpha/\beta} \\ &\ge \frac{(b-a)^\alpha}{2^{\alpha-1}} \frac{1}{8^{\alpha/\beta}} (\frac{\beta}{2(b-a)})^{\frac{\alpha}{\beta+1}} \varepsilon^{\frac{\alpha}{\beta+1}}, \end{aligned}$$

that is $|G_\varepsilon(y')| \ge c\varepsilon^{\alpha/(\beta+1)}$ for each $\varepsilon \in (0,\varepsilon_0)$, where

$$\varepsilon_0 = \frac{b-a}{\beta} \quad \text{and} \quad c = \frac{(b-a)^{\frac{\alpha\beta}{\beta+1}}}{2^{\alpha-1} 8^{\alpha/\beta}} (\frac{\beta}{2})^{\frac{\alpha}{\beta+1}}.$$

The main result of this section is the following.

Theorem 3.4. *Let the hypotheses of Theorem 2.7 be still valid, that is: let the sequence a_k satisfy* (2.1)*, let the functions $\theta(t)$ and $\omega(t)$, $\theta(t) \le \omega(t)$, $t \in [a,b]$, satisfy* (2.5), (2.12) *and* (2.18)*, and let the function $f(t,\eta,\xi)$ satisfy the assumptions* (2.13)*–*(2.16) *and* (2.19)*. Then each solution y of* (1.1) *such that $y \in C^1(a,b)$ satisfies*

$$\begin{gathered} |G_\varepsilon(y')| \ge h(\varepsilon) \quad \text{for each } \varepsilon \in (0,\varepsilon_0), \\ M^s(G(y')) \ge 2^{s-2} \limsup_{\varepsilon\to 0} \varepsilon^{s-2} h(\varepsilon) \quad \text{for any } s \in (1,2), \end{gathered} \tag{3.9}$$

where ε_0 is appearing in (2.1) *and*

$$h(\varepsilon) = \sum_{k=k(\varepsilon/2)}^{\infty} \big(\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \big).$$

Proof. As a consequence of the assumptions (2.13)–(2.16) and (2.18)–(2.19) we may use Lemma 2.5. Hence, each solution y of (1.1) is (θ,ω,a_k)-rapidly oscillating and so we may use Lemma 3.1 too. It gives us the inequality (3.1) from which immediately follows (3.9). □

Now, we are able to prove the inequalities (1.8) and (1.9).

Corollary 3.5. *Let the hypotheses of Corollary 2.8 be still valid, that is: for arbitrarily given real number $s \in (1,2)$, let a_k, θ and ω be given by* (2.21)*, and let the function $f(t,\eta,\xi)$ satisfies* (2.13)*–*(2.16)*, and* (2.19) *in respect to such given (θ,ω,a_k). Then each solution y of* (1.1) *such that $y \in C^1(a,b)$ satisfies*

$$\begin{gathered} |G_\varepsilon(y')| \ge \frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2} \quad \text{for each } \varepsilon \in (0,\varepsilon_0 = \tfrac{b-a}{\beta}), \\ M^{1+s/2}(G(y')) \ge \frac{1}{2^4}(b-a)^{s/2}. \end{gathered} \tag{3.10}$$

Proof. From the proof of Corollary 2.8, we know that the sequence a_k given in (2.21) satisfies the condition (2.1), where $k(\varepsilon)$ and ε_0 are determined in (2.11). Also, the functions θ and ω given in (2.21) satisfy the condition (2.5), (2.12) and (2.18). Therefore, we may apply Theorem 3.4 and to calculate the right hand side in (3.9), where θ, ω and a_k are given in (2.21). In this direction, by the help of (3.7) for $\alpha = 1$ and (3.8) we obtain

$$\begin{aligned} \sum_{k=k(\varepsilon/2)}^{\infty} \big(\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \big) &\ge (b-a) \sum_{k=k(\varepsilon/2)}^{\infty} \big(\frac{1}{2k}\big)^{1/\beta} \\ &\ge (b-a)\big(\frac{1}{2k(\varepsilon/2)}\big)^{1/\beta} \\ &\ge \big(\frac{\beta}{2}\big)^{\frac{1}{\beta+1}} \big(\frac{1}{8}\big)^{\frac{1}{\beta}} (b-a)^{\frac{\beta}{\beta+1}} \varepsilon^{\frac{1}{\beta+1}} \\ &\ge \frac{\sqrt{2}}{2^4}(b-a)^{\frac{s}{2}}\varepsilon^{1-\frac{s}{2}}, \end{aligned}$$

where (2.21) is used, that is, $\beta > 1$ and $2\beta/(\beta+1) = s$. Now from (3.9) immediately follows

$$|G_\varepsilon(y')| \ge \frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2} \quad \text{for each } \varepsilon \in (0,\varepsilon_0 = \frac{b-a}{\beta}).$$

Also, we have

$$\begin{aligned} M^{1+s/2}(G(y')) &\geq 2^{(1+s/2)-2} \limsup_{\varepsilon\to 0} \big[\varepsilon^{(1+s/2)-2}\frac{\sqrt{2}}{2^4}(b-a)^{s/2}\varepsilon^{1-s/2}\big] \\ &= \frac{2^{s/2+1/2}}{2^5}(b-a)^{s/2}\limsup_{\varepsilon\to 0}(\varepsilon^{s/2-1}\varepsilon^{1-s/2}) \geq \frac{1}{2^4}(b-a)^{s/2}. \end{aligned}$$

This proves the desired statement (3.10). □

4. Necessary conditions for rapid oscillations

Let us mention that the inequality (2.6), stated in Lemma 2.3, was useful in the calculation of a lower bound of $|G_\varepsilon(y)|$, where y is a (θ,ω,a_k)-rapidly oscillating function. Next, in Lemma 2.5 some (sufficient) conditions on the nonlinearity $f(t,\eta,\xi)$ were given such that each solution y of (1.1) is (θ,ω,a_k)-rapidly oscillating. In this section, we consider an inverse of Lemma 2.5. Precisely, supposing that there is at least one (θ,ω,a_k)-rapidly oscillating and smooth enough solution of (1.1) it is shown what type of (necessary) conditions on the nonlinearity $f(t,\eta,\xi)$ must be satisfied. Let us remark that in both cases we are working with solutions y of (1.1) which satisfy a basic condition

$$\begin{gathered} \tilde\theta_0 \leq y(t) \leq \tilde\omega_0 \quad \text{for each } t\in[a,b], \quad \text{where} \\ \tilde\theta_0 \leq \operatorname*{ess\,inf}_{(a,b)}\theta \quad\text{and}\quad \tilde\omega_0 \geq \operatorname*{ess\,sup}_{(a,b)}\omega. \end{gathered} \tag{4.1}$$

As we have seen in Theorem 2.7, the condition (4.1) can be easily verified if the assumption (2.19) is imposed on the nonlinear term $f(t,\eta,\xi)$.

