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A NEW LOOK AT BOUNDARY PERTURBATIONS OF GENERATORS

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Abstract. In this paper we show that Greiner's results on boundary perturbation can be obtained *systematically* and partially generalised by applying additive perturbation theorems to appropriate operator matrices.

1. Introduction

In a fundamental and much quoted paper [8] Günther Greiner developed a perturbation theory for generators of strongly continuous semigroups where the perturbation does *not* change the *mapping* but rather its *domain*. His approach is motivated by a semigroup approach to abstract boundary value problems of the form

$$\begin{aligned} \dot{f}(t) &= Af(t), \quad t \geq 0, \\ Lf(t) &= \Psi f(t), \quad t \geq 0, \\ f(0) &= f_0 \in X\,. \end{aligned} \tag{1.1}$$

for a linear operator $(A, D(A))$ defined on a Banach space X. This "maximal" operator is restricted by a "boundary condition" given by operators $L : D(A) \to \partial X$ and $\Psi : X \to \partial X$, where ∂X is another Banach space called "boundary space". If we assume the problem to be well posed for zero boundary condition, i.e., for $\Psi = 0$, then the following problem arises. *For which perturbation Ψ is the problem (BP) well posed again?*

In [8] it is shown how delay and other functional differential equations (see [17], [10]) and difference equations as well as diffusion equations fit into his abstract framework (see [8] and the references cited therein). In [9] this approach has been applied to semilinear problems and unbounded Ψ supposing additional analyticity conditions. A more recent application to age structured population equations may be found in [14], [16].

This approach is in contrast to the well established *additive* perturbation theory for generators (see [7, Chap. III]). In this paper, however, we show that Greiner's results can be obtained systematically and even generalised by applying additive perturbation theorems to appropriate operator matrices.

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The paper is organised as follows. In Section 2 we define wellposedness for boundary value problems in an abstract setting and characterise it by the generator property of a certain operator. Moreover, we state the general assumptions and a central lemma for the following approach. In Section 3 we use non-densely defined operator matrices to obtain wellposedness under specific conditions on Ψ. In the final Section 4 we show how the boundary value problem can be solved by associating a *dynamical* boundary value problem and then using one-sided coupled operator matrices as introduced by Engel [4], [5], [6].

2. Abstract boundary-value problems

We assume X to be a Banach space, called **(inner) state space**, and ∂X to be another Banach space, called **boundary space**. On X we consider a linear operator

$$A : D(A) \subset X \to X,$$

called the **maximal operator**, describing the (internal) dynamics of the system. The connection between the state space and the boundary space is given by a linear operator

$$L : D(A) \subset X \to \partial X,$$

the **boundary operator**, relating the state $f \in D(A)$ to its boundary value $x := Lf \in \partial X$. It is thus explicitly assumed that all $f \in D(A)$ "have" boundary values in ∂X.

Moreover, we consider a linear **boundary condition operator**

$$\Psi : X \to \partial X,$$

which can be interpreted as a "perturbation of the boundary condition." With the exception of Subsection 3.2, we will assume Ψ to be bounded.

In the abstract perspective we consider the following **boundary-value problem** $(BP)_{f_0}$,

$$\begin{aligned} \dot{f}(t) &= Af(t), \quad t \geq 0, \\ Lf(t) &= \Psi f(t), \quad t \geq 0, \\ f(0) &= f_0 \in X . \end{aligned}$$

Discussing wellposedness we write just (BP) if the initial value is not fixed.

We denote by $(A_0, D(A_0))$ the restriction of $(A, D(A))$ to zero boundary conditions, i.e.,

$$D(A_0) := \{f \in D(A) : Lf = 0\}, \quad A_0 f := Af. \tag{2.1}$$

We now give an example of a difference equation illustrating the abstract concepts.

Example 2.1. Let ∂X be a Banach space and $X_1 := L^1([-1,0], \partial X)$ be the state space. On this space we consider the maximal operator

$$Af := f'$$

defined on the domain

$$D(A) := W^{1,1}([-1,0], \partial X).$$

The boundary operator is given by

$$L : D(A) \to \partial X, \quad Lf := f(0).$$

The operator $(A_0, D(A_0))$ is then the generator of the (nilpotent) left shift semigroup $(T_0(t))_{t\geq 0}$ on X.

Finally, the boundary condition operator is $\Psi \in \mathcal{L}(X, \partial X)$. So, the boundary condition in (BP) means $f(0) = \Psi(f)$ and (BP) may be called "difference equation" (see, e.g., [10, Sect. I.1.]).

In the sequel we define and characterise wellposedness for (BP) by the generator property of an associated operator on X. To that purpose we use the following definition of classical solutions and wellposedness.

Definition 2.2. A function $f : \mathbb{R}_+ \to X$ is called a **classical solution** of (1.1) if

- (i) $f(\cdot) \in C^1(\mathbb{R}_+, X)$,
- (ii) $f(t) \in D(A)$ for every $t \geq 0$, and
- (iii) $f(\cdot)$ satisfies (1.1).

The problem (BP) is called **wellposed** if

- (i) for all $f_0 \in D(A) \cap D(\Psi)$ with $Lf_0 = \Psi f_0$ there exists a unique classical solution $f(\cdot, f_0)$ of (1.1),
- (ii) the set
$$\widetilde{D} := \{f \in D(A) \cap D(\Psi) : Lf = \Psi f\}$$
of initial values admitting classical solutions is dense in X, and
- (iii) the solutions depend continuously on the initial data, i.e., for every sequence of initial data $\widetilde{D} \supset (f_n)_n \to 0$ the corresponding solutions $f(\cdot, f_n)$ fulfill $\lim_{n\to\infty} f(t, f_n) = 0$ uniformly for t in compact subsets of $\mathbb{R}_+$.

