Electronic Journal of Differential Equations, Vol. 2004(2004), No. 108, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PERSISTENCE AND EXTINCTION OF SINGLE POPULATION IN A POLLUTED ENVIRONMENT

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Abstract. In this paper, we consider the ODE system corresponding to a diffusive-convective model for the dynamics of a population living in a polluted environment. Sufficient criteria on persistence and extinction of the population are derived.

1. Introduction

Today, the most threatening problem to the society is the change in environment caused by pollution, affecting the long term survival of species, human life style and biodiversity of the habitat. Therefore the study of the effects of toxicant on the population and the assessment of the risk to populations is becoming more important.

In the early eighties a deterministic modelling approach to the problem of assessing the effects of a pollutant on an ecological system was proposed by Hallam and his co-workers[4, 5, 6]. Since then, such models have been the subject of many investigations and improvements. Population-toxicant coupling has been applied in several contexts, including Lotka-Volterra and chemostat-like environments, resulting in ordinary, integro-differential and stochastic models. Usually a qualitative analysis was performed which focuses on the survival or extinction of populations [9, 10]. All these studies rely on the hypothesis of a complete spatially homogeneous environment.

Recently, a first attempt to consider a spatial structure has been carried out in [2, 3] where a reaction-diffusion model is proposed to describe the dynamics of a living population interacting with a toxicant present in the environment(external toxicant) through the amount of toxicant stored into the bodies of the living organisms(internal toxicant). However, as the authors pointed out, even if the resulting model presents many features which make stimulating its study, such a modelling approach is a rough approximation to the biological phenomena at hand. In 1999, Buonomo et.al. viewed the internal toxicant as drifted by the living population and then, by balance arguments, they derived a PDE system consisting into two reaction diffusion equations coupled with a first order convection equation, and the

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2000 *Mathematics Subject Classification.* 93C15.
Key words and phrases. Population dynamics; environment pollution; persistence.

Submitted January 9, 2004. Published Sept 12, 2004.
Supported by the National Natural Science Foundation of China (No.10171010), and by the Key Project on Science and Technology of the Education Ministry of China (No. Key 01061).

corresponding ODE system was obtained as well [1]. This model is the most realistic by now but the analysis of it is so difficult that they only used some analytic and numerical approaches. Obviously, the more clear work is deserved to do.

In this paper, we use some new methods to investigate the model made by Buonomo et al. and the conditions of survival and extinction are obtained.

2. The Model

We utilize a modified logistic equation [8] to model the effect of toxin on single species. We take

$n(t)$: concentration of the population biomass

$c(t)$: concentration of the toxicant in the environment

$z(t)$: concentration of the toxicant in the population.

We assume that there is a given(external) toxicant in the environment, and the living organisms absorb into their bodies part of this toxicant so that the dynamics of the population is affected by this(internal) toxicant. Concerning the growth rate of the population we assume that the birth rate is $b(n) = b_0 - fn$ and the death rate is $d(n, c) = d_0 + \alpha c$, where b_0,d_0 and α are positive constants. f is assumed to be a non-negative constant. Therefore we assume, in absence of toxicant, a malthusian($f = 0$) or a logistic growth rate($f > 0$). We can see that if $b_0 - d_0 - \alpha c \leq 0$, $n(t)$ will be extinct in the end, so we suppose

$$c < \frac{b_0 - d_0}{\alpha} := c_1. \tag{2.1}$$

We propose the following model governing the system

$$\begin{aligned} \frac{dn}{dt} &= n(b_0 - d_0 - \alpha c - fn) \\ \frac{dc}{dt} &= kz - (r + m + b_0 - fn)c \\ \frac{dz}{dt} &= -kzn + (r + d_0 + \alpha c)cn - hz + u(t). \end{aligned} \tag{2.2}$$

with initial data

$$n(0) = n_0 \geq 0; \quad c(0) = c_0 \geq 0; \quad z(0) = z_0 \geq 0.$$

Here α is the depletion rate coefficient of the population due to organismal pollutant concentration.

k is the depletion rate of toxicant in the environment due to its intake made by the population.

r is the depletion rate of toxicant in the population due to egestion.

m is the depletion rate of toxicant in the population due to metabolization processes.

h is the depletion rate of the toxicant in the environment.

u is the exogen+ous toxicant input rate which is assumed to be a smooth bounded non-negative function of t.