Theorem 4.1. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1). *Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t)\leq\omega(t)$, $t\in[a,b]$, which satisfy* (2.5) *and*

$$\operatorname*{ess\,sup}_{(a,b)}\omega > \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})}\theta,\qquad \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})}\omega > \operatorname*{ess\,inf}_{(a,b)}\theta,\quad k\geq 1.$$

If there is at least one solution $y\in C^2(a,b)$ of (1.1) *which is (θ,ω,a_k)-rapidly oscillating and satisfies* (4.1), *then for each $k\geq 1$, the Caratheodory function $f(t,\eta,\xi)$ needs to satisfy the following inequalities:*

$$\begin{aligned} &\int_{a_{2k+1}}^{a_{2k-2}} \operatorname*{ess\,sup}_{(\eta,\xi)\in I\times R} f^+(t,\eta,\xi)dt \\ &\geq \frac{1}{(a_{2k-2}-a_{2k+1})^{p-1}}\frac{(\operatorname{ess\,sup}_{(a_{2k},a_{2k-1})}\omega - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})}\theta)^p}{\operatorname{ess\,sup}_{(a,b)}\omega - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})}\theta}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\int_{a_{2k+2}}^{a_{2k-1}} \operatorname*{ess\,inf}_{(\eta,\xi)\in I\times R} f_-(t,\eta,\xi)dt \\ &\leq -\frac{1}{(a_{2k-1}-a_{2k+2})^{p-1}}\frac{(\operatorname{ess\,sup}_{(a_{2k+1},a_{2k})}\omega - \operatorname{ess\,inf}_{(a_{2k+1},a_{2k})}\theta)^p}{\operatorname{ess\,sup}_{(a_{2k+1},a_{2k})}\omega - \operatorname{ess\,inf}_{(a,b)}\theta}, \end{aligned} \tag{4.3}$$

where $I = (\operatorname{ess\,inf}_{(a,b)}\theta,\ \operatorname{ess\,sup}_{(a,b)}\omega)$, and $f_- = \min\{f,0\}\leq 0$, and $f^+ = \max\{f,0\}\geq 0$.

Taking in (2.12) that $\tilde\theta_0 = \operatorname{ess\,inf}_{(a,b)} \theta$ and $\operatorname{ess\,sup}_{(a,b)} \omega = \tilde\omega_0$ we see that both types of conditions, the sufficient conditions from Lemma 2.5 and the necessary conditions from Theorem 4.1 are similar each other, especially in the variable t.

Proof. Let y be a solution of the equation (1.1) which is (θ, ω, a_k)-rapidly oscillating and satisfies the condition (4.1). According to (2.2) and (2.5) it is easy to check that there are two numbers $s^* \in (a_{2k+1}, a_{2k-1})$ and $t^* \in (a_{2k}, a_{2k-2})$, $s^* < t^*$, such that

$$y'(s^*) \geq 0 \quad \text{and} \quad y(s^*) \geq \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega \geq \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta, \tag{4.4}$$

$$y(t^*) = \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta \quad \text{and} \quad y(t) \geq \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta, \quad t \in (s^*, t^*), \tag{4.5}$$

where k is a fixed natural number. It yields

$$\begin{aligned}
\operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta &\leq y(s^*) - y(t^*) \leq \int_{s^*}^{t^*} |y'| dt \\
&\leq (t^* - s^*)^{1/p'} \Big(\int_{s^*}^{t^*} |y'|^p dt\Big)^{1/p} \\
&\leq (a_{2k-2} - a_{2k+1})^{1/p'} \Big(\int_{s^*}^{t^*} |y'|^p dt\Big)^{1/p},
\end{aligned}$$

where $1/p + 1/p' = 1$. Thus, we have

$$\frac{(\operatorname{ess\,sup}_{(a_{2k},a_{2k-1})} \omega - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \theta)^p}{(a_{2k-2} - a_{2k+1})^{p-1}} \leq \int_{s^*}^{t^*} |y'|^p dt. \tag{4.6}$$

On the other hand, by means of (4.4) and (4.5) and multiplying the equation (1.1) by the test function $\varphi = y - \operatorname{ess\,inf}_{(a_{2k},a_{2k-1})} \theta$ and integrating both sides over $[s^*, t^*]$ we obtain

$$\begin{aligned}
&\int_{s^*}^{t^*} |y'|^p dt \\
&= \big(|y'|^{p-2} y' (y - \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta)\big)\Big|_{s^*}^{t^*} + \int_{s^*}^{t^*} f(t, y, y')(y - \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta) dt \\
&\leq \int_{s^*}^{t^*} f(t, y, y')(y - \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta) dt \\
&\leq (\operatorname*{ess\,sup}_{(a,b)} \omega - \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \theta) \int_{a_{2k+1}}^{a_{2k-2}} \operatorname*{ess\,sup}_{(\eta,\xi)\in I\times R} f^+(t, \eta, \xi) dt,
\end{aligned}$$

where $I = (\operatorname{ess\,inf}_{(a,b)} \theta, \operatorname{ess\,sup}_{(a,b)} \omega)$ and $f^+ = \max\{f, 0\} \geq 0$. Now, combining previous inequality with (4.6) we immediately derive the inequality (4.2).

Similar to (4.4) and (4.5) and by the help of (2.2) and (2.5) we get two numbers $s^* \in (a_{2k+2}, a_{2k})$ and $t^* \in (a_{2k+1}, a_{2k-1})$, $s^* < t^*$, such that

$$y'(s^*) \leq 0 \quad \text{and} \quad y(s^*) \leq \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \leq \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \omega, \tag{4.7}$$

$$y(t^*) = \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \omega \quad \text{and} \quad y(t) \leq \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \omega, \quad t \in (s^*, t^*). \tag{4.8}$$

Using the same observation as in the proof of (4.2), the inequalities (4.7) and (4.8) verify (4.3). □

5. Upper bounds for $|G_\varepsilon(y)|$ and $M^s(G(y))$

In this section, we will derive an upper bound for the behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$, where y is any smooth enough solution of (1.1). In this direction, we introduce the second type of rapid oscillations near the boundary point $t = a$. For the record, the function $y(t) = t^\alpha \cos(1/t^\beta)$,where $0 < \alpha < \beta < \infty$ has such type of rapid oscillations near $t = 0$.