Remark 2.3. Supposing (BP) to be well posed we immediately infer that — to avoid a trivial situation — at least one of the operators $(L, D(A))$ or $(\Psi, D(\Psi))$ has to be unbounded. Otherwise, we would obtain the relation $Lf = \Psi f$ on the dense subset $\{f \in D(A) \cap D(\Psi) : Lf = \Psi f\} \subset X$ implying $L = \Psi$ by the boundedness of both operators. This, however, implies that the boundary condition $Lf = \Psi f$ is trivially fulfilled for all $f \in X$, and problem (BP) is equivalent to an abstract Cauchy problem without boundary condition.

We now characterise wellposedness of (BP) by the generator property of a restriction of the maximal operator $(A, D(A))$ to an operator with "perturbed domain".

Definition 2.4. Consider the linear operator $(A_\Psi, D(A_\Psi))$ on X defined by
$$D(A_\Psi) := \{f \in D(A) \cap D(\Psi) : Lf = \Psi f\}, \quad A_\Psi f := Af, \tag{2.2}$$
and the associated abstract Cauchy problem
$$\begin{aligned} \dot{f}(t) &= A_\Psi f(t), \quad t \geq 0, \\ f(0) &= f_0 \in X\,. \end{aligned} \tag{2.3}$$

The following result connects between wellposedness of (BP) and the generator property of A_Ψ. The proof (using [7, Thm. II.6.7]) is straightforward and will be omitted.

Proposition 2.5. *The abstract boundary value problem (BP) is wellposed if and only if $(A_\Psi, D(A_\Psi))$ is the generator of a strongly continuous semigroup $(T_\Psi(t))_{t\geq 0}$ on X. In that case, $t \mapsto T_\Psi(t)f_0$ gives the classical solutions of* (1.1) *for all $f_0 \in D(A_\Psi)$.*

2.1. **Greiner's lemma and the Dirichlet operators.** In the spirit of Greiner's approach we assume the following properties of $(A, D(A))$ and L (see also [1]).

General Assumption 2.6. In the general setting of Section 2 we assume that

$(S1)$ the boundary operator $L : D(A) \subset X \to \partial X$ is surjective and the operator

$$\begin{pmatrix} A \\ L \end{pmatrix} : D(A) \to X \times \partial X, \quad D(A) \ni f \mapsto \begin{pmatrix} Af \\ Lf \end{pmatrix}$$

is closed.

$(S2)$ The operator $A_0 := A_{|\ker L}$ defined as the restriction of A to the kernel of L generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on the state space X.

The above assumptions imply a decomposition of the domain $D(A)$ which is fundamental for the following approach (see also [1, Lem. 2.2]).

Lemma 2.7 (Greiner [8, Lem. 1.2])**.** *Assume $(S1)$ and $(S2)$ of the General Assumptions 2.6 and take $\lambda \in \rho(A_0)$. Then the restriction of L to* $\ker(\lambda - A)$

$$L_\lambda := L|_{\ker(\lambda - A)} : \ker(\lambda - A) \to \partial X$$

is invertible with bounded inverse D_λ. Moreover, for all $\mu, \lambda \in \rho(A_0)$ we have

$$R(\mu, A_0)D_\lambda = R(\lambda, A_0)D_\mu, \tag{2.4}$$

$$D_\lambda = (1 - (\lambda - \mu)R(\lambda, A_0))D_\mu. \tag{2.5}$$

The operators $D_\lambda \in \mathcal{L}(\partial X, X)$ play a key role in our approach and correspond to the Dirichlet map in the case of boundary value problems for partial differential equations (see [1, Section 3]). Therefore we use the following terminology.

Definition 2.8. The operator

$$D_\lambda : \partial X \to \ker(\lambda - A) \subset X$$

is called **Dirichlet operator** corresponding to the boundary operator L, the maximal operator $(A, D(A))$ and the value $\lambda \in \mathbb{C}$.

Example 2.9. We consider the situation of Example 2.1. The Dirichlet operators corresponding to the boundary operator L and the maximal operator $(A, D(A))$ are given by

$$D_\lambda : \partial X \to L^1([-1, 0], \partial X), \quad x \mapsto D_\lambda x := \epsilon_\lambda x,$$

where

$$\epsilon_\lambda x(\tau) := e^{(\lambda\tau)}x, \quad \tau \in [-1, 0],$$

for all $\lambda \in \mathbb{C}$.

The characterisation of wellposedness of (BP) given in Proposition 2.5 does not contain explicit conditions on the operators $(A, D(A))$, $(L, D(A))$, and $(\Psi, D(\Psi))$. So, the following two sections are devoted to find conditions implying the generator property of $(A_\Psi, D(A_\Psi))$.

3. Wellposedness by non-densely defined operator matrices

The first question is whether, assuming the General Assumptions 2.6, (BP) is wellposed for *any* bounded operator $\Psi \in \mathcal{L}(X, \partial X)$. In fact, Greiner gives an example in [8] illustrating that this fails in general. However, assuming more on $(A, D(A))$ and L this holds true. In the following section we show these results (and a slight generalisation) by using operator matrices and additive perturbation.

