Electronic Journal of Differential Equations, Vol. 2004(2004), No. 110, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

UNIQUENESS FOR DEGENERATE ELLIPTIC SUBLINEAR PROBLEMS IN THE ABSENCE OF DEAD CORES

JORGE GARCÍA-MELIÁN

Abstract. In this work we study the problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(u)$$

in the unit ball of $\mathbb{R}^N$, with $u = 0$ on the boundary, where $p > 2$, and λ is a real parameter. We assume that the nonlinearity f has a zero $\bar{u}_0 > 0$ of order $k \geq p-1$. Our main contribution is showing that there exists a unique positive solution of this problem for large enough λ and maximum close to $\bar{u}_0$. This will be achieved by means of a linearization technique, and we also prove the new result that the inverse of the p-Laplacian is differentiable around positive solutions.

1. Introduction

In this paper we are concerned with the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda f(u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$, stands for the p-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a bounded domain, λ a real parameter and f a C^1 function with a positive zero $\bar{u}_0$ (see hypotheses (H) below).

In the semilinear case $p = 2$ (where Δ_p reduces to the usual Laplacian), problems like (1.1) have been widely considered in the literature. An important number of works (cf. for instance [2, 5, 13, 14, 15] and references therein) deal with nonlinearities $f(u)$ with a positive zero $\bar{u}_0$, and their interest is focused on positive solutions u with $u \leq \bar{u}_0$ and $\max u$ close to $\bar{u}_0$. The important matter is then to show that such solutions are unique for large λ, and to ascertain their qualitative behaviour as $\lambda \to +\infty$.

The results obtained in the semilinear case heavily rely on the use of linearization around positive solutions. However, when trying to use the same tools with problems like (1.1) we encounter an important difficulty: the formal linearization of Δ_p around a solution u becomes degenerate at points where ∇u vanishes. Since it is a really hard task for the moment to locate the set of critical points of positive

2000 *Mathematics Subject Classification.* 35J60, 35J70.
Key words and phrases. p-Laplacian; linearization; uniqueness.

Submitted July 5, 2004. Published September 21, 2004.
Supported by contract PB96-0621 from DGES.

solutions, we are restricting our attention to a symmetric situation, where $\Omega = B$, the unit ball in $\mathbb{R}^N$. That is, we will consider

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } B \\ u &= 0 \quad \text{on } \partial B. \end{aligned} \tag{1.2}$$

In this setting – actually the more general one of rotationally invariant domains of $\mathbb{R}^N$ – some problems slightly more general than (1.2) were studied in [7]. Concretely, the function f was allowed to depend on λ:

$$\begin{aligned} -\Delta_p u &= \lambda f(\lambda, u) \quad \text{in } B \\ u &= 0 \quad \text{on } \partial B. \end{aligned} \tag{1.3}$$

The main assumption on f was the existence of a zero $\bar{u}_0 > 0$ of order $k < p-1$. It was proved there the existence of a unique family of solutions $\{u_\lambda\}$ with the property that $u_\lambda \le \bar{u}_0$ and

$$\lim_{\lambda \to +\infty} u_\lambda = \bar{u}_0, \quad \text{uniformly on compacts.} \tag{1.4}$$

It is important to notice that condition (1.4) was crucial in [7] in order to obtain uniqueness. That is, it is possible to construct functions f in such a way that problem (1.2) admits *two* families of positive solutions, one of them not verifying (1.4) (see [8]).

We shall presently consider the complementary case in which f has a positive zero $\bar{u}_0$ of order $k \ge p-1$, therefore closing the analysis started in [7]. It turns out that this situation is similar to the semilinear phenomenology. Firstly, the solutions do not have a dead core (see Remark 1.2 (a)). And secondly, it is sufficient to search for families of solutions $\{u_\lambda\}$ with $\max u_\lambda \to \bar{u}_0$ as $\lambda \to +\infty$ to obtain uniqueness – and we have as a consequence condition (1.4).

The most important fact in this regard is that we can linearize around positive solutions, this being a novelty in the context of the p-Laplacian.

We will assume throughout that f satisfies the following hypotheses, which will be termed as hypotheses (H):

(H1) $f \in C^1(\mathbb{R})$

(H2) f has a zero $\bar{u}_0$ of order $k \ge p-1$; that is, for some positive constant γ,

$$\lim_{u \to \bar{u}_0 -} \frac{f(u)}{(\bar{u}_0 - u)^k} = \gamma\,.$$

(H3) $F(u) < F(\bar{u}_0)$ if $0 \le u < \bar{u}_0$, where $F(u) = \int_0^u f(s)ds$.

(H4) $f'(u) \le 0$ in $[\bar{u}_0 - \varepsilon, \bar{u}_0]$ for some $\varepsilon > 0$.

(H5) f has a finite number of zeros in the interval $[0, \bar{u}_0]$.

Note that condition (H3) is necessary in order to have a family of solutions $\{u_\lambda\}$ with $\max u_\lambda \to \bar{u}_0$ as $\lambda \to +\infty$.

Our main result can be stated in the following way:

Theorem 1.1. *Assume f verifies hypotheses (H). Then there exist $\eta_0 > 0$, $\lambda^* > 0$ such that the problem* (1.2)

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \textit{in } B \\ u &= 0 \quad \textit{on } \partial B \end{aligned}$$

has a unique positive solution u_λ with $\bar u_0-\eta_0\le \max u_\lambda<\bar u_0$, if $\lambda\ge\lambda^*$. Moreover, u_λ is a radial function, the family $\{u_\lambda\}$ verifies condition (1.4), and we obtain the following exact estimate for the boundary layer near ∂B:

$$\lim_{\lambda\to+\infty}\lambda^{-1/p}u_\lambda'(1)=-(p'F(\bar u_0))^{1/p}\,, \tag{1.5}$$

where $F(\bar u_0)=\int_0^{\bar u_0}f(s)ds$.

Remark 1.2. (a) Condition (H2) on f guarantees that solutions with $\max u\le\bar u_0$ also verify $\max u<\bar u_0$ (cf. [6]), that is, dead cores do not arise even for large λ. This case is complementary to the one treated in [7].

(b) The results in Theorem 1.1 are also valid when problem (1.2) in considered in an annulus. This will be shown elsewhere.

(c) The boundary layer estimate (1.5) can be shown to be valid even for general domains Ω (without any knowledge of uniqueness). See [6] and [7] for related situations.

(d) The symmetry of solutions for problems like (1.2) will play an important rôle. We refer to §2 for details.

This paper is organized as follows: section 2 is devoted to some symmetry considerations. In section 3, after proving the existence of a family of positive solutions, we obtain some precise estimates for all possible solutions. Sections 4 and 5 form the core of the paper: in §4 we show that it is possible to linearize problem (1.2) around positive solutions, using this fact in §5 to prove uniqueness.

2. Symmetry of solutions

In the semilinear case $p=2$, a well known theorem by Gidas, Ni and Nirenberg (see [10]) asserts that positive solutions to (1.2) are radially symmetric. Some attempts to generalize this result to degenerate operators have been made for instance in [3] and [11]. However, the possible presence of dead cores in the solutions prevents one to expect a quite general symmetry result.

In a recent paper of Brock ([4]) an almost complete answer to this problem has been given. Nevertheless it is necessary to impose additional assumptions on f in order to obtain symmetry of positive solutions. We are showing in this section how to use the results in [4] to obtain that all positive solutions to (1.2) in our setting are radially symmetric, without further conditions on the nonlinear term f. The following result contains Lemmas 1 and 4, and Remark 1 in [4].