Definition 5.1. Let a_k be a decreasing sequence of real numbers from interval (a, b) satisfying

$$\begin{gathered} a_k \searrow a \text{ and there is an } \varepsilon_1 > 0 \text{ such that for each } \varepsilon \in (0, \varepsilon_1) \\ \text{there is an } m(\varepsilon) \in \mathbb{N} \text{ such that } a_{j-1} - a_j > 4\varepsilon, \text{ for each } j \le m(\varepsilon). \end{gathered} \tag{5.1}$$

Let $\tilde\theta(t)$ and $\tilde\omega(t)$ be two measurable and bounded functions, both defined on $[a, b]$, such that $\tilde\theta(t) \le \tilde\omega(t)$, for each $t \in [a, b]$. We say that a real function y defined on $[a, b]$ has *convex-concave rapid oscillations* in respect to $(\tilde\theta, \tilde\omega, a_k)$ if there hold true:

$$y \text{ is concave in } (a_{2k}, a_{2k-1}) \text{ and } y \text{ is convex in } (a_{2k+1}, a_{2k}), \tag{5.2}$$

for each $k \ge 1$ and

$$\tilde\theta(t) \le y(t) \le \tilde\omega(t) \quad \text{for each } t \in [a, b]. \tag{5.3}$$

In contrast to (2.1) where the number $k(\varepsilon)$ was appearing like an index of ε-density of a_k, here in (5.1) the number $m(\varepsilon)$ could be taken as an index of ε-separation of the most finite numbers of a_k. Let us remark that there is a sequence a_k which satisfies both conditions (2.1) and (5.1). It will be the case of the following example, in which we show a calculation of the number $m(\varepsilon)$.

Example 5.2. Let a_k be a sequence of real numbers defined as in Example 2.2 above, that is

$$a_k = a + \frac{b-a}{2}\big(\frac{1}{k}\big)^{1/\beta}, \quad k \ge 1,\ 0 < \beta < \infty. \tag{5.4}$$

Let us take for $m(\varepsilon)$ any natural number which satisfies

$$m(\varepsilon) \le 2\big(\frac{b-a}{\beta 2^{4+2/\beta}}\big)^{\frac{\beta}{\beta+1}} \varepsilon^{-\frac{\beta}{\beta+1}} \qquad \text{for each } \varepsilon \in (0, \varepsilon_1) \text{ and } \varepsilon_1 > 0. \tag{5.5}$$

Using an elementary inequality

$$\frac{1}{\beta}\big(\frac{1}{j}\big)^{1+1/\beta} \le \big(\frac{1}{j-1}\big)^{1/\beta} - \big(\frac{1}{j}\big)^{1/\beta},$$

where $\beta > 0$ and $j \ge 2$, it is easy to show that the sequence a_k defined in (5.4) satisfies the condition (5.1) in respect to $m(\varepsilon)$ determined in (5.5).

It is clear now that by the help of Example 2.2 and Example 5.2 we have in (5.4) a sequence of real numbers which satisfies both conditions (2.1) and (5.1).

Now, we are interested to relate the convex-concave rapid oscillation of a function y with the asymptotic behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$.

Lemma 5.3. *Let a_k be a decreasing sequence of real numbers from (a, b) satisfying (5.1). Let $\tilde\theta(t)$ and $\tilde\omega(t)$ be two continuous functions on $[a, b]$ satisfying $\tilde\theta(a) = \tilde\omega(a) = 0$ and*

$$\tilde\theta \text{ is decreasing, and } \tilde\omega \text{ is increasing in } [a, b]. \tag{5.6}$$

Let $y \in C^2((a,b]) \cap C([a,b])$ have convex-concave rapid oscillations in respect to $(\tilde{\theta}, \tilde{\omega}, a_k)$. Then for each $\varepsilon \in (0, \varepsilon_1)$ we have

$$\begin{aligned}|G_\varepsilon(y|_{[a,a_1]})| &\le (a_{m(\varepsilon)} - a + 2\varepsilon)\big(\tilde{\omega}(a_{m(\varepsilon)}) - \tilde{\theta}(a_{m(\varepsilon)}) + 2\varepsilon\big) \\ &\quad + \varepsilon \sum_{j=2}^{m(\varepsilon)} \big[(6+\pi)\big(\tilde{\omega}(a_{j-1}) - \tilde{\theta}(a_{j-1})\big) + 2(a_{j-1} - a_j)\big] \\ &\quad + 2(\pi+4)\varepsilon^2 m(\varepsilon),\end{aligned} \tag{5.7}$$

where ε_1 and $m(\varepsilon)$ are defined in the assumption (5.1) *and where a_1 is the first member of a_k.*

The proof of this lemma is omitted because it is very similar to the proof of [9, Lemma 2.2, p. 273-277]

Next, we give an example for the calculation of the right hand side in (5.7).

Example 5.4. Let $y \in C^2((0,1]) \cap C([0,1])$ be a real continuous function on $[0,1]$ and let y have convex-concave rapid oscillations in respect to $(\tilde{\theta}, \tilde{\omega}, a_k)$, where $\tilde{\theta}$, $\tilde{\omega}$ and a_k are given by

$$\begin{gathered} a_k = \frac{1}{2}\Big(\frac{1}{k}\Big)^{1/\beta}, \quad k \ge 1, \\ \tilde{\theta}(t) = -2t^\alpha \quad \text{and} \quad \tilde{\omega}(t) = 2t^\alpha, \quad t \in [0,1], \\ \text{where } \alpha \text{ and } \beta \text{ satisfy } 0 < \alpha < \beta < \infty. \end{gathered} \tag{5.8}$$

We take for $m(\varepsilon)$ any number which satisfies

$$(\beta 2^{4+2/\beta}\varepsilon)^{-\frac{\beta}{\beta+1}} \le m(\varepsilon) \le 2(\beta 2^{4+2/\beta}\varepsilon)^{-\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_1), \tag{5.9}$$

where $\varepsilon_1 = 1/(\beta 2^{4+2/\beta})$. It is easy to check that $m(\varepsilon) \in \mathbb{N}$ for each $\varepsilon \in (0, \varepsilon_1)$. Since from (5.9) follows (5.5) we know that the sequence a_k defined in (5.8) satisfies the condition (5.1), where $m(\varepsilon)$ is determined by (5.9). Also, it is clear that the functions $\tilde{\theta}$ and $\tilde{\omega}$ defined in (5.8) satisfy the condition (5.6). Thus, the data $\tilde{\theta}$, $\tilde{\omega}$ and a_k from (5.8) satisfy the assumptions of Lemma 5.3 and therefore we may calculate (5.7), where $a = 0$.