To do so, we will now enlarge the state space by "adding" the boundary values, i.e., we consider the product space

$$\mathcal{X} := X \times \partial X,$$

and embed X as $\mathcal{X}_0 := X \times \{0\}$. The projections on the two factor spaces are denoted by $\Pi_1 : \mathcal{X} \to X$, $\Pi_1 \begin{pmatrix} f \\ x \end{pmatrix} := f$ and $\Pi_2 : \mathcal{X} \to \partial X$, $\Pi_2 \begin{pmatrix} f \\ x \end{pmatrix} := x$, respectively.

Definition 3.1. Consider on $\mathcal{X}$ the operator matrix $(\mathcal{L}, D(\mathcal{L}))$ defined by

$$\mathcal{L} := \begin{pmatrix} A & 0 \\ -L & 0 \end{pmatrix} \tag{3.1}$$

on the domain $D(\mathcal{L}) := D(A) \times \{0\} \subset \mathcal{X}$.

Remark 3.2. Clearly, $\mathcal{L}$ is not densely defined on the part $(\mathcal{L}_0, D(\mathcal{L}_0))$ of $(\mathcal{L}, D(\mathcal{L}))$ in $\mathcal{X}_0 := X \times \{0\}$, i.e.,

$$D(\mathcal{L}_0) := D(A_0) \times \{0\},$$
$$\mathcal{L}_0 \begin{pmatrix} f \\ x \end{pmatrix} := \begin{pmatrix} A_0 f \\ 0 \end{pmatrix},$$

can be identified with $(A_0, D(A_0))$.

Consider now the perturbed matrix

$$\mathcal{M} := \begin{pmatrix} A & 0 \\ \Psi - L & 0 \end{pmatrix} = \mathcal{L} + \begin{pmatrix} 0 & 0 \\ \Psi & 0 \end{pmatrix} =: \mathcal{L} + \mathcal{P}$$

still defined on $D(\mathcal{L}) \subset \mathcal{X}$. As before, the part of $\mathcal{M}$ in $\mathcal{X}_0$ is (isomorphic to) A_Ψ. Therefore, if we can show that this part in $\mathcal{X}_0$ generates a strongly continuous semigroup, we obtain the semigroup solving (BP). We state this observation explicitly.

Lemma 3.3. *Consider the operator matrix $(\mathcal{M}, D(\mathcal{M}))$ defined by*

$$\mathcal{M} := \begin{pmatrix} A & 0 \\ \Psi - L & 0 \end{pmatrix}$$

on the domain

$$D(\mathcal{M}) := D(A) \times \{0\}$$

and denote its part in $\mathcal{X}_0$ by $\mathcal{M}_0$, i.e.,

$$D(\mathcal{M}_0) := \Big\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A) \times \{0\} : \mathcal{M} \begin{pmatrix} f \\ x \end{pmatrix} \in X \times \{0\} \Big\}.$$

Then we have $D(\mathcal{M}_0) = D(A_\Psi) \times \{0\}$ and

$$\mathcal{M}_0 = \begin{pmatrix} A_\Psi & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{M}_0$ is a generator of a strongly continuous semigroup $(\mathcal{T}_\Psi(t))_{t\geq 0}$ if and only if A_Ψ is and the classical solutions of (1.1) *are obtained as*

$$\mathbb{R}_+ \ni t \mapsto \Pi_1 \Big[\mathcal{T}_\Psi(t) \begin{pmatrix} f_0 \\ 0 \end{pmatrix} \Big]$$

for every $f_0 \in D(A_\Psi)$.

This simple observation allows the use of powerful tools for showing wellposedness of (BP). We only have to show that $\mathcal{M}$ satisfies the Hille-Yosida estimates. Then it follows from [7, Cor. II.3.21] that its part in the closure of its domain is a generator. The following two subsections are devoted to follow this path.

3.1. **Hille-Yosida operator matrices.** In this section we assume $\Psi \in \mathcal{L}(X, \partial X)$ to be bounded, show that $\mathcal{L}$ is a Hille-Yosida operator, and then apply the bounded perturbation theorem for these operators. The idea of this approach is due to [14].

In addition to the General Assumptions 2.6 we now make an additional boundedness assumption on the operators D_λ.

Assumption 3.4.

$(S3)$ Assume that there exists $\omega_3 \in \mathbb{R}$ and $C \geq 0$ such that for all $\lambda > \omega_3$,

$$\|D_\lambda\|_{\mathcal{L}(\partial X, X)} \leq \frac{C}{(\lambda - \omega_3)} . \tag{3.2}$$

Lemma 3.5. *Under the Assumptions $(S1), (S2)$, and $(S3)$ the operator $(\mathcal{L}, D(\mathcal{L}))$ is a Hille-Yosida operator on the space $X \times \partial X$. Its resolvent is given by the operator matrix*

$$\mathcal{R}_\lambda := \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X}) \tag{3.3}$$

for all $\lambda \in \rho(A_0)$.