Lemma 2.1. *Let $u>0$ be a solution to* (1.2)*. Then there exists $m\in\mathbb{N}\cup\{+\infty\}$ so that B admits a decomposition:*

$$B=\cup_{k=1}^m C_k\cup\{x:\nabla u(x)=0\},$$

where $C_k=B_{R_k}(z_k)\setminus\overline{B_{r_k}(z_k)}$, for certain $z_k\in B$ and $0\le r_k<R_k$, such that

$$\begin{gathered} u(x)=u(\rho),\quad \rho=|x-z_k|\quad \text{in } C_k,\\ \frac{\partial u}{\partial\rho}<0\quad \text{in } C_k\,. \end{gathered} \tag{2.1}$$

Moreover, $u(x)\ge u_{|\partial B_{r_k}(z_k)}$ in $B_{r_k}(z_k)$ and $\nabla u=0$ on $\partial C_k\cap B$. On the other hand, $f(u)=0$ on $\partial B_{R_k}(z_k)\cap B$ and if $r_k>0$ then $f(u)=0$ on $\partial B_{r_k}(z_k)$.

Remark 2.2. This property of the solutions u to (1.2) is called *local symmetry.*

By means of this result, we are defining, for an arbitrary positive solution u to (1.2), a *rearrangement* u^*, that is, a positive radial solution u^*, with $\max u^* = \max u$.

Let u be a positive solution to (1.2). We claim that $\max u$ is attained at some z_k (z_k as in Lemma 2.1). Indeed, choose $x_0 \in B$ such that $u(x_0) = \max u$, and a sequence $x_j \in \partial C_j$ such that dist $(x_j, x_0) \to \inf \operatorname{dist}(C_k, x_0)$. It is easy to see that the $\{x_j\}$ can be chosen in such a way that (passing through a subsequence if necessary) one of the following situations holds: either $x_j \to x_0$ or the segment $[x_j, x_0]$ is contained in the set $\{x : \nabla u(x) = 0\}$.

In the first case, we have $f(u(x_j)) = 0$ (since $x_j \in \partial C_j$) and it follows that $u(x_0)$ is an accumulation point of zeros of f unless $u(x_j) = u(x_0)$ for infinitely many j's. Our hypotheses then imply that – passing through a subsequence – $u(x_j) = u(x_0)$. If $x_j \in \partial B_{R_j(z_j)}$, we obtain a contradiction to (2.1). Hence, $x_j \in \partial B_{r_j}(z_j)$. If $r_j = 0$ for some j, we have $x_j = z_j$, as was to be proved. If, on the contrary, $r_j > 0$, then $u(x) \geq u(x_j)$ in $B_{r_j}(z_j)$, and the maximum is attained in the whole $B_{r_j}(z_j)$, also showing the claim (notice in particular that $\{x_j\}$ reduces to a single point). In the second case, we also arrive at $u(x_0) = u(x_j)$ and the conclusion is the same.

Remark 2.3. The above reasoning also shows that if the maximum is achieved in z_k, then the solution u is radial in $B_{R_k}(z_k)$, which is not clear from Lemma 2.1 if $r_k > 0$.

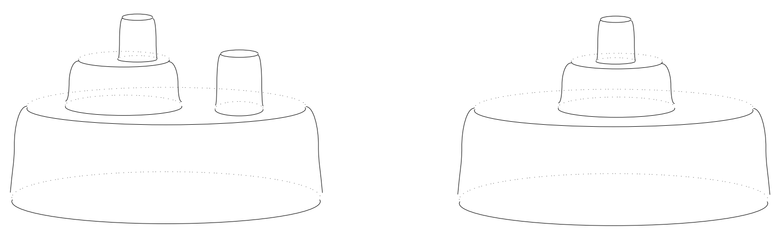

FIGURE 1. A locally symmetric solution u and its rearrangement u^*.

Without loss of generality, assume $u(z_1) = \max u$. In virtue of Lemma 2.1, $u_1 = u_{|\partial B_{R_1}(z_1)}$ is a zero of f, and $u_1 < u(z_1)$. Now we will define an auxiliary function $\bar{u}$ such that $\bar{u} \leq u_1$ in $B \setminus B_{R_1}(z_1)$, and show that it is possible to choose another annulus C_2 such that $r_2 > R_1$ and $\bar{u} = u_1$ on $\partial B_{r_2}(z_2)$. To this aim we are proceeding as follows: if there is a point $x \in C_k$, $k \neq 1$, such that $u(x) > u_1$ then set $\bar{u} = u_{|\partial B_{R_k}(z_k)}$ in $B_{R_k}(z_k)$. Otherwise define $\bar{u} = u$. Clearly, $\bar{u} \leq u_1$ in $B \setminus B_{R_1}(z_1)$.

Now take the sequences $x_j \in \partial B_{R_1}(z_1)$ and $\hat{x}_j \in \partial C_j$ in such a way that $\operatorname{dist}(x_j, \hat{x}_j) \to \inf_{k \neq 1} \operatorname{dist}\, (C_k, \partial B_{R_1}(z_1))$. As before, it is possible to choose these sequences such that two options may arise: $|x_j - \hat{x}_j| \to 0$ or $[x_j, \hat{x}_j] \subset \{x : \nabla u(x) = 0\}$. Both of them lead to $\bar{u}(\hat{x}_j) = \bar{u}(x_j) = u_1$ for a subsequence, and $\hat{x}_j \in \partial B_{r_j}(z_j)$. Thus, $\bar{u} = u_1$ in $B_{r_j}(z_j) \setminus B_{R_1}(z_1)$. As a conclusion, the sequence $\{\hat{x}_j\}$ reduces to a point, and $r_j > R_1$, $\bar{u} = u_1$ on $\partial B_{r_j}(z_j)$. Assume $j = 2$.

Repeating the above procedure, we arrive, after a finite number of steps, at $u_l = 0$ for some $l \in \mathbb{N}$. Denoting by χ the characteristic function of a set, we define:

$$u^*(x) = \sum_{i=1}^{l} \bar{u}(x + z_i)\chi_{\{R_{i-1} \le |x| < R_i\}}\,,$$

where $R_0 = 0$ (see Figure 1). The main property of the function u^* is the following:

Lemma 2.4. *Let u be a positive solution to* (1.2)*. Then the function u^* defined above is a radial positive solution to* (1.2)*. Moreover, if $\nabla u^* \neq 0$ in $B \setminus \{0\}$ then $u^* = u$ and u is a radial function.*

Proof. It is easy to check that u^* is a radial solution to (1.2) (see Remark 2.3). Thus, assume $\nabla u^* \neq 0$ in $B \setminus \{0\}$. According to the definition of u^*, consider $B_{R_1}(z_1)$. If this ball does not coincide with B then in virtue of Lemma 2.1 we obtain $\nabla u = 0$ on $\partial B_{R_1}(z_1)$, which is a contradiction. Thus, $B_{R_1}(z_1) = B$ and $u^* = u$. This proves the lemma. □

Remark 2.5. If we knew from the beginning that $\nabla u \neq 0$ in $B \setminus \{0\}$, then Theorem 1 in [3] could also be applied to conclude that u is radial.

3. Existence and estimates of solutions

In this section we are proving that, under the assumptions (H) on f (see §1), we can guarantee the existence of a family $\{u_\lambda\}$ of positive radial solutions to (1.2) which in addition verifies (1.4). The first important remark is that, due to hypotheses $(\mathrm{H})_2$ on the zero of f, positive solutions u to (1.2) with $u \le \bar{u}_0$ also satisfy $0 < u < \bar{u}_0$ (see [6]).