Let us remark that, in particular, from (5.9) we have

$$\frac{1}{m(\varepsilon)} \le (\beta 2^{4+2/\beta}\varepsilon)^{\frac{\beta}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_1). \tag{5.10}$$

From (5.8), and using the inequality (5.10), for each $\varepsilon \in (0, \varepsilon_1)$ we get

$$(a_{m(\varepsilon)} + 2\varepsilon)\big(\tilde{\omega}(a_{m(\varepsilon)}) - \tilde{\theta}(a_{m(\varepsilon)}) + 2\varepsilon\big) \le c_1 \varepsilon^{\frac{\alpha+1}{\beta+1}} + c_2 \varepsilon^{\frac{\beta+\alpha+1}{\beta+1}} + c_3 \varepsilon^{\frac{\beta+2}{\beta+1}} + 4\varepsilon^2, \tag{5.11}$$

where

$$c_1 = 2^{1-\alpha+(4+\frac{2}{\beta})\frac{\alpha+1}{\beta+1}} \beta^{\frac{\alpha+1}{\beta+1}}, \quad c_2 = 2^{3-\alpha+(4+\frac{2}{\beta})\frac{\alpha}{\beta+1}} \beta^{\frac{\alpha}{\beta+1}}, \quad c_3 = 2^{(4+\frac{2}{\beta})\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}}.$$

From (5.8), (5.9), and using the inequality

$$\sum_{j=1}^{n} \Big(\frac{1}{j}\Big)^H \le \frac{2}{1-H} n^{1-H} \tag{5.12}$$

for each $n \in \mathbb{N}$ and $H \in (0,1)$, we obtain

$$(6+\pi)\varepsilon \sum_{j=2}^{m(\varepsilon)} \big(\tilde{\omega}(a_{j-1}) - \tilde{\theta}(a_{j-1})\big) \le c_4 \varepsilon^{\frac{\alpha+1}{\beta+1}} \tag{5.13}$$

for each $\varepsilon \in (0, \varepsilon_1)$, where

$$c_4 = \frac{6+\pi}{\beta-\alpha} 2^{4-\alpha-\frac{\alpha}{\beta}-(4+\frac{2}{\beta})\frac{\beta-\alpha}{\beta+1}} \beta^{\frac{\alpha+1}{\beta+1}}.$$

From (5.8), and using the inequality

$$\Big(\frac{1}{j-1}\Big)^{1/\beta} - \Big(\frac{1}{j}\Big)^{1/\beta} \le \frac{1}{\beta}\Big(\frac{1}{j-1}\Big)^{1+1/\beta} \tag{5.14}$$

for each $j \in \mathbb{N}$ and $\beta > 0$, we get

$$2\varepsilon \sum_{j=2}^{m(\varepsilon)} (a_{j-1} - a_j) \le c_5 \varepsilon \quad \text{for each } \varepsilon \in (0, \varepsilon_1), \tag{5.15}$$

where

$$c_5 = \frac{1}{\beta} \sum_{j=1}^{\infty} \Big(\frac{1}{j}\Big)^{1+1/\beta}.$$

From (5.8), and using the right inequality from (5.9) we have

$$2(\pi+4)\varepsilon^2 m(\varepsilon) \le c_6 \varepsilon^{\frac{\beta+2}{\beta+1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_1), \tag{5.16}$$

where

$$c_6 = (\pi+4) 2^{2-(4+\frac{2}{\beta})\frac{\beta}{\beta+1}} \beta^{-\frac{\beta}{\beta+1}}.$$

Putting the inequalities (5.11), (5.13), (5.15), and (5.16) into (5.7) we obtain that for each $\varepsilon \in (0, \varepsilon_1)$ there holds

$$|G_\varepsilon(y|_{[0,a_1]})| \le (c_1 + c_4)\varepsilon^{\frac{\alpha+1}{\beta+1}} + c_5\varepsilon + (c_3 + c_6)\varepsilon^{\frac{\beta+2}{\beta+1}} + c_2 \varepsilon^{\frac{\beta+\alpha+1}{\beta+1}} + 4\varepsilon^2,$$

where the constants c_1, c_2, c_3, c_4, c_5, c_6 are determined in the process above.

Next, we give some sufficient conditions on the nonlinearity $f(t, \eta, \xi)$ such that each smooth enough solution can have convex-concave rapid oscillations in the sense of Definition 5.1.

Lemma 5.5. *Let a_k be a decreasing sequence of real numbers from (a, b) which satisfies* (5.1). *Let $\tilde{\theta}(t)$ and $\tilde{\omega}(t)$ be two continuous functions on $[a, b]$ satisfying $\tilde{\theta}(a) = \tilde{\omega}(a) = 0$ and*

$$\tilde{\theta} \text{ is decreasing and convex on } [a,b],\ \tilde{\omega} \text{ is increasing and concave on } [a,b]. \tag{5.17}$$

Let the Caratheodory function $f(t, \eta, \xi)$ satisfy

$$\begin{aligned} &f(t,\eta,\xi) < 0, \quad t \in (a,b),\ \eta > \tilde{\omega}(t),\ \xi \in \mathbb{R}, \\ &f(t,\eta,\xi) > 0, \quad t \in (a,b),\ \eta < \tilde{\theta}(t),\ \xi \in \mathbb{R}, \end{aligned} \tag{5.18}$$

and let for each $k \in \mathbb{N}$,

$$\begin{aligned} &f(t,\eta,\xi) > 0, \quad t \in (a_{2k}, a_{2k-1}),\ \eta \in (\tilde{\theta}_0, \tilde{\omega}(t)),\ \xi \in \mathbb{R}, \\ &f(t,\eta,\xi) < 0, \quad t \in (a_{2k+1}, a_{2k}),\ \eta \in (\tilde{\theta}(t), \tilde{\omega}_0),\ \xi \in \mathbb{R}, \end{aligned} \tag{5.19}$$

where $\tilde{\theta}_0$ and $\tilde{\omega}_0$ be two arbitrary given real numbers such that $\tilde{\theta}_0 \le \tilde{\theta}(b) < 0$ and $0 < \tilde{\omega}(b) \le \tilde{\omega}_0$. Then each solution y of (1.1) *such that $y \in C^2((a,b]) \cap C([a,b])$ has convex-concave rapid oscillations in respect to $(\tilde{\theta}, \tilde{\omega}, a_k)$.*

The assumption (5.17) can be avoided in a particular case as follows.

Lemma 5.6. *Let a_k be a decreasing sequence of real numbers from (a,b) which satisfies (5.1). Let $\tilde\theta(t)$ and $\tilde\omega(t)$ be two real functions on $[a,b]$ defined by*

$$\tilde\theta(t) = -2(t-a) \quad \text{and} \quad \tilde\omega(t) = 2(t-a) \quad \text{for each } t \in [a,b]. \tag{5.20}$$

Let the Caratheodory function $f(t,\eta,\xi)$ satisfy the hypotheses (5.18) and (5.19) in respect to $\tilde\theta$ and $\tilde\omega$ defined in (5.20). Then each solution y of (1.1) such that $y \in C^2((a,b]) \cap C([a,b])$ has convex-concave rapid oscillations in respect to $(\tilde\theta, \tilde\omega, a_k)$.

Sketch of the proofs of Lemmas 5.5 and 5.6. It is an elementary fact that the hypothesis (5.19) implies the condition (5.2), that is, the convex-concave property of smooth solutions of (1.1) is an easy consequence of (5.19) (see a discussion in [9, Appendix,p. 302]). But, it is not so easy to establish how the hypotheses (5.17) and (5.18) (or (5.18) and (5.20)) give us the condition (5.3) for any solution y of (1.1). A way to prove it, we suggest the argumentation presented in the proof of [9, Lemma 3.1, p. 278-279]. See also Lemma 6.4 below.