Proof. To show that $\mathcal{R}_\lambda$ is the resolvent of $\mathcal{L}$ for $\lambda \in \rho(A_0)$ we first remark that $\mathcal{R}_\lambda$ is a bounded operator on $\mathcal{X}$ by Lemma 2.7. Second, for all $\begin{pmatrix} f \\ x \end{pmatrix} \in \mathcal{X}$ we obtain

$$\mathcal{R}_\lambda \begin{pmatrix} f \\ x \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0)f + D_\lambda x \\ 0 \end{pmatrix} \in D(\mathcal{L}_0)$$

and

$$\begin{aligned} (\lambda - \mathcal{L})\mathcal{R}_\lambda \begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} \lambda - A & 0 \\ L & \lambda \end{pmatrix} \begin{pmatrix} D_\lambda x + R(\lambda, A_0)f \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - A)(D_\lambda x + R(\lambda, A_0)f) \\ LD_\lambda x + LR(\lambda, A_0)f \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - A_0)R(\lambda, A_0)f \\ LD_\lambda x \end{pmatrix} = \begin{pmatrix} f \\ x \end{pmatrix} . \end{aligned}$$

Moreover, for $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{L})$, i.e., $x = 0$ and $f \in D(A)$, we obtain

$$\begin{aligned} \mathcal{R}_\lambda(\lambda - \mathcal{L}) \begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\lambda - A)f \\ Lf \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda, A_0)(\lambda - A)f + D_\lambda Lf \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ x \end{pmatrix} \end{aligned}$$

since

$$\begin{aligned} R(\lambda, A_0)(\lambda - A)f + D_\lambda Lf &= (\lambda, A_0)(\lambda - A)[f - D_\lambda Lf] + D_\lambda Lf \\ &= R(\lambda, A_0)(\lambda - A_0)[f - D_\lambda Lf] + D_\lambda Lf \\ &= f - D_\lambda Lf + D_\lambda Lf = f. \end{aligned}$$

Third, the powers of $\mathcal{R}_\lambda$ can be obtained easily as

$$\mathcal{R}_\lambda^{n+1} = \begin{pmatrix} R(\lambda, A_0)^{n+1} & R(\lambda, A_0)^n D_\lambda \\ 0 & 0 \end{pmatrix}$$

for $n \in \mathbb{N}$. Thus for $\omega > \tilde{\omega} := \max\{\omega_0, \omega_3\}$ there exists $M \geq 1$ such that

$$\begin{aligned} \|\mathcal{R}_\lambda^{n+1}\| &\leq \max\{\|R(\lambda, A_0)^{n+1}\|, \|R(\lambda, A_0)^n D_\lambda\|\} \\ &\leq \max\{\frac{M}{(\lambda - \omega_0)^{n+1}}, \frac{CM}{(\lambda - \omega_3)(\lambda - \omega_0)^n}\} \leq \frac{\widetilde{M}}{(\lambda - \tilde{\omega})^{n+1}} \end{aligned}$$

by $(S3)$ for some $\widetilde{M} \geq 1$ and all $\lambda \geq \tilde{\omega}$. □

The bounded perturbation of a Hille-Yosida operator is again a Hille-Yosida operator (see [7, Thm. III.1.3]) and the part of a Hille-Yosida operator is a generator on the closure of its domain (see [7, Cor. II.3.21]). We thus immediately obtain one of Greiner's results.

Theorem 3.6 ([8, Thm. 2.1])**.** *Let the Assumptions $(S1), (S2)$, and $(S3)$ hold and assume $\Psi \in \mathcal{L}(X, \partial X)$. Then the matrix $(\mathcal{M}, D(\mathcal{L}))$ defined by*

$$\mathcal{M} := \begin{pmatrix} A & 0 \\ \Psi - L & 0 \end{pmatrix} =: \mathcal{L} + \mathcal{P}$$

is a Hille-Yosida operator on the space $X \times \partial X$. Thus, its part $(\mathcal{M}_0, D(\mathcal{M}_0))$ in $\mathcal{X}_0 := X \times \{0\}$ is the generator of a strongly continuous semigroup $(\mathcal{T}_\Psi(t))_{t\geq 0}$.

Summing up, we obtain the following wellposedness result for (BP).

Corollary 3.7. *Under the conditions of Theorem 3.6 the boundary value problem (BP) is wellposed.*

Example 3.8. We consider the situation of Example 2.1. An estimate of the norm of the Dirichlet operators yields

$$\|D_\lambda\| \leq \int_{-1}^{0} e^{(\lambda\tau)}\, d\tau = \frac{1}{\lambda}[1 - e^{-\lambda}] \leq \frac{1}{\lambda}$$

for all $\lambda > 0$.

Thus the conditions $(S1), (S2)$, and $(S3)$ are fulfilled and for every bounded operator $\Psi : L^1([-1, 0], \partial X) \to \partial X$ the "difference equation"

$$\begin{aligned} \dot{f}(t) &= f'(t), \quad t \geq 0, \\ f(t)(0) &= \Psi f(t), \quad t \geq 0, \\ f(0) &= f_0 \in L^1([-1, 0], \partial X) \end{aligned} \tag{3.4}$$

is well posed.

3.2. **Unbounded boundary condition Ψ with bounded extension $D_\lambda\Psi$.** In this subsection we do not assume boundedness of $(\lambda-\omega_3)D_\lambda$ as in $(S3)$ and boundedness of the operator Ψ, separately. Instead, we assume the following smallness condition of Ψ with respect to D_λ.

Assumption 3.9.

$(S4)$ Let $\Psi : D(A) \to \partial X$ be a linear operator and assume that each
$$D_\lambda\Psi : D(A) \to D(A) \subset X$$
can be extended continuously to bounded operators $D_\lambda\Psi : X \to X$ such that
$$\|\lambda D_\lambda \Psi\| \le C < \infty \tag{3.5}$$
for some $\omega_4 \in \mathbb{R}$ and all $\lambda > \omega_4$.