To begin with, notice that if u is a radial solution to (1.2), then it satisfies (see §4):

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' = r^{N-1}\lambda f(u) \\ u'(0) = 0\,, \quad u(1) = 0\,. \end{gathered}$$

Setting $v(r) = u(\lambda^{-1/p}r)$, this problem is equivalent to

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' = r^{N-1}f(u) \\ u'(0) = 0\,, \quad u(\lambda^{1/p}) = 0\,. \end{gathered} \tag{3.1}$$

Thus, it is apparent that the Cauchy problem

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' = r^{N-1}f(u) \\ u(0) = u_0\,, \quad u'(0) = 0\,, \end{gathered} \tag{3.2}$$

with u_0 in a left neighbourhood of $\bar{u}_0$, will play an important role. Let us just quote that this problem has a unique solution, denoted henceforth as $u(\cdot, u_0)$, in an interval of the form $[0, \delta]$ (see Theorem 2.1 in [7] and references therein). Moreover, this solution can be continued as long as $u'(\cdot, u_0) \neq 0$. It is also worthy of mention that $u(\cdot, \bar{u}_0) \equiv \bar{u}_0$ (Theorem 2.2 in [7]).

Our existence result is the following:

Theorem 3.1. *Assume f satisfies hypotheses (H). Then there exist $\eta > 0$, $\lambda_0 > 0$ such that for every $\lambda \ge \lambda_0$, problem* (1.2) *admits at least a radial positive solution u_λ verifying $\bar{u}_0 - \eta \le \max u_\lambda < \bar{u}_0$. In addition the family $\{u_\lambda\}$ satisfies (1.4) and $u'_\lambda(r) < 0$ for $r > 0$.*

Proof. Let us show that there exists $\eta > 0$ such that for the solution $u = u(\cdot, u_0)$ to the problem (3.2) with $\bar{u}_0 - \eta \le u_0 < \bar{u}_0$ there exists $T > 0$ with $u(T) = 0$.

First we claim that for every $\varepsilon_0 < \varepsilon$ (ε as in hypothesis (H4) on f), and $R > 0$, there exists $\eta = \eta(\varepsilon_0, R)$ such that $u(r, u_0)$ is defined in $[0, R]$, $u'(r, u_0) < 0$ and $u(r, u_0) \ge \bar{u}_0 - \varepsilon_0$ if $0 < r \le R$ whenever $\bar{u}_0 - \eta \le u_0 < \bar{u}_0$.

To prove the claim, assume first that there exist sequences $u_{0n} \to \bar{u}_0-$ and $r_n \le R$ such that $u'(r_n, u_{0n}) = 0$. Since

$$u'(r, u_{0n}) = -\varphi_{p'}\Big(\int_0^r \big(\frac{\rho}{r}\big)^{N-1} f(u(\rho, u_{0n}))\,d\rho\Big),$$

and $f > 0$ in $[\bar{u}_0 - \varepsilon, \bar{u}_0)$, it follows easily that $u(r_n, u_{0n}) < \bar{u}_0 - \varepsilon$. Thus, there exists $\hat{r}_n < r_n \le R$ such that $u(\hat{r}_n, u_{0n}) = \bar{u}_0 - \varepsilon_0$. Passing to a subsequence we can assume $\hat{r}_n \to \hat{r}_0$, this also implying that $u(\hat{r}_n, u_{0n}) \to u(\hat{r}_0, \bar{u}_0) = \bar{u}_0$. This clear contradiction proves the existence of $\eta > 0$ such that $u(r, u_0)$ is defined in $[0, R]$ and $u'(r, u_0) < 0$ in $[0, R]$ when $\bar{u}_0 - \eta \le u_0 < \bar{u}_0$. The remaining part of the claim is proved *exactly* in the same way.

Now notice that positive solutions to (3.1) together with their derivatives are uniformly bounded in $r \ge 0$. Indeed, multiplying the equation by u' and integrating in $[0, r]$ we arrive at the identity

$$|u'(r)|^p + p'(N-1)\int_0^r \frac{|u'(s)|^p}{s}ds = p'(F(u_0) - F(u(r))),$$

so that $|u'(r)|^p \le p'(F(u_0) - F(u(r)))$ if $r \ge 0$. Choose $\tau > 0$ and $R > 0$ to achieve

$$(N-1)\frac{|u'(r)|^{p-1}}{r} \le \tau, \quad r \ge R.$$

Thus, $-\varphi_p(u')' = f(u) + (N-1)\varphi_p(u')/r \ge f(u) - \tau$ if $r \ge R$ and, as long as $u' \le 0$, we obtain $-\varphi_p(u')'u' \le (f(u) - \tau)u'$, that is,

$$\big(|u'(r)|^p + p'F_\tau(u(r))\big)' \ge 0, \quad r \ge R, \tag{3.3}$$

where $F_\tau(u) = F(u) - \tau u$, and $F(u) = \int_0^u f(s)ds$. Notice that $f - \tau$ has a zero $\bar{u}_0(\tau) < \bar{u}_0$ with $\bar{u}_0(\tau) \to \bar{u}_0$ as $\tau \to 0$, verifying in addition the energy condition $F_\tau(u) < F_\tau(\bar{u})$, $0 \le u < \bar{u}$, if $\bar{u} \in [\bar{u}_0(\tau) - \delta(\tau), \bar{u}_0(\tau)]$, for some $\delta(\tau) > 0$.

Choose ε_0, τ small so that $\bar{u}_0(\tau) - \delta(\tau) < \bar{u}_0 - \varepsilon_0$. As the claim at the beginning of the proof shows, we have $\bar{u}_0(\tau) - \delta(\tau) \le u(R, u_0) < \bar{u}_0$. Thus if $r \ge R$, we have, in virtue of (3.3), $|u'(r)|^p \ge |u'(R)|^p + p'(F_\varepsilon(u(R)) - F_\varepsilon(u(r)))$, for $r \ge R$. In particular, $u'(r) < 0$ always holds, and then

$$u'(r) \le u'(R), \quad r \ge R.$$

Integrating this inequality,

$$u(r) \le u(R) + u'(R)(r - R), \quad r \ge R,$$

and we conclude that u has to vanish for some $T = T(u_0)$. We have defined in this way a continuous mapping $T : [\bar{u}_0 - \eta, \bar{u}_0) \to R^+$. To complete the proof of the Theorem it suffices to show that $T(u_0) \to +\infty$ as $u_0 \to \bar{u}_0-$ (then the construction of the family of solutions $\{u_\lambda\}_{\lambda \ge \lambda_0}$ is performed in a standard way). Assume on the contrary that there exists a sequence $u_{0n} \to \bar{u}_0$ such that $T(u_{0n})$ is bounded. Without loss of generality, we can assume $T(u_{0n}) \to T_0 > 0$, and it follows that $u_n := u(\cdot, u_{0n}) \to \bar{u}_0$ uniformly in $[0, T_0]$ as seen before. This contradicts the fact that $u(T(u_{0n}), u_{0n}) = 0$, finally proving the theorem. □

Corollary 3.2. *Let λ_0, η be as in Theorem 3.1. Then every positive solution u to (1.2) with $\bar{u}_0 - \eta \le \max u \le \bar{u}_0$ and $\lambda \ge \lambda_0$ is radial.*

Proof. Let u be a positive solution to (1.2) with $\bar{u}_0 - \eta \le \max u \le \bar{u}_0$ and $\lambda \ge \lambda_0$, where η, λ_0 are as in Theorem 3.1. Consider the function u^* defined in §2. In virtue of Lemma 2.4, u^* is a positive radial solution to (1.2) and $\max u^* = \max u$. Since the function $v(r) = u^*(\lambda^{-1/p} r)$ solves

$$\begin{gathered}
-(r^{N-1}\varphi_p(v'))' = r^{N-1} f(v)\\
v(0) = \max u\,, \quad u'(0) = 0\,,
\end{gathered}$$

it follows by the uniqueness of this Cauchy problem that $u^* = u_\mu$ for some $\mu \ge \lambda_0$, where $\{u_\mu\}$ is the family of functions given by Theorem 3.1. Thus $u^{*\prime}(r) < 0$ for $r > 0$, and Lemma 2.4 implies that $u = u^*$, and u is a radial function. $\square$

In the remaining part of the section we are obtaining estimates for the positive solutions u to (1.2) with $\max u$ close to $\bar{u}_0$ and large λ.

