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INTERNAL EXACT CONTROLLABILITY OF THE LINEAR POPULATION DYNAMICS WITH DIFFUSION

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Abstract. We consider the internal exact controllability of a linear age and space structured population model with nonlocal birth process. The control acts only in a spatial subdomain and only for small age classes. The methods we use combine the Carleman estimates for the backward adjoint system, some estimates in the theory of parabolic boundary value problems in L^k and the Banach fixed point theorem.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \leq 3)$ with a smooth boundary $\partial\Omega$. Assume that a biological population is free to move in the environment Ω. We denote by $y(a,t,x)$ the density of individuals of age $a \geq 0$ at time $t \geq 0$ and location $x \in \overline{\Omega}$ and assume that the flux of population takes the form $k\nabla y(a,t,x)$ with $k > 0$, where ∇ is the gradient vector with respect to the spatial variable. Let A be the life expectancy of an individual and T be a positive constant. Let $\beta(a)$ be the natural fertility rate and $\mu(a)$ the natural mortality rate corresponding to individuals of age a. The dynamics of the population is described by the following model

$$
\begin{gathered}
Dy + \mu(a)y - k\Delta y = f(a,x) + m(a,x)u(a,t,x), \quad (a,t,x) \in Q_T \\
\frac{\partial y}{\partial \nu}(a,t,x) = 0, \quad (a,t,x) \in \Sigma_T \\
y(0,t,x) = \int_0^A \beta(a)y(a,t,x)da, \quad (t,x) \in (0,T) \times \Omega \\
y(a,0,x) = y_0(a,x), \quad (a,x) \in (0,A) \times \Omega,
\end{gathered} \tag{1.1}
$$

where u is the control and m is the characteristic function of $(0,a^*) \times \omega$, f is the density of an infusion of population and y_0 is the initial population density. Here $a^* \in (0,A]$ and $\omega \subset\subset \Omega$ is a nonempty open subset, $Q_T = (0,A) \times (0,T) \times \Omega$, $\Sigma_T = (0,A) \times (0,T) \times \partial\Omega$.

We denote by

$$
Dy(a,t,x) = \lim_{\varepsilon \to 0} \frac{y(a+\varepsilon, t+\varepsilon, x) - y(a,t,x)}{\varepsilon}
$$

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the directional derivative of y with respect to the direction $(1,1,0)$. If y is smooth enough then

$$Dy = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a}.$$

The control acts only in the spatial set ω and for ages between 0 and a^*.

Let y_s be a nonnegative steady-state of (1.1), corresponding to $u \equiv 0$ and such that

$$y_s(a,x) \geq \rho_0 > 0 \quad \text{a.e. } (a,x) \in (0,a_1^*) \times \Omega, \tag{1.2}$$

where $\rho_0 > 0$ is constant and $a_1^* \in (0,A)$ is a constant which will be defined later.

The main goal of this paper is to prove the existence of a control u such that the solution y of (1.1) satisfies

$$\begin{gathered} y(a,T,x) = y_s(a,x) \quad \text{a.e. } (a,x) \in (0,A) \times \Omega, \\ y(a,t,x) \geq 0 \quad \text{a.e. } (a,t,x) \in Q_T. \end{gathered} \tag{1.3}$$

Condition (1.3) is natural because y represents the density of a population. We notice that if y is the solution to (1.1), then $y - y_s$ is the solution to

$$\begin{gathered} Dz + \mu(a)z - k\Delta z = m(a,x)u(a,t,x), \quad (a,t,x) \in Q_T \\ \frac{\partial z}{\partial \nu}(a,t,x) = 0, \quad (a,t,x) \in \Sigma_T \\ z(0,t,x) = \int_0^A \beta(a)z(a,t,x)da, \quad (t,x) \in (0,T) \times \Omega \\ z(a,0,x) = z_0(a,x), (a,x) \in (0,A) \times \Omega, \end{gathered} \tag{1.4}$$

where $z_0 = y_0 - y_s$.

The above formulated problem is equivalent to the exact null controllability problem with state constraints for (1.4). Indeed, if we denote now by z the solution to (1.4), then condition (1.3) becomes

$$z(a,t,x) \geq -y_s(a,x) \quad \text{a.e. } (a,t,x) \in Q_T.$$

We recall that the internal null controllability of the linear heat equation, when the control acts on a subset of the domain, was established by G. Lebeau and L. Robbiano [13] and was later extended to some semilinear equation by A.V. Fursikov and O.Yu. Imanuvilov [6], in the sublinear case and by V. Barbu [4] and E. Fernandez–Cara [5], in the superlinear case. The internal null controllability of the age-dependent population dynamics in the particular case when the control acts in a spatial subdomain ω but for all ages a (this is the particular case corresponding to $a^* = A$) was investigated by B. Ainseba and S. Aniţa [2].