Assuming $(S4)$ we obtain the following generation result.

Theorem 3.10. *Let the Assumptions $(S1), (S2)$, and $(S4)$ hold and consider the operator matrix $(\mathcal{M}, D(\mathcal{L}))$ defined by*
$$\mathcal{M} := \begin{pmatrix} A & 0 \\ \Psi - L & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ -L & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \Psi & 0 \end{pmatrix} =: \mathcal{L} + \mathcal{P}.$$
Then there exists $c > 0$ such that for all $\lambda > c$ we have $\lambda \in \rho(\mathcal{M})$ and
$$R(\lambda, \mathcal{M}) = \begin{pmatrix} \sum_{n=0}^{\infty} (D_\lambda\Psi)^n R(\lambda, A_0) & \sum_{n=0}^{\infty} (D_\lambda\Psi)^n D_\lambda \\ 0 & 0 \end{pmatrix}.$$
The part $(\mathcal{M}_0, D(\mathcal{M}_0))$ of $(\mathcal{M}, D(\mathcal{L}))$ in $\mathcal{X}_0 := X \times \{0\}$ is the generator of a strongly continuous semigroup and its resolvent is given by
$$R(\lambda, \mathcal{M}_0) = R(\lambda, \mathcal{M})_{|\mathcal{X}_0} = \begin{pmatrix} \sum_{n=0}^{\infty} (D_\lambda\Psi)^n R(\lambda, A_0) & 0 \\ 0 & 0 \end{pmatrix}$$
for all $\lambda > c$.

Proof. By rescaling (see [7, Ex. II.2.2]) and renorming the space X (see [7, Lem. II.3.10]) we may assume without loss of generality that $(A_0, D(A_0))$ generates a contraction semigroup. Condition (3.5) still holds with another constant $\widetilde{C}$.

For every $\lambda > 0$ we thus obtain $\lambda \in \rho(A_0) \cap \rho(\mathcal{L})$ and
$$R(\lambda, \mathcal{L}) = \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix}$$
by the same argument as in Lemma 3.5. We now write
$$(\lambda - \mathcal{M}) = (\lambda - \mathcal{L} - \mathcal{P}) = (\lambda - \mathcal{L})[\mathrm{Id} - R(\lambda, \mathcal{L})\mathcal{P}] \tag{3.6}$$
which is invertible if and only if $[\mathrm{Id} - R(\lambda, \mathcal{L})\mathcal{P}]$ is. We thus consider
$$R(\lambda, \mathcal{L})\mathcal{P} = \begin{pmatrix} D_\lambda\Psi & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X})$$
and calculate
$$[R(\lambda, \mathcal{L})\mathcal{P}]^n = \begin{pmatrix} (D_\lambda\Psi)^n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } n \ge 1.$$
For all $\lambda > c := \max\{\widetilde{C}, \omega_4\}$ we thus conclude from (3.5) that
$$\|R(\lambda, \mathcal{L})\mathcal{P}\| < \frac{\widetilde{C}}{\lambda} < 1$$

and

$$[\mathrm{Id} - R(\lambda, \mathcal{L})\mathcal{P}]^{-1} = \begin{pmatrix} \sum_{n=0}^{\infty} (D_\lambda \Psi)^n & 0 \\ 0 & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} [\mathrm{Id} - D_\lambda \Psi]^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

with

$$\|[\mathrm{Id} - R(\lambda, \mathcal{L})\mathcal{P}]^{-1}\| \le \frac{1}{1 - \widetilde{C}/\lambda}.$$

We therefore obtain the inverse of $(\lambda - \mathcal{M})$ as

$$R(\lambda, \mathcal{M}) = \begin{pmatrix} \sum_{n=0}^{\infty} (D_\lambda \Psi)^n R(\lambda, A_0) & \sum_{n=0}^{\infty} (D_\lambda \Psi)^n D_\lambda \\ 0 & 0 \end{pmatrix}.$$

Restricting $R(\lambda, \mathcal{M})$ to $\mathcal{X}_0 = X \times \{0\}$ the resolvent becomes

$$R(\lambda, \mathcal{M}_0) = R(\lambda, \mathcal{M})_{|\mathcal{X}_0} = \begin{pmatrix} \sum_{n=0}^{\infty} (D_\lambda \Psi)^n R(\lambda, A_0) & 0 \\ 0 & 0 \end{pmatrix},$$

which is the resolvent of the part $(\mathcal{M}_0, D(\mathcal{M}_0))$ of $(\mathcal{M}, D(\mathcal{L}))$ in $\mathcal{X}_0$. Its norm can now be estimated for all $\lambda > \widetilde{C}$ by

$$\|R(\lambda, \mathcal{M}_0)\| = \Big\| \sum_{n=0}^{\infty} (D_\lambda \Psi)^n R(\lambda, A_0) \Big\| \le \frac{1}{1 - \widetilde{C}/\lambda} \frac{1}{\lambda} = \frac{1}{\lambda - \widetilde{C}}.$$

The operator $(\mathcal{M}_0, D(\mathcal{M}_0))$ is densely defined since $D(A_0)$ is dense in X by assumption and since $\sum_{n=0}^{\infty} (D_\lambda \Psi)^n$ is invertible. We thus obtain $(\mathcal{M}_0, D(\mathcal{M}_0))$ as the generator of a (quasicontractive) strongly continuous semigroup. $\square$

Remark 3.11. It is easy to see that the condition $(S4)$ is weaker than condition $(S3)$. We thus obtained a generalization of Greiner's Theorem 2.1 [8] to unbounded operators $(\Psi, D(A))$.