First of all we are constructing a subsolution of (1.2), using ideas from [5]. To this aim, we are redefining f outside $[0, \bar{u}_0]$. More precisely, we can assume with no loss of generality that f is bounded, $f < 0$ in $[\bar{u}_0, +\infty)$, $f = 0$ in $(-\infty, -1]$ and $F(u) < F(\bar{u}_0)$ for $-1 \le u \le 0$.

Now let ε be as in hypothesis $(\mathrm{H})_4$. With no loss of generality, we can assume that $\varepsilon < \eta$. We can find a value $\lambda_1 > \lambda_0$ such that the solution u_{λ_1} given by Theorem 3.1 satisfies $u_{\lambda_1}(0) = \bar{u}_0 - \varepsilon/2$. Thanks to the condition verified by F (and diminishing ε again if necessary), we can produce u_{λ_1} to reach a value $r_0 > 1$ such that $u_{\lambda_1}(r_0) = -1$, and $u'_{\lambda_1}(r) < 0$ if $r \in (0, r_0]$. Moreover, since $f = 0$ in $(-\infty, -1]$, u_{λ_1} satisfies

$$\begin{gathered}
-(r^{N-1}\varphi_p(u'))' = 0, \quad r > r_0\\
u(r_0) = -1\,, \quad u'_{\lambda_1}(r_0) < 0\,,
\end{gathered}$$

that is

$$u(r) = \begin{cases} -1 + u'_{\lambda_1}(r_0)\frac{r_0^\theta r^{1-\theta} - r_0}{1-\theta}, & p \ne N\\ -1 + u'_{\lambda_1}(r_0) r_0 \log\big(\frac{r}{r_0}\big), & p = N\,, \end{cases}$$

for $r \ge r_0$, with $\theta = (N-1)/(p-1)$. In particular, $u_{\lambda_1}(r) < 0$ if $r > 1$. This function will allow us to obtain a subsolution to problem (1.2).

Lemma 3.3. *Let u_{λ_1} be as before, and define*

$$z_\lambda(r) = u_{\lambda_1}\Big(\big(\frac{\lambda}{\lambda_1}\big)^{1/p} r\Big)\,.$$

Then z_λ is a subsolution to (1.2) for $\lambda > \lambda_1$.

Proof. Clearly, z_λ verifies the equation. Moreover, $z_\lambda(1) = u_{\lambda_1}((\lambda/\lambda_1)^{1/p}) < 0$, since $u_{\lambda_1}(r) < 0$ if $r > 1$. $\square$

The existence of this subsolution is essential. Indeed, for large enough λ, every positive solution with maximum close to $\bar{u}_0$ lies above it.

Lemma 3.4. *There exist $0 < \eta_0 \le \eta$, $\lambda_2 > \lambda_1$ such that every positive solution u to (1.2) with $\lambda \ge \lambda_2$ and $\bar{u}_0 - \eta_0 \le \max u < \bar{u}_0$ verifies $u \ge z_\lambda$.*

Proof. In virtue of Corollary 3.2, u is a radial solution. Moreover, as seen there, $u = u_\mu$ for some $\mu = \mu(\lambda)$. It is not hard to show that $\mu(\lambda) \to +\infty$ as $\lambda \to +\infty$ and $\max u \to \bar{u}_0-$. Thus, for $\delta > 0$ fixed, there exist $\bar{\lambda}$, $0 < \eta_0 \le \eta$ such that $\lambda \ge \bar{\lambda}$ and $\bar{u}_0 - \eta_0 \le \max u < \bar{u}_0$ imply $u(r) \ge \bar{u}_0 - \varepsilon/2$, for every $r \in [0, 1-\delta]$ (this is a consequence of condition (1.4)). Since $\max z_\lambda = \bar{u}_0 - \varepsilon/2$, we obtain $u \ge z_\lambda$ in $[0, 1-\delta]$. In addition,

$$z_\lambda(r) \le z_\lambda(1-\delta) = u_{\lambda_1}\Big(\big(\frac{\lambda}{\lambda_1}\big)^{1/p}(1-\delta)\Big) \le 0\,,$$

if $\lambda \ge \lambda_1/(1-\delta)^p$, $r \ge 1-\delta$. Hence, $u_\lambda \ge z_\lambda$ in $[1-\delta, 1]$, and we can take $\lambda_2 = \max\{\lambda_1/(1-\delta)^p, \bar{\lambda}\}$. This concludes the proof of the Lemma. □

Lemma 3.5. *Let η_0, λ_2 be as in Lemma 3.4. Then there exists $\Lambda > 0$ such that every positive solution u to* (1.2) *with $\lambda \ge \lambda_2$ and $\bar{u}_0 - \eta_0 \le \max u < \bar{u}_0$ verifies $u(r) \ge \bar{u}_0 - \varepsilon$ if $0 \le r \le 1 - \Lambda\lambda^{-1/p}$.*

Proof. Choose $e \in \mathbb{R}^N$ with $|e| = 1$ and define the family of subsolutions

$$u_{t,\lambda}(x) = u_{\lambda_1}\Big(\big(\frac{\lambda}{\lambda_1}\big)^{1/p}|x - te|\Big)\,,$$

for $x \in B$ and $t \in [0, 1-(\lambda/\lambda_1)^{-1/p})$. Since, in virtue of Lemma 3.4, $u \ge u_t\big|_{t=0}$, we are in a position to apply the *sweeping principle* of the Appendix to conclude that $u \ge u_t\big|_{t=1-(\lambda/\lambda_1)^{-1/p}}$. Let $0 < R < 1$ be such that $u_{\lambda_1}(r) \ge \bar{u}_0 - \varepsilon$ if $0 \le r \le R$. Then $u(r) \ge \bar{u}_0 - \varepsilon$ for $0 \le r \le 1 - (1-R)(\lambda/\lambda_1)^{-1/p}$. Thus, we can take $\Lambda = (1-R)\lambda_1^{1/p}$. □

Remark 3.6. Notice that hypothesis $(\mathrm{H})_2$ suffices to guarantee that every family of positive solutions $\{u_\lambda\}$ to (1.2) such that $\lim_{\lambda\to+\infty} \max u_\lambda = \bar{u}_0$ verifies (1.4). This is in contrast with the case $0 < k < p-1$ treated in [7].

4. Differentiability properties

In this section we are showing some auxiliary results, which deal with the linearization of the inverse of the p-Laplacian under radial symmetry.