This paper is organized as follows. We first give the hypotheses and state the main result. The existence of a steady–state of (1.1) with $u \equiv 0$ is established in Section 3. The proof of the local exact null controllability is given in Section 4. The proof is based on Carleman's inequality for the backward adjoint system associated with (1.4).

2. Assumptions and the main result

Assume that the following hypotheses hold:

(H1) $\beta \in L^\infty(0,A)$, $\beta(a) \geq 0$ a.e. $a \in (0,A)$
There exists $a_0, a_1 \in (0,A)$, $a_0 < a_1$, such that $\beta(a) = 0$ a.e. $a \in (0,a_0) \cup (a_1,A)$ and $\beta(a) > 0$ a.e. in (a_0,a_1)

(H2) $\mu \in C([0, A))$, $\mu(a) \geq 0$ a.e. $a \in (0, A)$, $\int_0^A \mu(a)da = +\infty$
(H3) $y_0 \in L^\infty((0, A) \times \Omega)$, $y_0(a, x) \geq 0$ a.e. in $(0, A) \times \Omega$
$f \in L^\infty((0, A) \times \Omega)$, $f(a, x) \geq 0$ a.e. in $(0, A) \times \Omega$.

For the biological significance of the hypotheses and the basic existence results for the solution to (1.1) we refer to [3, 7, 8, 9, 11, 15].

Let y_s be a nonnegative steady-state of (1.1), corresponding to $u \equiv 0$ and such that

$$y_s(a, x) \geq \rho_0 > 0 \quad \text{a.e. } (a, x) \in (0, a_1) \times \Omega,$$

where $\rho_0 > 0$ is a constant.

Denote by $z_0 = y_0 - y_s$. Then we have the following internal controllability result

Theorem 2.1. *Let $T > A - a^*$ be arbitrary but fixed. If $\|y_0 - y_s\|_{L^\infty((0,A)\times\Omega)}$ is small enough, then there exists $u \in L^2(Q_T)$ such that the solution y of* (1.1) *satisfies*

$$\begin{gathered} y(a, T, x) = y_s(a, x) \quad \text{a.e. } (a, x) \in (0, A) \times \Omega \\ y(a, t, x) \geq 0 \quad \text{a.e. } (a, t, x) \in Q_T. \end{gathered} \tag{2.1}$$

If $T < A - a^*$ and if $\|y_0 - y_s\|_{L^\infty((a^*,A-T)\times\Omega)} > 0$, then there is no $u \in L^2(Q_T)$ such that the solution y of (1.1) *to satisfy* (2.1).

This result can be equivalently formulated as follows

Theorem 2.2. *Let $T > A - a^*$ be arbitrary but fixed. If $\|z_0\|_{L^\infty((0,A)\times\Omega)}$ is small enough, then there exists $u \in L^2(Q_T)$ such that the solution z of* (1.4) *satisfies*

$$\begin{gathered} z(a, T, x) = 0 \quad \text{a.e. } (a, x) \in (0, A) \times \Omega \\ z(a, t, x) \geq -y_s(a, x) \quad \text{a.e. } (a, t, x) \in Q_T. \end{gathered} \tag{2.2}$$

If $T < A - a^*$ and if $\|z_0\|_{L^\infty((a^*,A-T)\times\Omega)} > 0$, then there is no $u \in L^2(Q_T)$ such that the solution z of (1.4) *to satisfy* (2.2).

3. Existence of steady states for (1.1)

In this section we shall remind some results (see [2]) concerning the existence of y_s, a nonnegative steady-state of (1.1), corresponding to $u \equiv 0$, which satisfies (1.2). y_s should be a solution to

$$\begin{gathered} \frac{\partial y_s}{\partial a} + \mu(a)y_s - k\Delta y_s = f(a, x), \quad (a, x) \in (0, A) \times \Omega \\ \frac{\partial y_s}{\partial \nu}(a, x) = 0, \quad (a, x) \in (0, A) \times \partial\Omega \\ y_s(0, x) = \int_0^A \beta(a)y_s(a, x)da, \quad x \in \Omega\,. \end{gathered} \tag{3.1}$$

Denote by

$$R = \int_0^A \beta(a) \exp\big(-\int_0^a \mu(s)ds da\big)$$

the reproductive number and consider f_0 a nonnegative constant.

Theorem 3.1.
- *If $R < 1$ and $f(a, x) \geq f_0 > 0$ a.e. $(a, x) \in (0, A) \times \Omega$, then there exists a unique nonnegative solution to* (3.1)*, which in addition satisfies* (1.2).

- *If $R = 1$ and $f \equiv 0$, then there exist infinitely many nonnegative solutions to* (3.1)*, which satisfy* (1.2).
- *If $R > 1$, then there is no nonnegative solution to* (3.1)*, satisfying* (1.2).