Thus, the hypotheses (5.17)–(5.20) guarantee us that each solution y of (1.1) such that $y \in C^2((a,b]) \cap C([a,b])$ satisfies the conditions (5.2) and (5.3), that is to say, the solutions y of (1.1) have convex-concave rapid oscillations in respect to $(\tilde\theta, \tilde\omega, a_k)$. □

Now, we can state the main result of this section.

Theorem 5.7. *Let a_k be a decreasing sequence of real numbers from (a,b) which satisfies (5.1). Let $\tilde\theta(t)$ and $\tilde\omega(t)$ be two continuous functions on $[a,b]$, $\tilde\theta(a) = \tilde\omega(a) = 0$ which satisfy (5.6) and either (5.17) or (5.20). Next, let the Caratheodory function $f(t,\eta,\xi)$ satisfy the hypotheses (5.18) and (5.19). Then each solution y of (1.1) such that $y \in C^2((a,b]) \cap C([a,b])$ satisfies*

$$\begin{aligned} M^s(G(y)) \le 2^{s-2} \limsup_{\varepsilon\to 0} \varepsilon^{s-2} \big\{ & (a_{m(\varepsilon)} - a)\big(\tilde\omega(a_{m(\varepsilon)}) - \tilde\theta(a_{m(\varepsilon)})\big) \\ & + \varepsilon \sum_{j=2}^{m(\varepsilon)} \big[(6+\pi)\big(\tilde\omega(a_{j-1}) - \tilde\theta(a_{j-1})\big) + 2(a_{j-1} - a_j)\big] \\ & + 2(\pi+4)\varepsilon^2 m(\varepsilon)\big\}, \end{aligned} \tag{5.21}$$

where $s \in (1,2)$.

Proof. ¿From Lemma 5.5 or Lemma 5.6 we have that each solution y of (1.1) such that $y \in C^2((a,b]) \cap C([a,b])$ has convex-concave rapid oscillations in respect to given $(\tilde\theta, \tilde\omega, a_k)$. Now, from Lemma 5.3 we have that such solutions y of (1.1) satisfy the inequality (5.7). Multiplying both side in (5.7) by $(2\varepsilon)^{s-2}$ and taking $\limsup$ as $\varepsilon \to 0$ we immediately obtain the inequality (5.21). Note that it is not necessary to consider the part $y|_{[a_1,b]}$, because $y \in W^{1,p}(a_1,b)$. □

At the end of this section, we are able to derive the constant m_s appearing in (1.6).

Corollary 5.8. *For arbitrarily given $s \in (1,2)$ let a_k, $\tilde\theta(t)$, and $\tilde\omega(t)$ be defined by*

$$\begin{gathered} a_k = a + \frac{b-a}{2}\Big(\frac{1}{k}\Big)^{1/\beta}, \quad k \ge 1, \\ \tilde\theta(t) = -\tilde\omega(t) \quad \text{and} \quad \tilde\omega(t) = 2(t-a), \quad t \in (a,b), \\ \text{where } 1 < \beta < \infty \text{ and } \tfrac{2\beta}{\beta+1} = s. \end{gathered} \tag{5.22}$$

Let the Caratheodory function $f(t,\eta,\xi)$ satisfy the hypotheses (5.18) *and* (5.19) *in respect to such given $(\tilde\theta,\tilde\omega,a_k)$. Then each solution y of* (1.1) *such that $y\in C^2(a,b)$ satisfies*

$$M^s(G(y))\le m_s(b-a)^s, \tag{5.23}$$

where

$$m_s=2^6+2^2(6+\pi)\frac{2-s}{s-1}. \tag{5.24}$$

Proof. As in Example 5.4, we take for $m(\varepsilon)$ any number that satisfies

$$\big(\frac{b-a}{\beta 2^{4+2/\beta}}\big)^{\frac{\beta}{\beta+1}}\varepsilon^{-\frac{\beta}{\beta+1}}\le m(\varepsilon)\le 2\big(\frac{b-a}{\beta 2^{4+2/\beta}}\big)^{\frac{\beta}{\beta+1}}\varepsilon^{-\frac{\beta}{\beta+1}}, \tag{5.25}$$

where $\varepsilon\in(0,\varepsilon_1=\frac{b-a}{\beta 2^{4+2/\beta}})$. From (5.22) and (5.25) we have

$$\begin{aligned}
&(a_{m(\varepsilon)}-a)\big(\tilde\omega(a_{m(\varepsilon)})-\tilde\theta(a_{m(\varepsilon)})\big)\\
&=4(\frac{b-a}{2})^2(\frac{1}{m(\varepsilon)})^{2/\beta}\\
&\le 2^{(4+\frac{2}{\beta})\frac{2}{\beta+1}}\beta^{\frac{2}{\beta+1}}(b-a)^{\frac{2\beta}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}}\le 2^6(b-a)^s\varepsilon^{2-s}\quad\text{for each }\varepsilon\in(0,\varepsilon_1),
\end{aligned}$$

where $s\in(1,2)$. This implies

$$2^{s-2}\limsup_{\varepsilon\to0}\varepsilon^{s-2}\big[(a_{m(\varepsilon)}-a)\big(\tilde\omega(a_{m(\varepsilon)})-\tilde\theta(a_{m(\varepsilon)})\big)\big]\le 2^6(b-a)^s. \tag{5.26}$$

Using inequality (5.12), from (5.22) and (5.25) we have

$$\begin{aligned}
&\varepsilon\sum_{j=2}^{m(\varepsilon)}(6+\pi)\big(\tilde\omega(a_{j-1})-\tilde\theta(a_{j-1})\big)\\
&=\frac{b-a}{2}(6+\pi)4\varepsilon\sum_{j=2}^{m(\varepsilon)}(\frac{1}{j-1})^{1/\beta}\\
&\le 2(6+\pi)(b-a)\frac{2}{1-1/\beta}\varepsilon(m(\varepsilon))^{1-1/\beta}\\
&\le(6+\pi)2^{3-\frac1\beta-(4+\frac2\beta)\frac{\beta-1}{\beta+1}}\frac{\beta^{\frac{2}{\beta+1}}}{\beta-1}(b-a)^{\frac{2\beta}{\beta+1}}\varepsilon^{\frac{2}{\beta+1}}\\
&\le 2^2(6+\pi)\frac{2-s}{s-1}(b-a)^s\varepsilon^{2-s}\quad\text{for each }\varepsilon\in(0,\varepsilon_1).
\end{aligned}$$

For any $s\in(1,2)$, this implies

$$2^{s-2}\limsup_{\varepsilon\to0}\varepsilon^{s-2}\big[\varepsilon\sum_{j=2}^{m(\varepsilon)}(6+\pi)\big(\tilde\omega(a_{j-1})-\tilde\theta(a_{j-1})\big)\big]\le\frac{2^2(6+\pi)(2-s)}{s-1}(b-a)^s. \tag{5.27}$$