By the same argument as in Section 3.1 we obtain a wellposedness result for the boundary value problem.

Corollary 3.12. *If the conditions of Theorem 3.10 are fulfilled, then (BP) is well posed.*

4. Wellposedness by reduction to dynamic boundary-value problems

In this section we show how wellposedness can also be obtained by associating to the boundary value problem a *dynamic* boundary value problem and then solve it by operator matrix techniques as developed, e.g., in [6], [1], [11].

If $\Psi : X \to \partial X$ is a bounded operator and $f(\cdot) \in C^1(\mathbb{R}_+, X)$, then the function $\Psi f(\cdot)$ is also differentiable. Therefore, if $f(\cdot)$ solves (1.1), then also $Lf(\cdot) = \Psi f(\cdot)$ is differentiable, and $f(\cdot)$ solves the *dynamic* boundary value problem

$$\begin{aligned} \dot{f}(t) &= Af(t), \quad t \ge 0, \\ x(t) &:= Lf(t), \quad t \ge 0, \\ \dot{x}(t) &= (\Psi A) f(t), \quad t \ge 0, \\ f(0) = f_0 \in X, &\qquad x(0) = \Psi f_0 \in \partial X\,. \end{aligned} \tag{4.1}$$

This observation leads to the following approach using a characterisation for wellposedness of (DBP) by the generator property of an operator matrix with coupled domain.

Definition 4.1. On $\mathcal{X} := X \times \partial X$ we define the operator matrix

$$\mathcal{A}_\Psi := \begin{pmatrix} A & 0 \\ \Psi A & 0 \end{pmatrix} \tag{4.2}$$

with domain

$$D(\mathcal{A}_\Psi) := \Big\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A) \times \partial X : Lf = x \Big\}. \tag{4.3}$$

Moreover, we consider the corresponding abstract Cauchy problem

$$\begin{aligned} \dot{U}(t) &= \mathcal{A}_\Psi U(t), \quad t \geq 0, \\ U(0) &= \begin{pmatrix} f_0 \\ x_0 \end{pmatrix} \in \mathcal{X}\,. \end{aligned} \tag{4.4}$$

with initial values $f_0 \in X$ and $x_0 \in \partial X$.

As suggested by the above observation connecting (BP) and (DBP), the generator property of $\mathcal{A}_\Psi$ implies the generator property of A_Ψ.

Proposition 4.2. *Assume $D(A_\Psi)$ to be dense in X. If the matrix $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ generates a strongly continuous semigroup on $\mathcal{X}$ then so does $(A_\Psi, D(A_\Psi))$ on X.*

Proof. Let $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ be a generator and consider $f_0 \in D(A_\Psi)$. Then

$$\begin{pmatrix} f_0 \\ \Psi f_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ L f_0 \end{pmatrix} \in D(\mathcal{A}_\Psi),$$

and we consequently obtain a classical solution for (4.1). Denote its first component by $f(\cdot)$. Then $f(0) = f_0$, $Lf(0) = \Psi f(0)$ and the equations

$$\dot{f}(t) = Af(t), \quad t \geq 0,$$

and

$$\frac{d}{dt} Lf(t) = \Psi Af(t) = \Psi \frac{d}{dt} f(t) = \frac{d}{dt} \Psi f(t), \quad t \geq 0$$

hold. This in turn implies $Lf(t) = \Psi f(t)$ for all $t \geq 0$ and thus $f(\cdot)$ is a classical solution of (1.1). Continuous dependence of the solutions is obtained easily. Moreover, the closedness of $\mathcal{A}_\Psi$ implies the closedness of A_Ψ. To see this, consider $D(A_\Psi) \supset f_n \to f_0 \in X$ and $A_\Psi f_n = A f_n \to g \in X$. Then we infer $\Psi A f_n \to \Psi g$ and $Lf_n = \Psi f_n \to \Psi f_0 \in X$ by the boundedness of Ψ. Since $\mathcal{A}_\Psi$ is closed and $\mathcal{A}_\Psi \begin{pmatrix} f_n \\ Lf_n \end{pmatrix} \to \begin{pmatrix} g \\ \Psi g \end{pmatrix}$, this implies $\begin{pmatrix} f_0 \\ \Psi f_0 \end{pmatrix} \in D(\mathcal{A}_\Psi)$ and $\mathcal{A}_\Psi \begin{pmatrix} f_0 \\ \Psi f_0 \end{pmatrix} = \begin{pmatrix} g \\ \Psi g \end{pmatrix}$. Explicitly this means that $f_0 \in D(A)$, $Af_0 = g$, and $Lf_0 = \Psi f_0$. Thus $f_0 \in D(A_\Psi)$ and $A_\Psi f_0 = g$ which means that A_Ψ is closed. By a well known theorem (see [7, Thm. II.6.7] we infer that $(A_\Psi, D(A_\Psi))$ is a generator. □

With respect to wellposedness the two systems are, however, *not* equivalent. This is due to the fact that there are mild solutions for $(DBP)_{f_0,x_0}$ for *all* $f_0 \in X$ and $x_0 \in \partial X$, while for (1.1) the condition $x_0 = \Psi f_0$ must always hold. So there are more mild solutions for (DBP) and wellposedness of (DBP) implies wellposedness of (BP), but not conversely. Here is an example.