Proof. If $R < 1$, then there exists a unique and nonnegative solution to (3.1) (this follows by Banach's fixed point theorem). Since $f(a, x) \geq f_0 > 0$ a.e. $(a, x) \in (0, A) \times \Omega$, then by the comparison result in [7](see also [3]) we get that

$$y_s(a, x) \geq y_i(a, t, x) \quad \text{a.e. } (a, t, x) \in Q = (0, A) \times (0, +\infty) \times \Omega,$$

where y_i is the solution to

$$\begin{gathered} Dy_i + \mu y_i - k\Delta y_i = f_0, \quad (a, t, x) \in Q \\ \frac{\partial y_i}{\partial \nu} = 0, \quad (a, t, x) \in \Sigma \\ y_i(0, t, x) = \int_0^A \beta(a) y_i(a, t, x) da, \quad (t, x) \in (0, +\infty) \times \Omega \\ y_i(a, 0, x) = 0, \quad (a, x) \in (0, A) \times \Omega \end{gathered}$$

Note that $\Sigma = (0, A) \times (0, +\infty) \times \partial\Omega)$; y_i does not explicitly depend on x. So, we shall write $y_i(a, t)$ instead of $y_i(a, t, x)$. It means that

$$y_s(a, x) \geq y_i(a, t) \quad \forall t \in [0, +\infty), \quad \text{a.e.}(a, x) \in (0, A) \times \Omega,$$

and that y_i is the solution of

$$\begin{gathered} Dy_i + \mu y_i = f_0, \quad (a, t) \in (0, A) \times (0, +\infty) \\ y_i(0, t) = \int_0^A \beta(a) y_i(a, t) da, \quad t \in (0, +\infty) \\ y_i(a, 0) = 0, \quad a \in (0, A). \end{gathered}$$

For $t > A$ we have $y_i(0, t) > 0$ and $y_i(0, t)$ is continuous with respect to t (see [3]). As a consequence we obtain that there exists $\rho_0 > 0$ such that, for t large enough, and for any $a \in (0, a_1^*)$,

$$y_i(a, t) > \rho_0,$$

and in conclusion we get that y_s satisfies (1.2).

If $R = 1$ and $f \equiv 0$, then all the solutions of (3.1) which are satisfying (1.2) are given by

$$y(a, x) = ce^{-\int_0^a \mu(s) ds}, \quad (a, x) \in (0, A) \times \Omega,$$

where $c \in \mathbb{R}_+^*$ is an arbitrary constant. The conclusion is now obvious.

If $R > 1$ and if it would exist a nonnegative solution y_s to (3.1) satisfying (1.2), then $y(a, t, x) = y_s(a, x)$, $(a, t, x) \in \overline{Q}$ is the solution to

$$\begin{gathered} Dy + \mu y - k\Delta y = f(a, x), \quad (a, t, x) \in Q \\ \frac{\partial y}{\partial \nu} = 0, \quad (a, t, x) \in \Sigma \\ y(0, t, x) = \int_0^A \beta(a) y(a, t, x), \quad (t, x) \in (0, +\infty) \times \Omega \\ y(a, 0, x) = y_s(a, x), \quad (a, x) \in (0, A) \times \Omega \end{gathered}$$

and for $t \to +\infty$ we have (see [3, 12])

$$\lim_{t \to +\infty} \|y(t)\|_{L^2((0,A) \times \Omega)} = +\infty.$$

On the other hand
$$\|y(t)\|_{L^2((0,A)\times\Omega)} = \|y_s\|_{L^2((0,A)\times\Omega)},$$
and so $\|y_s\|_{L^2((0,A)\times\Omega)} = +\infty$, which is absurd. $\square$

4. Proof of the main result

We shall prove Theorem 2.2 (which is equivalent to Theorem 2.1). We intend to use the general Carleman inequality for linear parabolic equations given in [6]. Namely, let $\widetilde{\omega} \subset\subset \omega$ be a nonempty bounded set, $T_0 \in (0,+\infty)$ and $\psi \in C^2(\overline{\Omega})$ be such that
$$\psi(x) > 0,\ \forall x \in \Omega, \quad \psi(x) = 0,\ \forall x \in \partial\Omega, \quad |\nabla\psi(x)| > 0,\ \forall x \in \overline{\Omega} \setminus \widetilde{\omega}$$
and set
$$\alpha(t,x) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T_0-t)},$$
where λ is an appropriate positive constant. Denote by $D_{T_0} = (0,T_0)\times\Omega$.

Lemma 4.1. *There exist positive constants C_1, s_1 such that*
$$\begin{aligned}
&\frac{1}{s}\int_{D_{T_0}} t\,(T_0-t)\,e^{2s\alpha}\left(|w_t|^2 + |\Delta w|^2\right)dx\,dt \\
&+ s\int_{D_{T_0}} \frac{e^{2s\alpha}}{t\,(T_0-t)}\,|\nabla w|^2\,dx\,dt + s^3\int_{D_{T_0}} \frac{e^{2s\alpha}}{t^3\,(T_0-t)^3}\,|w|^2\,dx\,dt \\
&\le C_1\Big[\int_{D_{T_0}} e^{2s\alpha}\,|w_t+\Delta w|^2\,dx\,dt + s^3\int_{(0,T_0)\times\omega} \frac{e^{2s\alpha}}{t^3\,(T_0-t)^3}\,|w|^2\,dx\,dt\Big],
\end{aligned} \tag{4.1}$$
for all $w \in C^2(\overline{D}_{T_0})$, $\frac{\partial w}{\partial\nu}(t,x) = 0$, $\forall (t,x) \in (0,T_0)\times\partial\Omega$ and $s \ge s_1$.