Using inequality (5.14), from (5.22) and (5.25) we have

$$\begin{aligned}
2\varepsilon\sum_{j=2}^{m(\varepsilon)}(a_{j-1}-a_j)&=(b-a)\varepsilon\sum_{j=2}^{m(\varepsilon)}\big((\frac{1}{j-1})^{1/\beta}-(\frac1j)^{1/\beta}\big)\\
&\le\varepsilon\frac{b-a}{\beta}\sum_{j=2}^{\infty}(\frac{1}{j-1})^{1+1/\beta}\quad\text{for each }\varepsilon\in(0,\varepsilon_1).
\end{aligned}$$

For any $s \in (1,2)$, this implies

$$2^{s-2} \limsup_{\varepsilon\to 0} \varepsilon^{s-2}\big[2\varepsilon \sum_{j=2}^{m(\varepsilon)} (a_{j-1}-a_j)\big] = 0. \tag{5.28}$$

Finally, from (5.22) and (5.25), we have

$$2(\pi+4)\varepsilon^2 m(\varepsilon) \le 4(\pi+4)\big(\frac{b-a}{\beta 2^{4+2/\beta}}\big)^{\frac{\beta}{\beta+1}} \varepsilon^{2-\frac{\beta}{\beta+1}}.$$

For any $s \in (1,2)$, this implies

$$2^{s-2} \limsup_{\varepsilon\to 0} \varepsilon^{s-2}\big[2(\pi+4)\varepsilon^2 m(\varepsilon)\big] = 0. \tag{5.29}$$

Putting (5.26), (5.27), (5.28), and (5.29) into (5.21) we obtain (5.23) and (5.24). □

At the end of this section we consider both inequalities in (1.6). In this direction, let us remark that both types of hypotheses on the nonlinear term $f(t,\eta,\xi)$ appearing in Corollary 2.8 and Corollary 5.8 are completely harmonized. It is because the data θ, ω, $\tilde\theta$, and $\tilde\omega$ defined in (2.21) and (5.22) satisfy

$$\begin{gathered} \tilde\theta(t) < \theta(t) < 0 < \omega(t) < \tilde\omega(t), \\ \operatorname*{ess\,inf}_{(a_{2k+1},a_{2k})} \theta \ge \operatorname*{ess\,sup}_{(a_{2k+1},a_{2k})} \tilde\theta, \\ \operatorname*{ess\,sup}_{(a_{2k},a_{2k-1})} \omega \le \operatorname*{ess\,inf}_{(a_{2k},a_{2k-1})} \tilde\omega. \end{gathered} \tag{5.30}$$

Therefore, we may combine Corollary 2.8 and Corollary 5.8 to obtain the following consequence.

Corollary 5.9. *For arbitrarily $s \in (1,2)$ let a_k, θ, ω, $\tilde\theta(t)$, and $\tilde\omega(t)$ be given by* (2.21) *and* (5.22). *Let the Caratheodory function $f(t,\eta,\xi)$ satisfy the hypotheses* (2.13)–(2.16)*, and* (2.19)*, and* (5.18)–(5.19) *in respect to such given $(\theta,\omega,\tilde\theta,\tilde\omega,a_k)$. Then each solution y of* (1.1) *such that $y \in C^2(a,b)$ satisfies*

$$\frac{1}{2^7}(b-a)^s \le M^s(G(y)) \le m_s(b-a)^s, \tag{5.31}$$

where the constant m_s is defined in (5.24).

Thus, because of (5.31), we have proved the desired statement (1.6).

6. Lower bound for the s-dimensional upper density

In this section, we derive the inequalities (1.10) and (1.11). In this direction, we need some preliminaries. The first one is a version of Lemma 2.3 above.

Lemma 6.1. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1). *Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t) \le \omega(t)$, $t \in [a,b]$, which satisfy* (2.5). *Let y be (θ,ω,a_k)-rapidly oscillating function on $[a,b]$ and let $y \in C((a,b])$. Let $k(\varepsilon)$ and ε_0 be from* (2.1). *Then for any $c \in (a,b)$ such that*

$$\text{there exists } \varepsilon_c \in (0,\varepsilon_0) \text{ satisfying } a_{k(\varepsilon)-1} \in (a,c) \text{ for each } \varepsilon \in (0,\varepsilon_c), \tag{6.1}$$

we have

$$|G_\varepsilon(y|_{[a,c]})| \ge \int_a^{a_{k(\varepsilon)}} (\omega(t)-\theta(t))dt \quad \text{for each } \varepsilon \in (0,\varepsilon_c). \tag{6.2}$$

Proof. Regarding to the last line in the proof of Lemma 2.3 above and using the same notations, we have already proved that

$$A(\varepsilon,\theta,\omega) \subseteq G_\varepsilon\big(\cup_{k=k(\varepsilon)+1}^{\infty} y|_{[\sigma_k,\sigma_{k-1}]}\big) \quad \text{for each } \varepsilon \in (0,\varepsilon_0), \tag{6.3}$$

where $k(\varepsilon)$ and ε_0 are appearing in (2.1). Let $c \in (a,b)$ be a real number which satisfies (6.1). It is clear that

$$G(y|_{(a,a_{k(\varepsilon)-1}]}) \subseteq G(y|_{(a,c]}) \quad \text{for each } \varepsilon \in (0,\varepsilon_c). \tag{6.4}$$

Since $\varepsilon_c \in (0,\varepsilon_0)$ and $\sigma_k \in (a_k, a_{k-1})$, combining (6.3) and (6.4) we obtain:

$$\begin{aligned} A(\varepsilon,\theta,\omega) &\subseteq G_\varepsilon\big(\cup_{k=k(\varepsilon)+1}^{\infty} y|_{[\sigma_k,\sigma_{k-1}]}\big) \\ &= G_\varepsilon(y|_{(a,\sigma(k(\varepsilon))]}) \subseteq G_\varepsilon(y|_{(a,a_{k(\varepsilon)-1}]}) \subseteq G_\varepsilon(y|_{(a,c]}); \end{aligned}$$

that is,

$$A(\varepsilon,\theta,\omega) \subseteq G_\varepsilon(y|_{(a,c]}) \quad \text{for each } \varepsilon \in (0,\varepsilon_c). \tag{6.5}$$

Taking the Lebesgue measure of the both sides in (6.5) we get (6.2). □

A choice for the number ε_c appearing in (6.1) and (6.2) will be given in Corollary 6.3 below. Analogously to Theorem 2.7 we can state the following result.