3. The conditions of survival and extinction for the population when $u(t) = Q > 0$

We now recall the definitions of persistence and extinction. A component $n(t)$ of a given ODE system is said to be persistent if for any $n(0) > 0$ it follows that $n(t) > 0, t > 0$ and $\liminf_{t\to\infty} n(t) > 0$. If there exists $\delta > 0$ (independent of $n(0)$)

such that $n(t)$ is persistent and is bounded and $\liminf_{t\to\infty} n(t) \geq \delta$, then $n(t)$ is said to be uniformly strongly persistent. If there exists $\delta > 0$ (independent of $n(0)$) such that $\limsup_{t\to\infty} n(t) \geq \delta$, then $n(t)$ is said to be uniformly weakly persistent. If $\limsup_{t\to\infty} n(t) \leq 0$, then $n(t)$ is said to be extinct.

Theorem 3.1. *The system* (2.2) *is uniformly strongly persistent if and only if*

$$\alpha k Q < h(b_0 - d_0)(r + m + b_0).$$

Proof. First, we deduce that the system is uniformly weakly persistent. Assume that the system is not uniformly weakly persistent then there exists a sequence of initial value $(n_k(0), c_k(0), z_k(0)) \in (0, +\infty) \times R_+^2$, such that

$$\limsup_{t\to\infty} n_k(t) = \varepsilon_k \to 0, \quad \text{as } k \to +\infty.$$

Then there exist $T_k > 0$, such that

$$n_k(t) < 2\varepsilon_k \quad \text{for } t \geq T_k. \tag{3.1}$$

From (2.1) and (3.1) we have

$$\dot{z_k} \leq (r + d_0 + \alpha c) c n_k - h z_k + Q \leq (r + b_0) c_1 2\varepsilon_k - h z_k + Q \quad \text{for } t \geq T_k\,.$$

Using the Comparison Theorem we have

$$\limsup_{t\to\infty} z_k(t) \leq \frac{c_1(r + b_0)2\varepsilon_k + Q}{h}.$$

So for all $\varepsilon > 0$, there exists $H_k > T_k > 0$, such that

$$z_k(t) \leq \frac{c_1(r + b_0)2\varepsilon_k + Q}{h} + \varepsilon =: \overline{z}_k \quad \text{for } t \geq H_k. \tag{3.2}$$

From (3.1) and (3.2) we see that

$$\begin{aligned}
\dot{c_k} &\leq k\overline{z}_k - (r + m + b_0)c_k + f 2\varepsilon_k c_k \\
&= k\overline{z}_k - (r + m + b_0 - f 2\varepsilon_k)c_k \ for\ t \geq H_k\,.
\end{aligned}$$

Similarly by the Comparison Theorem and let $\varepsilon \to 0$, we have

$$\limsup_{t\to\infty} c_k(t) \leq \frac{k c_1(r + b_0)2\varepsilon_k + kQ}{h(r + m + b_0 - f 2\varepsilon_k)}\,.$$

Then, for $t \geq H_k$,

$$c_k(t) \leq \frac{k c_1(r + b_0)2\varepsilon_k + kQ}{h(r + m + b_0 - f 2\varepsilon_k)} := \beta(\varepsilon_k). \tag{3.3}$$

Now we consider the first equation of the model (2.2), from (3.3) it is easy to see there exists $S_k > H_k > 0$, such that

$$\dot{n}_k \geq n_k(b_0 - d_0 - \alpha\beta(\varepsilon_k) - f n_k) \ for\ t \geq S_k.$$

Using the Comparison Theorem again we have

$$\liminf_{t\to\infty} n_k(t) \geq \frac{b_0 - d_0 - \alpha\beta(\varepsilon_k)}{f}. \tag{3.4}$$

By (3.4) and the assumption $\limsup_{t\to\infty} n_k(t) = \varepsilon_k$, we obtain

$$\limsup_{t\to\infty} n_k(t) = \varepsilon_k \geq \liminf_{t\to\infty} n_k(t) \geq \frac{b_0 - d_0 - \alpha\beta(\varepsilon_k)}{f}.$$