The proof of this result can be found in [6].

If $a^* = A$, the result has already been proved in [2]. We shall treat now the case $a^* \in (0,A)$. Consider $a_1^* := a^*$. Let us choose $T_0 \in (0, \min\{a_0, a^*, A-a^*, T-A+a^*, A-a_1\})$. Define
$$K = L^\infty\left((0, A-a^*+T_0)\times\Omega\right).$$
In what follows we shall denote by the same symbol C, several constants independent of z_0 and all other variables. For $b \in K$ arbitrary but fixed and for any $\varepsilon > 0$, consider the following optimal control problem:
Minimize
$$\Big\{\int_G\int_\Omega \varphi(a,t,x)|u(a,t,x)|^2 dx\,dt\,da + \frac{1}{\varepsilon}\int_{\Gamma_0}\int_\Omega |z(a,t,x)|^2\,dx\,dl\Big\}, \tag{4.2}$$
subject to (4.3) ($u \in L^2(G\times\Omega)$ and z is the solution of (4.3) corresponding to u). Here
$$\begin{gathered}
G = (0,a^*)\times(0,T_0) \cup (0,T_0)\times(0,A-a^*+T_0), \\
\Gamma_0 = \{T_0\}\times(T_0, A-a^*+T_0) \cup (T_0,a^*)\times\{T_0\}, \\
\varphi(a,t,x) = \begin{cases} e^{-2s\alpha(t,x)}t^3(T_0-t)^3, & \text{if } t<a,\ (a,t)\in G \\ e^{-2s\alpha(a,x)}a^3(T_0-a)^3, & \text{if } a<t,\ (a,t)\in G \end{cases}
\end{gathered}$$

(See figure 1).

$$\begin{gathered} Dz + \mu z - k\Delta z = m(a,x)u(a,t,x), \quad (a,t,x) \in G \times \Omega \\ \frac{\partial z}{\partial \nu} = 0, \quad (a,t,x) \in G \times \partial\Omega \\ z(0,t,x) = b(t,x), \quad (t,x) \in (0, A - a^* + T_0) \times \Omega \\ z(a,0,x) = z_0(a,x), \quad (a,x) \in (0,a^*) \times \Omega. \end{gathered} \tag{4.3}$$

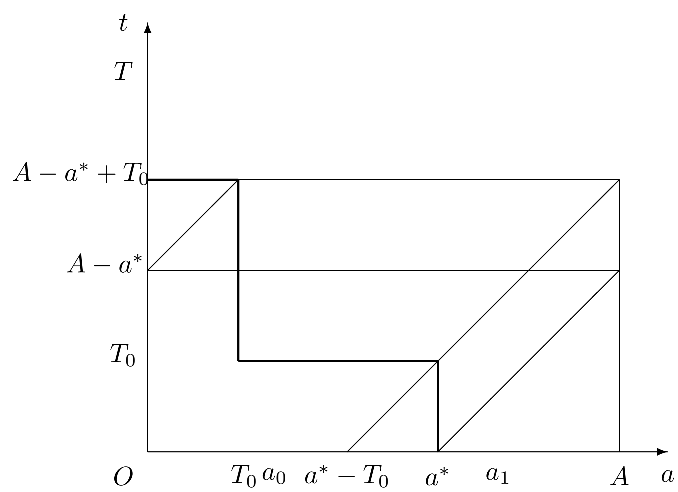

FIGURE 1.

Denote by $\Psi_\varepsilon(u)$ the value of the cost function in u. Since the cost function $\Psi_\varepsilon : L^2(G \times \Omega) \to \mathbb{R}^+$ is convex, continuous and

$$\lim_{\|u\|_{L^2(G\times\Omega)} \to +\infty} \Psi_\varepsilon(u) = +\infty,$$

then it follows that there exists at least one minimum point for Ψ_ε and consequently an optimal pair $(u_\varepsilon, z_\varepsilon)$ for $(\mathbf{P}_\varepsilon)$. By standard arguments we have

$$u_\varepsilon(a,t,x) = \tilde{m}(x) q_\varepsilon(a,t,x)\varphi^{-1}(a,t,x) \text{ a.e. } (a,t,x) \in G \times \Omega, \tag{4.4}$$

where $\tilde{m}$ is the characteristic function of ω and q_ε is the solution of

$$\begin{gathered} Dq - \mu q + k\Delta q = 0, \quad (a,t,x) \in G \times \Omega \\ \frac{\partial q}{\partial \nu} = 0, \quad (a,t,x) \in G \times \partial\Omega \\ q(a,t,x) = 0, \quad (a,t,x) \in (\Gamma \setminus \Gamma_0) \times \Omega \\ q(a,t,x) = -\frac{1}{\varepsilon} z_\varepsilon(a,t,x), \quad (a,t,x) \in \Gamma_0 \times \Omega. \end{gathered} \tag{4.5}$$

Here $\Gamma = (0,T_0) \times \{A - a^* + T_0\} \cup \{a^*\} \times (0,T_0) \cup \Gamma_0$.