Let $f \in C(\overline{B})$ be radially symmetric. Since $p > 2$, it is well known that for every $m \ge 0$ there exists a unique weak solution u to the equation

$$\begin{gathered} -\Delta_p u + mu = f \quad \text{in } B \\ u = 0 \quad \text{on } \partial B\,, \end{gathered}$$

which is $C^{1,\beta}(\overline{B})$ for some $0 < \beta < 1$ (cf. [16]). Moreover, u is a radial function, so that $u \in C^1[0,1]$, $r^{N-1}\varphi_p(u') \in C^1[0,1]$ and u solves

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' + mr^{N-1}u = r^{N-1}f(r) \quad 0 < r < 1 \\ u'(0) = 0\,, \quad u(1) = 0\,, \end{gathered}$$

where $' = d/dr$. Thus we can define an operator $K_m : C[0,1] \to C^1[0,1]$ given by $u = K_m(f)$, which is compact. For $m = 0$ it is easily seen that

$$K(f)(r) := K_0(f)(r) = \int_r^1 \varphi_{p'}\Big(\int_0^s \big(\frac{\rho}{s}\big)^{N-1} f(\rho)\,d\rho\Big)\,ds\,.$$

For simplicity, we still denote K_m to the restriction of those operators to $C^1[0,1]$.

Back to problem (1.2), taking $m > 0$ such that $f'(u) + m > 0$ in $[0, \bar{u}_0]$, then radial solutions to (1.2) coincide with fixed points of the operator equation

$$u = K_{\lambda m}(\lambda f(u) + \lambda m u)\,.$$

Denote $T_\lambda(u) = K_{\lambda m}(\lambda f(u) + \lambda m u)$. T_λ is a compact operator in $C^1[0,1]$, and it is increasing in the order interval $[0, \bar{u}_0]$. Our first objective is to show that T_λ is differentiable in a neighbourhood of its fixed points in the interval $[z_\lambda, \bar{u}_0]$ (see §3). With this in mind, it is convenient to consider first the case $m = 0$. See Theorem 2.1 in [8] for a related result.

Theorem 4.1. *Assume $f \in C^1[0,1]$ verifies $f(0) \neq 0$ and $u'(r) \neq 0$ if $0 < r \le 1$, where $u = Kf$. Then K, as an operator defined in $C^1[0,1]$, is Fréchet-differentiable on f, and*

$$DK(f)g = \frac{1}{p-1}\int_r^1 \frac{1}{|u'(s)|^{p-2}}\int_0^s \big(\frac{\rho}{s}\big)^{N-1} g(\rho)\,d\rho\,ds \tag{4.1}$$

for every $g \in C^1[0,1]$. In particular, $w = DK(f)g$ is a solution to the equation

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}w')' = \frac{r^{N-1}}{p-1} g(r) \quad 0 < r < 1 \\ w'(0) = 0\,, \quad w(1) = 0\,. \end{gathered} \tag{4.2}$$

Proof. Without loss of generality, assume $f(0) > 0$. Thanks to Theorem 2.1 in [7], we have

$$u'(r) \sim -C r^{\frac{1}{p-1}}, \quad r \to 0+$$

with a certain constant $C > 0$. Thus there exists a constant $c > 0$ such that $|u'(s)|/s^{\frac{1}{p-1}} \ge c$ for $0 < s \le 1$, and the expression (4.1) makes sense. Denote it by $R(g)$. Then

$$\begin{aligned} &|(K(f+g) - K(f) - R(g))'(s)| \\ &\le \Big|\varphi_{p'}\Big(\int_0^s \big(\frac{\rho}{s}\big)^{N-1}(f(\rho) + g(\rho))\,d\rho\Big) - \varphi_{p'}\Big(\int_0^s \big(\frac{\rho}{s}\big)^{N-1} f(\rho)\,d\rho\Big) \\ &\quad - \frac{1}{p-1}\frac{1}{|u'(s)|^{p-2}}\int_0^s \big(\frac{\rho}{s}\big)^{N-1} g(\rho)\,d\rho\Big| \\ &\le \frac{1}{p-1}\Big||\xi(s)|^{p'-2} - \frac{1}{|u'(s)|^{p-2}}\Big|\int_0^s \big(\frac{\rho}{s}\big)^{N-1}|g(\rho)|\,d\rho \\ &\le \frac{1}{p-1}\Big|s|\xi(s)|^{p'-2} - \frac{s}{|u'(s)|^{p-2}}\Big||g|_1\,, \end{aligned}$$

where $\xi(s)$ is an intermediate function comprised between $\int_0^s (\frac{\rho}{s})^{N-1}(f(\rho)+g(\rho))\,d\rho$ and $\int_0^s (\frac{\rho}{s})^{N-1} f(\rho)\,d\rho$. Thus, as $|g|_1 \to 0$ we obtain

$$\xi(s) \to \int_0^s \big(\frac{\rho}{s}\big)^{N-1} f(\rho)\,d\rho = -|u'(s)|^{p-1}$$

uniformly in $[0,1]$. Let us see that $\xi(s)/s \to -|u'(s)|^{p-1}/s$ uniformly in $[0,1]$ as $|g|_1 \to 0$. Indeed,

$$\Big|\frac{1}{s}\int_0^s \big(\frac{\rho}{s}\big)^{N-1}(f(\rho) + g(\rho))\,d\rho - \frac{1}{s}\int_0^s \big(\frac{\rho}{s}\big)^{N-1} f(\rho)\,d\rho\Big| \le |g|_1\,, \tag{4.3}$$

and since $\xi(s)/s$ is an intermediate value, we are done. This implies

$$\begin{aligned}
&|(K(f+g)-K(f)-R(g))'(s)|\\
&\le \frac{1}{p-1}s^{\frac{1}{p-1}}\Big|\big(\frac{|\xi(s)|}{s}\big)^{p'-2}-\big(\frac{|u'(s)|^{p-1}}{s}\big)^{p'-2}\Big||g|_1=o(|g|_1)\,,
\end{aligned}$$

uniformly as $|g|_1\to 0$. In the same way $|K(f+g)-K(f)-R(g)|_\infty=o(|g|_1)$ as $|g|_1\to 0$. This proves the differentiability assertion. That $w=R(g)$ is a solution to (4.2) is a direct consequence of expression (4.3). □

Corollary 4.2. *Assume $f\in C^1[0,1]$ verifies $f(0)\ne 0$ and $u'(r)\ne 0$ if $0<r\le 1$, where $u=Kf$. Then K is C^1 in a neighbourhood of f in $C^1[0,1]$.*

Proof. Let us show first that K is differentiable in a neighbourhood of f. Notice that, in virtue of (4.3), if $|g|_1$ is small, we have that

$$\frac{|v'(s)|^{p-1}}{s}\ge c>0$$

in $(0,1]$, where $v=K(f+g)$. As the proof of Theorem 4.1 shows, this condition turns out to be sufficient for the differentiability of K on $f+g$. Moreover, for $h\in C^1[0,1]$,

$$\begin{aligned}
&|(DK(f+g)h-DK(f)h)'(s)|\\
&\le \frac{1}{p-1}\Big|\frac{1}{|u'(s)|^{p-2}}-\frac{1}{|v'(s)|^{p-2}}\Big|\int_0^s\big(\frac{\rho}{s}\big)^{N-1}|h(\rho)|d\rho\\
&\le \frac{1}{p-1}s\Big|\frac{1}{|u'(s)|^{p-2}}-\frac{1}{|v'(s)|^{p-2}}\Big||h|_1\\
&\le \frac{1}{p-1}\Big|\big(\frac{s^{\frac{1}{p-1}}}{|u'(s)|}\big)^{p-2}-\big(\frac{s^{\frac{1}{p-1}}}{|v'(s)|}\big)^{p-2}\Big||h|_1
\end{aligned}$$

and we obtain, in virtue of (4.3),

$$\sup_h\frac{|DK(f+g)h-DK(f)h|_1}{|h|_1}\to 0$$

as $g\to 0$ in $C^1[0,1]$. This proves the corollary. □

As a consequence of the implicit function theorem, we can now obtain the differentiability of the operator T_λ on its fixed points in the interval $[z_\lambda,\bar u_0]$.