Theorem 6.2. *Let a_k be a decreasing sequence of real numbers from interval (a,b) satisfying* (2.1)*. Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a,b]$, $\theta(t) \le \omega(t)$, $t \in [a,b]$, which satisfy* (2.5)*,* (2.12) *and* (2.18)*. Next, let the Caratheodory function $f(t,\eta,\xi)$ satisfy* (2.13)*–*(2.16)*, and* (2.19)*. Let $c \in (a,b)$ and ε_c be numbers satisfying* (6.1)*. Then each solution y of the equation* (1.1) *satisfies*

$$\begin{gathered} |G_\varepsilon(y|_{[a,c]})| \ge \int_a^{a_{k(\varepsilon)}} (\omega(t)-\theta(t))dt \quad \text{for each } \varepsilon \in (0,\varepsilon_c), \\ M^s(G(y|_{[a,c]})) \ge 2^{s-2} \limsup_{\varepsilon\to 0} \varepsilon^{s-2} \int_a^{a_{k(\varepsilon)}} (\omega(t)-\theta(t))dt, \end{gathered}$$

for any $s \in (1,2)$.

The proof of this theorem is the same as the proof of Theorem 2.7, but using Lemma 6.1 instead of Lemma 2.3. Now, from Theorem 6.2 we obtain the following resutls.

Corollary 6.3. *For an arbitrarily real number $s \in (1,2)$, let a_k, θ and ω be given by* (2.21)*. Let the Caratheodory function $f(t,\eta,\xi)$ satisfy* (2.13)*–*(2.16)*, and* (2.19) *in respect to such given (θ,ω,a_k). Then for each $c \in (a,b)$ and for each solution y of the equation* (1.1) *there holds*

$$\begin{gathered} |G_\varepsilon(y|_{[a,c]})| \ge \frac{1}{2^6}(c-a)^s \varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0,\varepsilon_c), \\ M^s(G(y|_{[a,c]})) \ge \frac{1}{2^7}(c-a)^s, \end{gathered} \tag{6.6}$$

where

$$\varepsilon_c = \min\{\varepsilon_0, \frac{1}{\beta}\frac{(c-a)^{\beta+1}}{(b-a)^\beta}\} \quad \text{and } \varepsilon_0 = \frac{b-a}{\beta}. \tag{6.7}$$

Proof. It is simple to check that every $c \in (a,b)$ satisfies the condition (6.1) in respect to a_k, $k(\varepsilon)$ and ε_c given in (2.21), (2.11) and (6.7), respectively. Next, using the same calculation as in the proof of Corollary 2.8, the statement (6.6) immediately follows from Theorem 6.2. □

To derive the inequalities (1.10) and (1.11) we need a comparison result for solutions of (1.1) which is a modification of [9, Lemma 3.1, p. 278].

Lemma 6.4. *Let $\tilde\theta(t) = -2(t-a)$ and $\tilde\omega(t) = 2(t-a)$. Let the Caratheodory function $f(t,\eta,\xi)$ satisfy*

$$\begin{gathered} f(t,\eta,\xi) < 0, \quad t \in (a,b),\ \eta > \tilde\omega(t),\ \xi \in \mathbb{R}, \\ f(t,\eta,\xi) > 0, \quad t \in (a,b),\ \eta < \tilde\theta(t),\ \xi \in \mathbb{R}. \end{gathered}$$

Then each solution y of (1.1) *such that $y \in C^2(a,b)$ satisfies*

$$\tilde\theta(t) \le y(t) \le \tilde\omega(t) \quad \text{for each } t \in [a,b]. \tag{6.8}$$

The proof of this lemma is omitted because it is the same as the proof of [9, Lemma 3.1, p. 278]. Finally, according to Corollary 6.3 and Lemma 6.4 we are able to state the main result of this section.

Corollary 6.5. *For an arbitrarily $s \in (1,2)$, let the hypotheses of Corollary 6.3 and Lemma 6.4 be still valid. Then each solution y of* (1.1) *such that $y \in C^2(a,b)$ satisfies*

$$\begin{gathered} M^s(G(y) \cap B_r(a,0)) \ge \frac{1}{2^7}\Big(\frac{r}{\sqrt5}\Big)^s \quad \text{for each } r \in (0,b-a), \\ D^s(G(y); t=a) \ge \frac{1}{2^7}\Big(\frac{1}{2\sqrt5}\Big)^s; \end{gathered} \tag{6.9}$$

that is, the constant d_s appearing in (1.10) *and* (1.11) *satisfies*

$$d_s = \frac{1}{2^7}\Big(\frac{1}{2\sqrt5}\Big)^s.$$

Proof. Let us remark that because of (5.30) the assumptions of Corollary 6.3 and Lemma 6.4 are completely harmonized, where a_k, θ and ω be given by (2.21) and where $\tilde\theta(t) = -2(t-a)$ and $\tilde\omega(t) = 2(t-a)$. Therefore, the main conclusions of Corollary 6.3 and Lemma 6.4 may be used together.

By (6.8) and making intersections of $\tilde\theta(t)$ and $\tilde\omega(t)$ with $B_r(a,0)$, it is clear that for any $r \in (0, \sqrt5(b-a))$ we have

$$G(y|_{[a,a+\frac{r}{\sqrt5}]}) \subseteq G(y) \cap B_r(a,0),$$

where y is any smooth enough solution of (1.1). Since M^s is a monotone set function, it yields

$$M^s(G(y|_{[a,a+\frac{r}{\sqrt5}]})) \le M^s(G(y) \cap B_r(a,0)). \tag{6.10}$$

Next, we apply Corollary 6.3 to $y|_{[a,a+\frac{r}{\sqrt5}]}$ and from (6.6) we derive

$$M^s(G(y|_{[a,a+\frac{r}{\sqrt5}]})) \ge \frac{1}{2^7}\Big(\frac{r}{\sqrt5}\Big)^s \quad \text{for any } r \in (0,b-a). \tag{6.11}$$

Combining (6.10) and (6.11) we conclude that

$$M^s(G(y) \cap B_r(a,0)) \ge \frac{1}{2^7}\Big(\frac{r}{\sqrt5}\Big)^s \text{ for any } r \in (0,b-a).$$

Multiplying both inequalities by $1/(2r)^s$ and taking $\limsup$ as $r \to 0$ we immediately derive the desired statement (6.9). □

7. Appendix

In this appendix, we sketch the proof of Lemma 2.5. Under the assumptions of Lemma 2.5, we verify that each solution y of the equation (1.1) which satisfies (2.17) is (θ, ω, a_k)-rapidly oscillating on $[a, b]$. In this proof, a simple method of the localisation of integration in (1.1) is exploited, often used in analysis of local regular properties of solutions of PDE's (see for instance [4, 6, 12, 13]).