Example 4.3. Let $(A, D(A))$ be a generator on X and $L = \Psi \in \mathcal{L}(X, \partial X)$ be any bounded operator. Then $A_\Psi = A$ and the boundary value problem (BP) is equivalent to the abstract Cauchy problem for the generator A, thus wellposed. However, the matrix

$$\mathcal{A}_\Psi := \begin{pmatrix} A & 0 \\ \Psi A & 0 \end{pmatrix} \tag{4.5}$$

with domain

$$D(\mathcal{A}_\Psi) := \Big\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A) \times \partial X : Lf = x \Big\} \tag{4.6}$$

is not even densely defined, thus not a generator, and the coresponding dynamical boundary value problem (DBP) is not wellposed.

In view of the preceding Proposition 4.2 we have to find conditions implying the matrix $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ to be the generator of a strongly continuous semigroup. This situation has been studied in [11] based on the theory of one-sided coupled operator matrices developed by Engel (see [6]). We will now apply these results, in part [11, Prop. 4.3] to our situation.

We sketch the proof and refer to [6] and [11] for more details. It turns out that the condition for the generator property of the matrix $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ is exactly the condition obtained by applying multiplicative perturbation theory (see Remark 4.9 below).

Theorem 4.4. *Assume the General Assumptions 2.6 and let $\Psi : X \to \partial X$ be a bounded operator. Moreover, assume that ΨA_0 is relatively $(Id - \Psi D_\lambda)A_0$-bounded. (1) Then the matrix $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ of Definition 4.1 is the generator of a strongly continuous (analytic) semigroup if and only if the operator $(A_0 - D_\lambda \Psi A_0, D(A_0))$ is the generator of a strongly continuous (analytic) semigroup for some $\lambda \in \rho(A_0)$. (2) In that case, the operator $(A_\Psi, D(A_\Psi))$ is the generator of a strongly continuous (analytic) semigroup.*

Proof. (1) For any fixed $\lambda \in \rho(A_0)$ we can factor the matrix $\mathcal{A}_\Psi - \lambda$ as

$$\mathcal{A}_\Psi - \lambda = \begin{pmatrix} A_0 - \lambda & 0 \\ \Psi A_0 & \lambda \Psi D_\lambda - \lambda \end{pmatrix} \begin{pmatrix} \mathrm{Id}_X & -D_\lambda \\ 0 & \mathrm{Id}_{\partial X} \end{pmatrix} =: \mathcal{A}_d \mathcal{D}_\lambda \tag{4.7}$$

with the bounded and invertible operator $\mathcal{D}_\lambda \in \mathcal{L}(\mathcal{X})$ and an operator matrix $(\mathcal{A}_d, D(\mathcal{A}_d))$ with diagonal domain $D(\mathcal{A}_d) := D(A_0) \times \partial X$. To verify this factorisation we first remark that $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A}_\Psi)$ is equivalent to $f \in D(A)$, $x \in \partial X$, and $x = Lf$. This in turn is equivalent to $f \in X$, $x \in \partial X$, and $f - D_\lambda x \in D(A_0)$, i.e., $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A}_d \mathcal{D}_\lambda)$. The equality (4.7) is now obtained by considering $\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A}_\Psi)$ and calculating

$$\begin{aligned}
\mathcal{A}_d \mathcal{D}_\lambda \begin{pmatrix} f \\ x \end{pmatrix} &= \mathcal{A}_d \begin{pmatrix} f - D_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} (A_0 - \lambda)(f - D_\lambda x) \\ \Psi A_0 (f - D_\lambda x) + (\lambda \Psi D_\lambda - \lambda) x \end{pmatrix} \\
&= \begin{pmatrix} (A - \lambda) f \\ \Psi A f - \lambda x \end{pmatrix} \\
&= (\mathcal{A}_\Psi - \lambda) \begin{pmatrix} f \\ x \end{pmatrix}
\end{aligned}$$

since $A_0(f - D_\lambda f) = Af - \lambda f$.

Due to the invertibility of $\mathcal{D}_\lambda$ the matrix $\mathcal{A}_\Psi - \lambda$ is similar to

$$\tilde{\mathcal{A}} := \mathcal{D}_\lambda \mathcal{A}_d = \begin{pmatrix} A_0 - \lambda - D_\lambda \Psi A_0 & \lambda D_\lambda - \lambda D_\lambda \Psi D_\lambda \\ \Psi A_0 & \lambda \Psi D_\lambda - \lambda \end{pmatrix}$$

on the diagonal domain $D(\tilde{\mathcal{A}}) := D(A_0) \times \partial X$. This operator is a bounded perturbation of the operator

$$\mathcal{G} := \begin{pmatrix} A_0 - D_\lambda \Psi A_0 & 0 \\ \Psi A_0 & 0 \end{pmatrix}$$

on the domain $D(\tilde{\mathcal{A}})$. Observe further that the lower left entry is, by assumption, relatively bounded with respect to the upper left entry. Hence, by well-known results on matrices with diagonal domain (see, e.g., [12, Cor. 3.2 and Cor. 3.3]) we finally conclude that $\mathcal{G}$ (thus $\mathcal{A}_\Psi$) generates a strongly continuous (analytic) semigroup on $\mathcal{X}$ if and only if $(A_0 - D_\lambda \Psi A_0, D(A_0))$ does so on X.