Corollary 4.3. *Let u be a fixed point of the operator T_λ in the interval $[z_\lambda,\bar u_0]$, with $f(u(0))\ne 0$, and $u'(r)\ne 0$ if $0<r\le 1$. Then T_λ is C^1 in a neighbourhood of u and we have, for every $g\in C^1[0,1]$, that $w=DT_\lambda(u)g$ solves*

$$\begin{gathered}
-(r^{N-1}|u'|^{p-2}w')'+\frac{r^{N-1}}{p-1}\lambda mw=\frac{r^{N-1}}{p-1}(\lambda f'(u)+\lambda m)g\\
w'(0)=0\,,\quad w(1)=0\,.
\end{gathered}$$

Proof. Let us prove that $K_{\lambda m}$ is differentiable in a neighbourhood of $\lambda f(u)+\lambda mu$. Denote $\hat f=\lambda f(u)+\lambda mu$. Solving the equation $-\Delta_p v+\lambda mv=f$ is equivalent to solving

$$\mathcal{F}(v,f):=v-K(f-\lambda mv)=0\,.$$

We will show that the implicit function theorem can be applied in this case. Indeed, $\mathcal{F}(u,\hat f)=0$, and in virtue of Corollary 4.2, K is C^1 in a neighbourhood of $\hat f-$

$\lambda mu = \lambda f(u)$ (since $f(u(0)) \neq 0$ and $u'(r) \neq 0$ for $0 < r \leq 1$). Thus, $\mathcal{F}$ is C^1 with respect to both variables. Assume there exists g such that $D_v\mathcal{F}(u,\hat{f})g = g + DK(\hat{f} - \lambda mu)\lambda mg = 0$. Then g solves

$$
\begin{gathered}
-(r^{N-1}|u'|^{p-2}g')' + \frac{r^{N-1}}{p-1}\lambda mg = 0 \\
g'(0) = 0\,, \quad g(1) = 0\,,
\end{gathered}
$$

and we obtain $g \equiv 0$. Hence, $D_v\mathcal{F}(u,\hat{f})$ is an isomorphism (notice that $D_v\mathcal{F}(u,\hat{f})$ is a compact perturbation of the identity).

Implicit function theorem guarantees the existence of a unique C^1 function R, such that $\mathcal{F}(v,f) = 0$ implies $v = R(f)$ in a neighbourhood of $(u,\hat{f})$. Uniqueness yields $R(f) = K_{\lambda m}(f)$, hence $K_{\lambda m}$ is C^1. In addition,

$$
D_v\mathcal{F}(u,\hat{f})DK_{\lambda m}(\hat{f}) + D_f\mathcal{F}(u,\hat{f}) = 0\,,
$$

so that $w = DK_{\lambda m}(\hat{f})g$ is a solution to the equation

$$
\begin{gathered}
-(r^{N-1}|u'|^{p-2}w')' + \frac{r^{N-1}}{p-1}\lambda mw = \frac{r^{N-1}}{p-1}g \\
w'(0) = 0\,, \quad w(1) = 0\,.
\end{gathered}
$$

The conclusion of this theorem is then a direct consequence of the chain rule. □

5. Uniqueness

This section is devoted to the proof of Theorem 1.1. First of all, we need a result about the eigenvalues of the linearization of the operator T_λ. The spectral radius of a bounded linear operator L will be denoted by $\operatorname{spr}(L)$. We recall that $|\mu| \leq \operatorname{spr}(L)$ for every spectral value of L, in particular for possible eigenvalues μ (cf. [17]). We have the following lemma.

Lemma 5.1. *Let u be a fixed point of T_λ in the interval $[z_\lambda, \bar{u}_0]$. If $\sigma = \operatorname{spr}(DT_\lambda(u))$ is positive, then σ is an eigenvalue of $DT_\lambda(u)$ which admits an eigenfunction v such that $v(r) > 0$ for $0 \leq r < 1$.*

Proof. Let us see that the operator $DT_\lambda(u)$ is positive. That is, $g \geq 0$ implies $DT_\lambda(u)g \geq 0$. Let $w = DT_\lambda(u)g$. Then

$$
\begin{gathered}
-(r^{N-1}|u'|^{p-2}w')' + \frac{r^{N-1}}{p-1}\lambda mw = \frac{r^{N-1}}{p-1}(\lambda f'(u) + \lambda m)g \\
w'(0) = 0\,, \quad w(1) = 0\,.
\end{gathered}
$$

Multiplying this equation by $w^- = \max\{0, -w\}$, integrating in $(0,1)$ and performing an integration by parts in the left-hand side, we obtain

$$
\int_{w\leq 0} \big(r^{N-1}|u'|^{p-2}(w')^2 + \frac{r^{N-1}}{p-1}\lambda mw^2\big)dr = \int_{w\leq 0} \frac{r^{N-1}}{p-1}(\lambda f'(u) + \lambda m)wg\,dr \leq 0\,,
$$

hence $w^- \equiv 0$. Thus $w \geq 0$ follows.

Since $DT_\lambda(u)$ is a compact operator, Krein-Rutman's theorem [1, Theorem 3.1] guarantees the existence of an eigenfunction v associated to σ such that $v \geq 0$. Let us show that $v > 0$.

Assume on the contrary that $v(r_0) = 0$ for some $0 \le r_0 < 1$. Since $v \ge 0$, we have $v'(r_0) = 0$. Moreover, v satisfies

$$-v'(r) = \frac{1}{p-1}\frac{\lambda}{\sigma}\frac{1}{|u'(r)|^{p-2}}\int_{r_0}^{r}\Big(\frac{\rho}{r}\Big)^{N-1}(f'(u)+m(1-\sigma))v(\rho)d\rho\,.$$

Then, letting $|v|_{\infty,\delta} = \sup_{|r-r_0|\le\delta}|v(r)|$, we obtain

$$|v'(r)| \le C|v|_{\infty,\delta}\,.$$

for a certain constant $C > 0$. After an integration we arrive at $|v|_{\infty,\delta} \le C\delta|v|_{\infty,\delta}$, and thus $v \equiv 0$ in $|r - r_0| \le \delta$ if δ is small. A continuation argument gives $v \equiv 0$ in $[0, 1]$, which is clearly impossible. Thus $v(r) > 0$ if $r \in [0, 1)$. □

Remark 5.2. Note that the conclussion of Lemma 5.1 cannot be achieved by means of the strong maximum principle, since the operator becomes degenerate for $r = 0$.

Proof of Theorem 1.1. As seen in §3, every positive solution to (1.2) with large λ and maximum close to $\bar u_0$ lies in the ordered interval $[z_\lambda, \bar u_0]$. Since the operator T_λ is increasing, $z_\lambda \le T_\lambda(z_\lambda)$ and $T_\lambda(\bar u_0) \le \bar u_0$, it follows that T_λ leaves the interval $[z_\lambda, \bar u_0]$ invariant.

Furthermore, T_λ is compact, and does not have fixed points in the boundary of the interval. Thus, the Leray-Schauder degree of $I - T_\lambda$ makes sense. We will denote it by $d(I - T_\lambda, (z_\lambda, \bar u_0), 0)$. As usual, the local index of a fixed point u will be denoted by $i(I - T_\lambda, u, 0)$.

Since $(z_\lambda, \bar u_0)$ is convex, we have ([1])

$$d(I - T_\lambda, (z_\lambda, \bar u_0), 0) = 1\,.$$

Let us show that, for large enough λ, every fixed point of T_λ in the interval $[z_\lambda, \bar u_0]$ is isolated, and has index 1. This will conclude the proof of the uniqueness assertion in Theorem 1.1.