Regarding Definition 2.1, it is sufficient to prove that for any fixed $k \in \mathbb{N}$ the hypotheses (2.13), (2.14) and (2.17) verify that

$$\exists \sigma_{2k} \in (a_{2k}, a_{2k-1}) \text{ such that } y(\sigma_{2k}) \geq \operatorname*{ess\,sup}_{(a_{2k}, a_{2k-1})} \omega, \tag{7.1}$$

and on the other hand, that the hypotheses (2.15), (2.16) and (2.17) verify that

$$\exists \sigma_{2k+1} \in (a_{2k+1}, a_{2k}) \text{ such that } y(\sigma_{2k+1}) \leq \operatorname*{ess\,inf}_{(a_{2k+1}, a_{2k})} \theta. \tag{7.2}$$

Since (7.2) is the dual statement of (7.1), in order to simplify the proof, we will prove only the statement (7.1) for a fixed $k \in \mathbf{N}$. In this direction, let σ and r be two real numbers defined by

$$\sigma = \frac{a_{2k} + a_{2k-1}}{2} \quad \text{and} \quad r = \frac{1}{4}(a_{2k-1} - a_{2k}).$$

Let $B_r = B_r(\sigma)$ denote a ball with radius $r > 0$ centered at the point σ. Then we have:

$$B_{2r} = B_{2r}(\sigma) = (a_{2k}, a_{2k-1}),$$
$$B_r = B_r(\sigma) = (a_{2k} + \frac{1}{4}(a_{2k-1} - a_{2k}), a_{2k-1} - \frac{1}{4}(a_{2k-1} - a_{2k})).$$

Also, let $\tilde{\theta}_0 = \operatorname{ess\,inf}_{(a,b)} \theta$, and $\tilde{\omega}_0 = \operatorname{ess\,sup}_{(a,b)} \omega$, and $\omega_{2r} = \operatorname{ess\,sup}_{B_{2r}} \omega$, and $J_{2r} = (\tilde{\theta}_0, \omega_{2r})$. Because of (2.5) and (2.12) we have that $\tilde{\theta}_0 < \omega_{2r} < \tilde{\omega}_0$. Using the preceding notation, we can rewrite the main assumptions (2.13) and (2.14) in the form

$$f(t, \eta, \xi) \geq 0, \quad t \in B_{2r},\ \eta \in J_{2r},\ \xi \in \mathbb{R}, \tag{7.3}$$

$$\begin{aligned} \int_{B_r} \operatorname*{ess\,inf}_{(\eta,\xi) \in J_{2r} \times R} f(t, \eta, \xi) dt &> \frac{c(p)}{4^{p-1}} \frac{1}{r^{p-1}} \frac{(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{\tilde{\omega}_0 - \omega_{2r}} \\ &= (p-1)^{p-1} \frac{(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{\tilde{\omega}_0 - \omega_{2r}} \frac{|B_r|}{r^p}, \end{aligned} \tag{7.4}$$

where we have used $|B_r| = 2r$ and $c(p) = 2[4(p-1)]^{p-1}$. Next, let y be a solution of (1.1) which satisfies (2.17). Let us suppose the contrary statement to (7.1), that is

$$y(t) < \omega_{2r} = \operatorname*{ess\,sup}_{B_{2r}} \omega \quad \text{for each } t \in B_{2r}. \tag{7.5}$$

According to (2.17) and (7.3)–(7.5) we have

$$f(t, y, y') \geq 0 \quad \text{in } B_{2r}, \tag{7.6}$$

$$\int_{B_r} f(t, y, y') dt > (p-1)^{p-1} \frac{(\tilde{\omega}_0 - \tilde{\theta}_0)^p}{\tilde{\omega}_0 - \omega_{2r}} \frac{|B_r|}{r^p}. \tag{7.7}$$

Now we can repeat a similar argument as in [4, Theorem 5, p. 256] or [9, Lemma 4.1, p. 280]. It is known that for any $c_0 > 1$ there exists a function $\Phi \in C_0^\infty(\mathbb{R})$,

$0 \le \Phi \le 1$ in $\mathbb{R}$ such that the following properties are fulfilled, see [4, Lemma 5, pp. 267],

$$\begin{gathered}\Phi(t)=1 \quad \text{for } t\in B_r \quad \text{and} \quad \Phi(t)=0 \quad \text{for } t\in\mathbb{R}\setminus B_{2r},\\ \Phi(t)>0 \quad \text{for } t\in B_{2r} \quad \text{and} \quad |\Phi'(t)|\le \frac{c_0}{r}, \quad t\in\mathbb{R}.\end{gathered} \tag{7.8}$$

For any $c_0>1$, we take a test function φ defined by

$$\varphi(t)=\begin{cases}(y(t)-\tilde\omega_0)\Phi^p(t) & \text{if } t\in B_{2r},\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that $\varphi\in W_0^{1,p}(B_{2r})\cap L^\infty(B_{2r})$. Putting in (1.1) this test function we obtain

$$\begin{aligned}&\int_{B_{2r}}|y'|^p\Phi^p dt\\ &\le p\int_{B_{2r}}|y'|^{p-1}\Phi^{p-1}(\tilde\omega_0-y(t))|\Phi'|dt-\int_{B_{2r}}f(t,y,y')(\tilde\omega_0-y(t))\Phi^p dt.\end{aligned} \tag{7.9}$$

Next, using that $\tilde\omega_0-y(t)\le\tilde\omega_0-\tilde\theta_0$, and $(p-1)p'=p$, and $\delta_1(p\delta_2)\le\delta_1^{p'}+(\frac{p}{p'})^{p-1}\delta_2^p$ in particular for

$$\delta_1=|y'|^{p-1}\Phi^{p-1} \quad \text{and} \quad \delta_2=(\tilde\omega_0-y(t))|\Phi'|,$$

from (7.9) we obtain

$$\begin{aligned}0&=\big[1-\frac{p'}{p'}\big]\int_{B_{2r}}|y'|^p\Phi^p dt\\ &\le\big(\frac{p}{p'}\big)^{p-1}(\tilde\omega_0-\tilde\theta_0)^p\int_{B_{2r}}|\Phi'|^p dt-\int_{B_{2r}}f(t,y,y')(\tilde\omega_0-y(t))\Phi^p dt.\end{aligned}$$

Now, by (7.5), (7.6) and (7.8), we have

$$0\le\big(\frac{p}{p'}\big)^{p-1}(\tilde\omega_0-\tilde\theta_0)^p|B_{2r}\setminus B_r|\big(\frac{c_0}{r}\big)^p-(\tilde\omega_0-\omega_{2r})\int_{B_r}f(t,y,y')dt.$$

Since $|B_{2r}\setminus B_r|=|B_r|$ and passing to the limit as $c_0\to1$ we obtain

$$\int_{B_r}f(t,y,y')dt\le(p-1)^{p-1}\frac{|B_r|}{r^p}\frac{(\tilde\omega_0-\tilde\theta_0)^p}{\tilde\omega_0-\omega_{2r}}.$$

This inequality contradicts (7.7) and so the assumption (7.5) is not possible. Thus, the statement (7.1) is proved.

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Mervan Pašić
Department of Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail address: mervan.pasic@fer.hr

Vesna Županović
Department of Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail address: vesna.zupanovic@fer.hr