Multiplying the first equation in (4.5) by z_ε and integrating on $G\times\Omega$ we obtain after some calculation (and using (4.3) and (4.4)) that

$$\begin{aligned}&\int_G\int_\omega \varphi(a,t,x)|u_\varepsilon(a,t,x)|^2dx\,da\,dt+\frac{1}{\varepsilon}\int_{\Gamma_0}\int_\Omega |z_\varepsilon(a,t,x)|^2dx\,dl\\&=-\int_0^{A-a^*+T_0}\int_\Omega b(t,x)q_\varepsilon(0,t,x)dx\,dt-\int_0^{a^*}\int_\Omega z_0(a,x)q_\varepsilon(a,0,x)dx\,da.\end{aligned}$$

Let S be an arbitrary characteristic line of equation

$$S=\{(\gamma+t,\theta+t);\ t\in(0,T_0),\ (\gamma,\theta)\in(0,a^*-T_0)\times\{0\}\cup\{0\}\times(0,A-a^*)\}\,.$$

Define

$$\begin{gathered}\widetilde{u}(t,x)=u(\gamma+t,\theta+t,x),\quad (t,x)\in(0,T_0)\times\Omega\\ \widetilde{z}_\varepsilon(t,x)=z_\varepsilon(\gamma+t,\theta+t,x),\quad (t,x)\in(0,T_0)\times\Omega\\ \widetilde{q}_\varepsilon(t,x)=q_\varepsilon(\gamma+t,\theta+t,x),\quad (t,x)\in(0,T_0)\times\Omega\\ \widetilde{\mu}(t)=\mu(\gamma+t),\quad t\in(0,T_0).\end{gathered}$$

Note that $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)$ satisfies

$$\begin{gathered}(\widetilde{z}_\varepsilon)_t+\widetilde{\mu}\widetilde{z}_\varepsilon-k\Delta\widetilde{z}_\varepsilon=\tilde{m}(x)\widetilde{u}_\varepsilon(t,x),\quad (t,x)\in(0,T_0)\times\Omega\\ \frac{\partial\widetilde{z}_\varepsilon}{\partial\nu}=0,\quad (t,x)\in(0,T_0)\times\partial\Omega\\ \widetilde{z}_\varepsilon(0,x)=\begin{cases}b(\theta,x) & \gamma=0,\ x\in\Omega\\ z_0(\gamma,x) & \theta=0,\ x\in\Omega\end{cases}\end{gathered}\tag{4.6}$$

By (4.4) we get that

$$\widetilde{u}_\varepsilon(t,x)=\tilde{m}(x)\widetilde{q}_\varepsilon(t,x)\cdot\frac{e^{2s\alpha(t,x)}}{t^3(T_0-t)^3}\tag{4.7}$$

a.e. $(t,x)\in(0,T_0)\times\Omega$,

$$\begin{gathered}(\widetilde{q}_\varepsilon)_t+k\Delta\widetilde{q}_\varepsilon=\widetilde{\mu}\widetilde{q}_\varepsilon,\quad (t,x)\in(0,T_0)\times\Omega\\ \frac{\partial\widetilde{q}_\varepsilon}{\partial\nu}=0,\quad (t,x)\in(0,T_0)\times\partial\Omega\\ \widetilde{q}_\varepsilon(T_0,x)=-\frac{1}{\varepsilon}\widetilde{z}_\varepsilon(T_0,x)\quad x\in\Omega.\end{gathered}\tag{4.8}$$

Multiplying the first equation in (4.8) by $\widetilde{z}_\varepsilon$ and integrating on D_{T_0}, we obtain that

$$\begin{aligned}&\int_0^{T_0}\int_\omega e^{-2s\alpha(t,x)}t^3(T_0-t)^3|\widetilde{u}_\varepsilon(t,x)|^2dx\,dt+\frac{1}{\varepsilon}\int_\Omega|\widetilde{z}_\varepsilon(T_0,x)|^2dx\\&=-\int_\Omega\widetilde{z}_\varepsilon(0,x)\widetilde{q}_\varepsilon(0,x)dx.\end{aligned}\tag{4.9}$$