(2) By Proposition 4.2 it remains to show that $D(A_\Psi)$ is dense in X. Assume without restriction that $0 \in \rho(A_0)$ and suppose the condition in (1). We first remark that $D(A_\Psi)$ can be written as

$$D(A_\Psi) = \{f \in X : (\mathrm{Id} - D_\lambda \Psi) f \in D(A_0)\}$$

with the bounded operator $P_\lambda := \mathrm{Id} - D_\lambda \Psi \in \mathcal{L}(X)$. Since $(P_\lambda A_0, D(A_0))$ and $(A_0, D(A_0))$ are generators on X, we infer that $D(A_0)$ is a Banach space with respect to the norms $\|\cdot\|_{P_\lambda A_0}$ and $\|\cdot\|_{A_0}$ while $\|\cdot\|_{A_0}$ is finer than $\|\cdot\|_{P_\lambda A_0}$. By the open mapping theorem both norms are equivalent and thus $D((P_\lambda A_0)^2)$ is dense in $(D(A_0), \|\cdot\|_{A_0})$. Take now $f \in X$ and $\epsilon > 0$. Then $A_0^{-1} f \in D(A_0)$ and there exists $g_\epsilon \in D((P_\lambda A_0)^2)$ with $\|A_0^{-1} f - g_\epsilon\|_{A_0} \le \epsilon$. This implies $P_\lambda A_0 g_\epsilon \in D(A_0)$ and thus $f_\epsilon := A_0 g_\epsilon \in D(A_\Psi)$ and, finally,

$$\|f - f_\epsilon\| = \|f - A_0 g_\epsilon\| = \|A_0[A_0^{-1} f g_\epsilon]\| \le \|A_0^{-1} f - g_\epsilon\|_{A_0} \le \epsilon.$$

Thus $D(A_\Psi)$ is dense in X. □

The following results are immediate consequences of this theorem and cover all of Greiner's results not yet contained in the preceding Section 3. Corollary 4.5 follows by the bounded perturbation theorem, Corollary 4.7 by a perturbation result for analytic semigroups, see, e.g. [7, Cor. 2.17 (ii)]. Remark that $\Psi : X \to \partial X$ is automatically compact if the boundary space ∂X is finite dimensional. Corollary 4.6 follows by the perturbation theorem for analytic semigroups, see, e.g., [7, Thm. 2.10]. It is a slight generalization of [8, Thm. 2.1']. Finally, Corollary 4.8 follows by Rellich's perturbation theorem for selfadjoint operators and Stone's theorem.

In all four situations, (BP) is wellposed.

Corollary 4.5 ([8, Thm. 2.3])**.** *In the situation of Theorem 4.4 assume that* $(A_0, D(A_0))$ *is the generator of a strongly continuous semigroup and assume that* $(\Psi A_0, D(A_0))$ *has a bounded extension. Then the operator* $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ *is the generator of a strongly continuous semigroup. If* $(A_0, D(A_0))$ *generates an analytic (compact) semigroup,* $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ *generates an analytic (compact) semigroup.*

Corollary 4.6 ([8, Thm. 2.1'])**.** *In the situation of Theorem 4.4 assume that* $(A_0, D(A_0))$ *is the generator of an analytic semigroup. Moreover, assume that*

$$\inf_{\lambda \in \rho(A_0)} \|D_\lambda \Psi\| = 0.$$

Then also $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ *generates an analytic semigroup.*

Corollary 4.7 ([8, Thm. 2.4]). *In the situation of Theorem 4.4 assume that* $(A_0, D(A_0))$ *is the generator of an analytic semigroup. Moreover, assume that* $\Psi : X \to \partial X$ *is a compact operator. Then also* $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ *generates an analytic semigroup.*

Corollary 4.8. *In the situation of Theorem 4.4 assume that* $(iA_0, D(A_0))$ *is a self-adjoint operator on a Hilbert space* X. *Moreover, assume that* $(iD_\lambda \Psi A_0, D(A_0)))$ *is symmetric and*

$$\|D_\lambda \Psi\| < 1$$

for some $\lambda \in \rho(A_0)$. *Then also the operator* $(i(A_0 - D_\lambda \Psi A_0), D(A_0))$ *is selfadjoint on* X. *Thus* $(\mathcal{A}_\Psi, D(\mathcal{A}_\Psi))$ *generates a strongly continuous (semi)group.*

Remark 4.9. The content of Theorem 4.4 can also be obtained by using the theory of multiplicative perturbations developed, e.g., in [2], [3], [15].

We take any $\lambda \in \rho(A_0)$ and observe that

$$D(A_\Psi) = \widetilde{D} := \{f \in X : (\mathrm{Id} - D_\lambda \Psi) f \in D(A_0)\}$$

and

$$A_\Psi f = A_0(\mathrm{Id} - D_\lambda \Psi) f + \lambda D_\lambda \Psi f$$

for $f \in \widetilde{D}$. Since the operator $\lambda D_\lambda \Psi$ is bounded on X, the operator A_Ψ is a generator if and only if $(A_0(\mathrm{Id} - D_\lambda \Psi), D(A_\Psi))$ is a generator. Applying a result on multiplicative perturbation [7, Thm. III.3. 20] we finally draw the following consequence. If the operator $((\mathrm{Id} - D_\lambda \Psi) A_0, D(A_0))$ is the generator of a strongly continuous (analytic) semigroup, the same holds for A_Ψ.

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