Let $k \to +\infty$, it follows that $\varepsilon_k \to 0$ and $\beta(\varepsilon_k) \to \frac{kQ}{h(r+m+b_0)}$. Hence,

$$0 \geq \frac{b_0 - d_0 - \alpha\beta(\varepsilon_k)}{f} \to \frac{h(b_0 - d_0)(r + m + b_0) - \alpha kQ}{fh(r + m + b_0)}.$$

That is, $\alpha kQ \geq h(b_0 - d_0)(r + m + b_0)$. So there is uniformly weakly persistent if

$$\alpha kQ < h(b_0 - d_0)(r + m + b_0).$$

Using the well known result which says that uniform weak persistence implies uniform strong persistence [7, Section 2], then the proof is completed. □

Moreover, if we look at the system restricted to $\{0\} \times R^2_+$, then there is a unique equilibrium

$$\overline{X} = (0, \overline{c}, \overline{z})$$

with

$$\overline{z} = \frac{Q}{h} \quad \text{and} \quad \overline{c} = \frac{k\overline{z}}{(r + m + b_0)}$$

Then an easy investigation of the linearized equation at $\overline{X}$ shows that when $\alpha kQ > h(b_0 - d_0)(r + m + b_0)$, $\overline{X}$ is locally asymptotically stable. In particular the system is not uniformly persistent anymore.

Theorem 3.2. *Consider the system* (2.2). *If* $Q > \frac{(r+m+b_0)(b_0-d_0)(fh+k(b_0-d_0))}{\alpha kf}$, *then the population is extinct.*

Proof. From (3.1) we know $n(t) \leq n_1$, for $t > t_1$. So from the last equation of the model (2.2) , we can obtain

$$\frac{dz}{dt} > Q - hz - kn_1 z \quad \text{for } t > t_1.$$

Now we use the Comparison Theorem again, then we have

$$\liminf_{t\to+\infty} z(t) \geq \frac{Q}{h + kn_1}.$$

Let $\varepsilon \to 0$, we have

$$\liminf_{t\to+\infty} z(t) \geq= \frac{fQ}{fh + k(b_0 - d_0)} := m_z.$$

That is to say for all $\varepsilon > 0, \exists T_1$, such that $z(t) > m_z - \varepsilon$, for all $t > T_1$. From the second equation of the model (2.2) , if $t > T_1$, it is easy to see

$$\frac{dc}{dt} > k(m_z - \varepsilon) - (r + m + b_0)c.$$

Similarly by the Comparison Theorem,

$$\liminf_{t\to+\infty} c(t) \geq \frac{km_z - \varepsilon)}{r + m + b_0}.$$

Let $\varepsilon \to 0$. Then we have

$$\liminf_{t\to+\infty} c(t) \geq \frac{k(m_z}{r + m + b_0} = \frac{kfQ}{(r + m + b_0)(fh + k(b_0 - d_0))} =: m_c.$$

Then for all $\varepsilon > 0$ there exists $T_2 > T_1$, such that $c(t) > m_c - \varepsilon$ for all $t > T_2$. Obviously, from the first equation of the model (2.2), if $t > T_2$, we have

$$\frac{dn}{dt} < n(b_0 - d_0 - \alpha(m_c - \varepsilon) - fn).$$

By the Comparison Theorem, we have

$$\limsup_{t\to+\infty} n(t) \le \frac{b_0 - d_0 - \alpha(m_c - \varepsilon)}{f}.$$

Let $\varepsilon \to 0$. Then we obtain

$$\limsup_{t\to+\infty} n(t) \le \frac{b_0 - d_0 - \alpha m_c}{f} =: M_n\,.$$

Clearly,

$$\limsup_{t\to+\infty} n(t) \le M_n \le \frac{b_0 - d_0}{f}.$$

If $M_n < 0$, that is $Q > \frac{(r+m+b_0)(b_0-d_0)(fh+k(b_0-d_0))}{\alpha kf}$, the population will be extinct. □

Acknowledgements. We thank the anonymous referee for his/her remarks that helped us improving Theorem 3.1.

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