By Carleman's inequality (4.1) we infer that

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega e^{2s\alpha}[\frac{t(T_0-t)}{s}\Big(|(\widetilde{q}_\varepsilon)_t|^2+|\Delta\widetilde{q}_\varepsilon|^2\Big)+\frac{s}{t(T_0-t)}|\nabla\widetilde{q}_\varepsilon|^2\\
&+\frac{s^3}{t^3(T_0-t)^3}|\widetilde{q}_\varepsilon|^2]dx\,dt\\
&\le C_1\Big[\int_0^{T_0}\int_\Omega e^{2s\alpha}\|\widetilde{\mu}\|^2_{C([0,T_0])}\cdot|\widetilde{q}_\varepsilon|^2\,dx\,dt+s^3\int_{(0,T_0)\times\omega}\frac{e^{2s\alpha}}{t^3(T_0-t)^3}|\widetilde{q}_\varepsilon|^2\,dx\,dt\Big]
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega e^{2s\alpha}[\frac{t(T_0-t)}{s}\left(|(\widetilde{q}_\varepsilon)_t|^2+|\Delta\widetilde{q}_\varepsilon|^2\right)+\frac{s}{t(T_0-t)}|\nabla\widetilde{q}_\varepsilon|^2\\
&+\frac{s^3}{t^3(T_0-t)^3}|\widetilde{q}_\varepsilon|^2]dx\,dt\\
&\le C\int_0^{T_0}\int_\omega e^{2s\alpha}\frac{s^3}{t^3(T_0-t)^3}|\widetilde{q}_\varepsilon|^2\,dx\,dt,
\end{aligned}\tag{4.10}$$

for $s\ge\max(s_1,\,C\|\mu\|^{\frac{2}{3}}_{C([0,a^*])})$. Multiplying the first equation in (4.8) by $\widetilde{q}_\varepsilon$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\widetilde{q}_\varepsilon(t,x)|^2\,dx-k\int_\Omega|\nabla\widetilde{q}_\varepsilon(t,x)|^2\,dx-\int_\Omega\widetilde{\mu}(t)\,|\widetilde{q}_\varepsilon(t,x)|^2\,dx=0$$

and

$$\frac{d}{dt}\int_\Omega|\widetilde{q}_\varepsilon(t,x)|^2\,dx\ge 0\quad\text{a.e. } t\in(0,T_0).$$

Integrating the last inequality we get that

$$\int_\Omega|\widetilde{q}_\varepsilon(0,x)|^2\,dx\le C\int_0^{T_0}\int_\Omega|\widetilde{q}_\varepsilon(t,x)|^2\,\frac{e^{2s\alpha(x,t)}}{t^3\,(T_0-t)^3}dx.$$

and by Carleman's inequality we have that

$$\int_\Omega|\widetilde{q}_\varepsilon(0,x)|^2\,dx\le C\int_0^{T_0}\int_\omega|\widetilde{q}_\varepsilon(t,x)|^2\cdot\frac{e^{2s\alpha(x,t)}}{t^3(T_0-t)^3}dx\,dt.\tag{4.11}$$

By Young's inequality, (4.9), (4.11) and (4.7) we obtain that

$$\begin{aligned}
&\int_{(0,T_0)\times\omega}e^{-2s\alpha}t^3(T_0-t)^3\,|\widetilde{u}_\varepsilon(t,x)|^2\,dx\,dt+\frac{1}{\varepsilon}\int_\Omega|\widetilde{z}_\varepsilon(T_0,x)|^2\,dx\\
&\le C\|\widetilde{z}_\varepsilon(0)\|^2_{L^2(\Omega)},
\end{aligned}$$

for $s\ge\max(s_1,\,C\|\mu\|^{\frac{2}{3}}_{C([0,a^*])})$. Using now (4.10) we get

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega e^{2s\alpha}[\frac{t(T_0-t)}{s}\left(|(\widetilde{q}_\varepsilon)_t|^2+|\Delta\widetilde{q}_\varepsilon|^2\right)\\
&+\frac{s}{t(T_0-t)}|\nabla\widetilde{q}_\varepsilon|^2+\frac{s^3}{t^3(T_0-t)^3}|\widetilde{q}_\varepsilon|^2]dx\,dt\le C\|\widetilde{z}_\varepsilon(0)\|^2_{L^2(\Omega)},
\end{aligned}$$

for any $\varepsilon>0$ and consequently

$$\|\widetilde{v}_\varepsilon\|^2_{W_2^{1,2}((0,T_0)\times\Omega)}\le C\|\widetilde{z}_\varepsilon(0)\|^2_{L^2(\Omega)},$$

where $\widetilde{v}_\varepsilon(t,x) = \frac{e^{2s\alpha(t,x)}}{t^3(T_0-t)^3}\widetilde{q}_\varepsilon$, $(t,x)\in(0,T_0)\times\Omega$. As

$$W_2^{1,2}\left((0,T_0)\times\Omega\right)\subset L^l((0,T_0)\times\Omega)$$

(where $l=+\infty$ for $N=1,2$ and $l=10$ for $N=3$), we may infer that

$$\|\widetilde{u}_\varepsilon\|^2_{L^{10}((0,T_0)\times\Omega)} = \|m\widetilde{v}_\varepsilon\|^2_{L^{10}((0,T_0)\times\Omega)} \le C\|\widetilde{z}_\varepsilon(0)\|^2_{L^2(\Omega)}, \tag{4.12}$$

for any $\varepsilon>0$ and $s\ge\max(s_1,\, C\|\mu\|^{\frac{2}{3}}_{C([0,a^*])})$.