Lemma 5.3. *There exists $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$, every fixed point u of T_λ in the interval $[z_\lambda, \bar u_0]$ is isolated, and $i(I - T_\lambda, u, 0) = 1$.*

Proof. Let $u \in (z_\lambda, \bar u_0)$ be a fixed point of T_λ. In virtue of Corollary 4.3, T_λ is differentiable on u. To prove the theorem it will suffice with showing that $\operatorname{spr}(DT_\lambda(u)) < 1$, for large λ (this implies in particular the isolation of u). Then since

$$i(I - T_\lambda, u, 0) = (-1)^\chi\,,$$

where χ stands for the sum of multiplicities of the eigenvalues of $DT_\lambda(u)$ greater than 1 (cf. [1, Theorem 11.4]), the conclusion follows.

Assume on the contrary that there exist sequences $\lambda_n \to +\infty$, $u_n > 0$ in such a way that $\sigma_n = \operatorname{spr}(DT_{\lambda_n}(u_n)) \ge 1$. In virtue of Lemma 5.1, σ_n has an associated eigenfunction $v_n > 0$, which will be normalized by $|v_n|_\infty = 1$. Notice that $f'(u_n) \le 0$ in $[0, 1 - \Lambda\lambda_n^{-1/p}]$, in virtue of Lemma 3.5. Then v_n verifies

$$-(r^{N-1}|u_n'|^{p-2}v_n')' \le \frac{r^{N-1}}{p-1}f'(u_n)v_n \le 0$$

in $[0, 1 - \Lambda\lambda_n^{-1/p}]$. The maximum principle is then applicable to conclude that v_n attains its maximum in $[1 - \Lambda\lambda_n^{-1/p}, 1]$. Choose $r_n \in [1 - \Lambda\lambda_n^{-1/p}, 1]$ such that

$v_n(r_n) = 1 = \max v_n$. We introduce the functions

$$
\begin{gathered}
U_n(x) = u_n(1 - \lambda_n^{-1/p} x) \\
V_n(x) = v_n(1 - \lambda_n^{-1/p} x)\,,
\end{gathered}
$$

if $0 \le x \le \lambda_n^{1/p}$. Since u_n is a solution to (1.2), we have that

$$
U_n(x) = \int_0^x \varphi_{p'}\Big(\int_s^{\lambda_n^{1/p}} \Big(\frac{\lambda_n^{1/p} - \rho}{\lambda_n^{1/p} - s} \Big)^{N-1} f(U_n(\rho))\, d\rho \Big)\, ds\,.
$$

Similarly,

$$
\begin{aligned}
V_n(x) = \frac{1}{p-1} \int_0^x \frac{1}{|U_n'(s)|^{p-2}} \int_s^{\lambda_n^{1/p}} & \Big(\frac{\lambda_n^{1/p} - \rho}{\lambda_n^{1/p} - s} \Big)^{N-1} \\
& \times \Big[\frac{1}{\sigma_n} f'(U_n(\rho)) + m\big(\frac{1}{\sigma_n} - 1\big) \Big] V_n(\rho)\, d\rho\, ds\,.
\end{aligned}
$$

Now note that $U_n' \ne 0$ in $[0, \lambda_n^{1/p})$. Hence $U_n,\ V_n \in C^3[0, \lambda_n^{1/p})$. Moreover, $\{U_n\}$, $\{V_n\}$ are precompact in $C^2[0,T]$ for every $T > 0$. Thus, we can assume $U_n \to \overline{U}$, $V_n \to \overline{V}$ in $C^2_{\rm loc}[0,+\infty)$, where

$$
\begin{gathered}
\overline{U}(x) = \int_0^x \varphi_{p'}\Big(\int_0^{+\infty} f(\overline{U}(\rho))\, d\rho \Big)\, ds, \\
\overline{V}(x) = \frac{1}{p-1} \int_0^x \frac{1}{|\overline{U}'(s)|^{p-2}} \int_s^{+\infty} \Big[\frac{1}{\overline{\sigma}} f'(\overline{U}(\rho)) + m\big(\frac{1}{\overline{\sigma}} - 1\big) \Big] \overline{V}(\rho)\, d\rho\, ds\,,
\end{gathered}
$$

and $\overline{\sigma} = \lim_{n\to\infty} \sigma_n$ ($\overline{\sigma} = +\infty$ is not excluded and then we should set $1/\overline{\sigma} = 0$). Thus, $\overline{U}$, $\overline{V}$ are solutions to the one-dimensional problems

$$
\begin{gathered}
-\varphi_p(\overline{U}')' = f(\overline{U}) \\
\overline{U}(0) = \overline{U}'(+\infty) = 0\,,
\end{gathered}
\tag{5.1}
$$

and

$$
\begin{gathered}
-(|\overline{U}'|^{p-2} \overline{V}')' = \frac{1}{p-1} \Big(\frac{1}{\overline{\sigma}} f'(\overline{U}) + m\big(\frac{1}{\overline{\sigma}} - 1\big) \Big) \overline{V} \\
\overline{V}(0) = \overline{V}'(+\infty) = 0\,,
\end{gathered}
\tag{5.2}
$$

while $0 \le \overline{U} \le \bar u_0$, $0 \le \overline{V} \le 1$. Since the functions V_n attain their maxima in $x_n = \lambda_n^{1/p}(1 - r_n) \le \Lambda$, we obtain that $\overline{V} \not\equiv 0$. Thus $\overline{V} > 0$. On the other hand, notice that $\overline{U}$ verifies $|\overline{U}'|^p = p'(F(\bar u_0) - F(\overline{U}))$, together with $\overline{U}' > 0$ in $[0,+\infty)$. Taking derivatives in (5.1), it follows that $\overline{U}'$ solves the second order equation

$$
-(|\overline{U}'|^{p-2} (\overline{U}')')' = \frac{1}{p-1} f'(\overline{U}) \overline{U}'\,,
$$

and consequently $\overline{U}' \in C^2[0,+\infty)$.

Choose the least $C > 0$ such that $W := C\overline{U}' - \overline{V} \ge 0$ in $[0, \Lambda+1]$. W has to vanish in some point of $[0, \Lambda+1]$. Furthermore, $W \in C^2[0,+\infty)$ and satisfies the equation

$$
-(|\overline{U}'|^{p-2} W')' = \frac{1}{p-1} \Big(f'(\overline{U}) C \overline{U}' - \frac{1}{\overline{\sigma}} f'(\overline{U}) \overline{V} - m\big(\frac{1}{\overline{\sigma}} - 1\big) \overline{V} \Big)\,.
$$

The choice of m implies $f'(\overline{U})+m>0$, and, since $\overline{\sigma}\geq 1$,

$$-(|\overline{U}'|^{p-2}W')'\geq \frac{1}{p-1}f'(\overline{U})W$$

in $[0,\Lambda+1]$. This implies

$$-(|\overline{U}'|^{p-2}W')'+\frac{m}{p-1}W\geq \frac{1}{p-1}(f'(\overline{U})+m)W\geq 0 \tag{5.3}$$

in $[0,\Lambda+1]$. Notice that $\overline{U}'\neq 0$ in $[0,\Lambda+1]$, and the operator in (5.3) becomes nondegenerate. The strong maximum principle gives us that $W>0$ in $(0,\Lambda+1)$. Moreover, $W(0)>0$ and we obtain $W(\Lambda+1)=0$. Hopf's boundary lemma then provides with $W'(\Lambda+1)<0$.