The last estimate and the existence theory of parabolic boundary value problems in L^r (see [10]) imply that on a subsequence (also denoted by $(\tilde{u}_\varepsilon)$) we have that

$$\widetilde{u}_\varepsilon\to\widetilde{u}\quad\text{weakly in } L^{10}\left((0,T_0)\times\Omega\right)$$
$$\widetilde{z}_\varepsilon\to\widetilde{z}^{\widetilde{u}}\quad\text{weakly in } W_{10}^{1,2}\left((0,T_0)\times\Omega\right),$$

where $\left(\widetilde{u},\widetilde{z}^{\widetilde{u}}\right)$ satisfies (4.6) and

$$\widetilde{z}^{\widetilde{u}}(T_0,x)=0\quad\text{a.e. } x\in\Omega.$$

By (4.6) we get that

$$\|\widetilde{z}^{\widetilde{u}}\|^2_{L^\infty\left((0,T_0)\times\Omega\right)} \le C\left(\|\widetilde{z}^{\widetilde{u}}(0)\|^2_{L^\infty(\Omega)} + \|m\widetilde{u}\|^2_{L^3((0,T_0)\times\Omega)}\right)$$

(we recall that $W_3^{1,2}\left((0,T_0)\times\Omega\right)\subset L^\infty\left((0,T_0)\times\Omega\right)$ for $N\in\{1,2,3\}$; see [1, 10]). So by (4.12) we have

$$\|\widetilde{z}^{\widetilde{u}}\|^2_{L^\infty((0,T_0)\times\Omega)} \le C\|\widetilde{z}^{\widetilde{u}}(0)\|^2_{L^\infty(\Omega)}.$$

We extend u given by $\widetilde{u}$ (on each characteristic line) by 0. In this manner we get that $u\in L^2(Q_T)$.

Let z^u be the solution to

$$\begin{gathered}
Dz+\mu z-k\Delta z = m(a,x)u(a,t,x),\quad (a,t,x)\in(0,A)\times(0,A-a^*+T_0)\times\Omega\\
\frac{\partial z}{\partial\nu}=0,\quad (a,t,x)\in(0,A)\times(0,A-a^*+T_0)\times\partial\Omega\\
z(0,t,x)=b(t,x),\quad (t,x)\in(0,A-a^*+T_0)\times\Omega\\
z(a,0,x)=z_0(a,x),\quad (a,x)\in(0,A)\times\Omega.
\end{gathered}$$

Since $z^u=0$ on $\Gamma_0\times\Omega$ and $u=0$ outside $G\times\Omega$ we conclude that $z^u(a,t,x)=0$ a.e. in $\{(a,t,x); t\in(T_0,A-a^*+T_0), T_0<a<t+a^*-T_0,\ x\in\Omega\}$, $z^u(a,A-a^*+T_0,x)=0$ a.e. $(a,x)\in(T_0,A)\times\Omega$ and that

$$\|z^u\|_{L^\infty(Q_{A-a^*+T_0})} \le C(\|z_0\|_{L^\infty((0,A)\times\Omega)} + \|b\|_{L^\infty((0,A-a^*+T_0)\times\Omega)}). \tag{4.13}$$

We are now ready to prove the exact null controllability result. For any $b\in K$, we denote by $\Phi(b)\subset L^2((0,A-a^*+T_0)\times\Omega)$ the set of all $\int_0^A\beta(a)z^u(a,t,x)da$, such that $u\in L^2(Q_{A-a^*+T_0})$, $u=0$ outside $G\times\Omega$, where z^u satisfies (4.13) and

$$z^u(a,t,x)=0$$

a.e. in $\{(a,t,x); t\in(T_0,A-a^*+T_0),\ T_0<a<t+a^*-T_0,\ x\in\Omega\}$,

$$z^u(a,A-a^*+T_0,x)=0,\quad\text{a.e. } (a,x)\in(T_0,A)\times\Omega.$$

There exists an element in $\Phi(b)$ which does not depend on b:
If $t>T_0$, then $\int_0^A\beta(a)z^u(a,t,x)da=\int_t^A\beta(a)z^u(a,t,x)da$ and does not depend on b.

If $t \in (0, T_0)$, then $\int_0^A \beta(a) z^u(a,t,x)da = \int_{T_0}^{A-T_0} \beta(a) z^u(a,t,x)da$, and this depends only on z_0 and not on b.

We also have that $z^u(a, A - a^* + T_0, x) = 0$ a.e. $(a,x) \in (T_0, A) \times \Omega$ and

$$\Big| \int_{T_0}^{A-T_0} \beta(a) z^u(a,t,x)da \Big| \le C \|\beta\|_{L^\infty(0,A)} \cdot \|z_0\|_{L^\infty((0,A)\times\Omega)} \tag{4.14}$$

a.e. in $(0, A - a^* + T_0) \times \Omega$. It also follows that

$$\int_0^A \beta(a) z^u(a,t,x)da = \int_0^{T_0} \beta(a) z^u(a,t,x)da + \int_{A-T_0}^A \beta(a) z^u(a,t,x)da = 0$$

a.e. $(t,x) \in (A - a^*, A - a^* + T_0)$ (because $\beta(a) = 0$ on $(0, T_0) \cup (A - T_0, A)$). So, for any u as above we can take

$$b(t,x) = \begin{cases} 0 & \text{a.e. } (t,x) \in (A - a^*, A - a^* + T_0) \times \Omega \\ \int_0^A \beta(a) z^u(a,t,x)da & \text{a.e. } (t,x) \in (0, A - a^*) \times \Omega \end{cases}$$

a fixed point of the multivalued function Φ. In addition, by (4.13) and (4.14) we have

$$\|z^u\|_{L^\infty(Q_{A-a^*+T_0})} \le C \|z_0\|_{L^\infty((0,A)\times\Omega)}.$$

So, if $\|z_0\|_{L^\infty((0,A)\times\Omega)}$ is small enough, there exists $u \in L^2(Q_{A-a^*+T_0})$, $u = 0$ on $(a^*, A) \times (A - a^*, A - a^* + T_0) \times \Omega$, such that z, the solution of (1.4) (with $T := A - a^* + T_0$) satisfies

$$z(a, A - a^* + T_0, x) = 0 \quad \text{a.e. } (a,x) \in (0, A) \times \Omega,$$
$$\|z\|_{L^\infty(Q_{A-a^*+T_0})} \le C \|z_0\|_{L^\infty((0,A)\times\Omega)} \le \rho_0 \ .$$

In conclusion $z(a,t,x) \ge -\rho_0$ a.e. $(a,t,x) \in Q_{A-a^*+T_0}$. This implies (via Theorem 3.1) that

$$z(a,t,x) \ge -y_s(a,x) \quad \text{a.e. } (a,t,x) \in (0, a^*) \times (0,T) \times \Omega.$$

On the other hand $mu = 0$ on $(a^*, A) \times (0,T) \times \Omega$. The comparison principle for parabolic equations allows us to conclude that

$$z(a,t,x) \ge -y_s(a,x) \quad \text{a.e. } (a,t,x) \in (a^*, A) \times (0,T) \times \Omega.$$

For the second assertion of Theorem 2.2 we assume by contradiction that $T < A - a^*$ (this also implies that $a^* < A$), $\|z_0\|_{L^\infty((a^*, A-T)\times\Omega)} > 0$ and there exists $u \in L^2(Q_T)$ such that z^u the solution of (1.4) satisfies (2.2) (see figure 2).

Since $mu = 0$ on $(a^*, A) \times (0,T) \times \Omega$ we may conclude that z^u does not explicitly depend on u on $\mathcal{S} \times \Omega$, where $\mathcal{S} = \{(a,t); a \in (a^*, A), t \in (0,T), t < a - a^*\}$. However we have that z^u satisfies

$$Dz^u + \mu(a) z^u - k\Delta z^u = 0, \quad (a,t,x) \in \mathcal{S} \times \Omega$$
$$\frac{\partial z^u}{\partial \nu}(a,t,x) = 0, \quad (a,t,x) \in \mathcal{S} \times \partial\Omega$$
$$z^u(a,0,x) = z_0(a,x), \quad (a,x) \in (a^*, A) \times \Omega,$$

and since $\|z_0\|_{L^\infty((a^*, A-T)\times\Omega)} > 0$, we conclude that $\|z^u(\cdot, T, \cdot)\|_{L^\infty((0,A)\times\Omega)} > 0$ (this follows via the backward uniqueness theorem); which is in contradiction to (2.2). So, we get the conclusion.

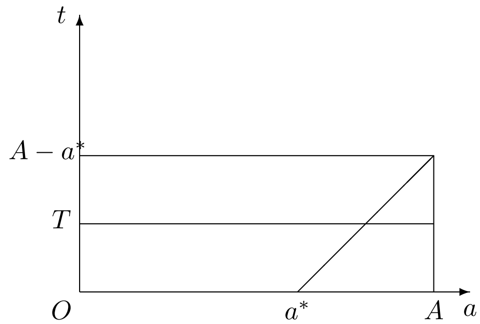

FIGURE 2.

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