Thus, we can choose $\delta>0$ small so that $W<0$ in $(\Lambda+1,\Lambda+1+\delta)$. We claim that this inequality holds in $(\Lambda+1,+\infty)$. If, on the contrary we have $\delta_0=\sup\{\delta>0 : W<0 \quad \text{in} \quad (\Lambda+1,\Lambda+1+\delta)\}<+\infty$, we obtain

$$\begin{gathered}-(|\overline{U}'|^{p-2}W')'\geq \frac{1}{p-1}f'(\overline{U})W\\ W(\Lambda+1)=W(\Lambda+1+\delta_0)=0\,.\end{gathered}$$

Since $f'(\overline{U})\leq 0$ in $[\Lambda+1,+\infty)$, maximum principle implies $W\geq 0$ in $[\Lambda+1,\Lambda+1+\delta]$ – impossible. Thus $W<0$ in $[\Lambda+1,+\infty)$. This leads us to

$$-(|\overline{U}'|^{p-2}W')'\geq 0$$

in $[\Lambda+1,+\infty)$. Integrating this inequality we arrive at

$$W(x)\leq W'(\Lambda+1)|\overline{U}'(\Lambda+1)|^{p-2}\int_{\Lambda+1}^{x}\frac{ds}{|\overline{U}'(s)|^{p-2}}\,, \tag{5.4}$$

and since $p>2$, we have $\lim_{x\to+\infty}W(x)=-\infty$, contradicting $W\geq -1$. This finishes the proof. □

It only remains to prove estimate (1.5). Let $\lambda_n\to+\infty$ be an arbitrary sequence and define U_n as in Lemma 5.3. As already seen, we can assume $U_n\to\overline{U}$ in $C^2_{\rm loc}[0,+\infty)$. Thus, $U_n'(0)\to\overline{U}'(0)$. The proof is concluded by noticing that $\overline{U}'(0)=(p'F(\bar{u}_0))^{1/p}$ and $U_n'(0)=-\lambda_n^{1/p}u_n'(1)$. □

6. Appendix

In this Appendix we are providing a generalization of Serrin's sweeping principle adequate for our purposes.

Theorem 6.1 (Sweeping principle). *Let $\{u_t\}_{t\in[0,a]}\subset W_0^{1,p}(\Omega)\cap C^{1,\beta}(\overline{\Omega})$ be a family of subsolutions to the problem*

$$\begin{gathered}-\Delta_p u=f(u)\quad \text{in }\Omega\\ u=0\quad \text{on }\partial\Omega\,,\end{gathered} \tag{6.1}$$

where f is a C^1 function and Ω a smooth bounded domain of $\mathbb{R}^N$. Let $u>0$ be a solution to (6.1). Assume that $\{u_t\}$ verifies

(i) *$u_t<0$ on $\partial\Omega$.*
(ii) *The mapping $t\to u_t\in C(\overline{\Omega})$ is continuous.*
(iii) *The set $\{x:\nabla u_t(x)=0\}$ reduces to a single point x_t, and for every t we have $\nabla u(x_t)\neq 0$ or $u(x_t)>u_t(x_t)$.*

(iv) $u \geq u_{t}\Big|_{t=0}$.

Then $u \geq u_{t}\Big|_{t=a}$.

Proof. Consider the set $E = \{t \in [0,a] : u \geq u_t \quad \text{in} \ \ \Omega\}$. Hypotheses (ii) and (iv) imply that E is closed and nonempty. Let us show that it is also open.

Indeed, assume $t_0 \in E$, and define $B_{t_0} := \{x \in \Omega \setminus \{x_{t_0}\} : u(x) = u_{t_0}(x)\}$. The set B_{t_0} is closed with respect to $\Omega \setminus \{x_{t_0}\}$. To prove it is also open, let $x_0 \in B_{t_0}$. Since $u \geq u_{t_0}$, $u(x_0) = u_{t_0}(x_0)$ and $x_0 \neq x_{t_0}$, we obtain $\nabla u(x_0) = \nabla u_{t_0}(x_0) \neq 0$. Thus, choosing $m > 0$ so that $f(u) + mu$ is increasing in a neighbourhood of $u(x_0)$,

$$-\Delta_p u + \Delta_p u_{t_0} + m(u - u_{t_0}) \geq 0 \quad \text{in} \ \ \Omega\,,$$

and since the gradients of u and u_{t_0} do not vanish, we arrive at $L(u - u_{t_0}) \geq 0$, where L is an uniformly elliptic operator in a neighbourhood of x_0 (cf. Appendix in [12]). This implies $u \equiv u_{t_0}$ in that neighbourhood, and B_{t_0} is open.

Since $\Omega \setminus \{x_{t_0}\}$ is connected, we should have $B_{t_0} = \Omega \setminus \{x_{t_0}\}$ or $B_{t_0} = \emptyset$. The first possibility implies $u \equiv u_{t_0}$ in $\overline{\Omega}$, which is impossible since $u_{t_0} < 0$ on $\partial\Omega$. The second leads to $u > u_{t_0}$ in $\Omega \setminus \{x_{t_0}\}$. Hypothesis (iii) then gives $u > u_{t_0}$ in Ω, and then $u > u_t$ in Ω for $t \sim t_0$, that is, E is open.

Finally, the connectedness of $[0,a]$ implies $E = [0,a]$, and $u \geq u_{t}\Big|_{t=a}$ follows. This proves the sweeping principle. $\square$

Acknowledgements. The author would like to thank Professor José Sabina de Lis for his careful reading of this paper and also for some very helpful comments.

References

[1] Amann H., *Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces*, SIAM Review, **18** (1976), 620-709.

[2] Angenent S. B., *Uniqueness of the solution of a semilinear boundary value problem*, Math. Ann. **272** (1985), 129-138.

[3] Badiale M., Nabana E., *A note on radiality of solutions of p-Laplacian equation*, Appl. Anal. **52** (1994), 35-43.

[4] Brock F., *Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian*, in Progress in Partial Differential Equations. Pont-à-Mousson 1997, vol. 1. Eds. H. Amman et al. Pitmann Research Notes in Math. **383**, 46-58.

[5] Clément P., Sweers G., *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 97-121.

[6] García Melián J., Sabina de Lis J., *Stationary Profiles of Degenerate Problems when a Parameter is Large*, Differential Integral Equations **13** (10-12) (2000), 1201-1232.

[7] García Melián J., Sabina de Lis J., *Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains,* Nonlinear Anal. T.M.A. **43** (7) (2001), 803-835.

[8] García Melián J., Sabina de Lis J., *A local bifurcation theorem for degenerate equations with radial symmetry,* J. Diff. Eqns. **179** (1) (2002), 27-43.

[9] García Melián J., Sabina de Lis J., Multiplicity of solutions to a degenerate diffusion problem, Nonlinear Anal. T.M.A. **53** (7-8) (2003), 1063-1077.

[10] Gidas B., Ni W. M., Nirenberg L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[11] Kesavan S., Pacella F., *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities*, Appl. Anal. **54** (1994), 27-37.

[12] Sakaguchi S., *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) **14** (1987), 403-421.

[13] Smoller J., Wasserman A., *An existence theorem for positive solutions of semilinear elliptic equations*, Arch. Rational Mech. Anal. **95** (1986), 211-216.

[14] Smoller J., Wasserman A., *On the monotonicity of the Time-Map*, J. Differential Equations **77** (1989), 287-303.
[15] Sweers G., *On the maximum of solutions for a semilinear elliptic equation*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 357-370.
[16] Tolksdorf P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), 773-817.
[17] Yosida K., *Functional Analysis*, Springer Verlag Berlin, 1965.

Jorge García-Melián
Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna - Tenerife, Spain
E-mail address: jjgarmel@ull.es