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APPROXIMATIONS OF SOLUTIONS TO RETARDED INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider a retarded integrodifferential equation and prove existence, uniqueness and convergence of approximate solutions. We also give some examples to illustrate the applications of the abstract results.

1. Introduction

Consider the semilinear retarded differential equation with a nonlocal history condition in a separable Hilbert space H:

$$\begin{gathered} u'(t) + Au(t) = Bu(t) + Cu(t-\tau) + \int_{-\tau}^{0} a(\theta)Lu(t+\theta)d\theta, \quad 0 < t \le T < \infty, \tau > 0, \\ u(t) = h(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{1.1}$$

Here u is a function defined from $[-\tau, \infty)$ to the space H, $h : [-\tau, 0] \to H$ is a given function and the kernel $a \in L^p_{\rm loc}(-\tau, 0)$. For each $t \ge 0$, $u_t : [-\tau, 0] \to H$ is defined by $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0]$ and the operators $A : D(A) \subseteq H \to H$, is a linear operator. The operators $B : D(B) \subseteq H \to H$, $C : D(C) \subseteq H \to H$, and $L : D(L) \subseteq H \to H$ are non-linear continuous operators.

For $t \in [0, T]$, we shall use the notation $\mathcal{C}_t := C([-\tau, t]; H)$ for the Banach space of all continuous functions from $[-\tau, t]$ into H endowed with the supremum norm

$$\|\psi\|_t := \sup_{-\tau \le \eta \le t} \|\psi(\eta)\|.$$

The existence, uniqueness and regularity of solutions of (1.1) under different conditions have been considered by Blasio [8] and Jeong [14] and some of the papers cited therein. For the initial work on the existence, uniqueness and stability of various types of solutions of differential and functional differential equations, we refer to Bahuguna [1, 2], Balachandran and Chandrasekaran [5], Lin and Liu [15]. The related results for the approximation of solutions may be found in Bahuguna, Srivastava and Singh [4] and Bahuguna and Shukla [3].

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Initial study concerning existence, uniqueness and finite-time blow-up of solutions for the following equation,

$$\begin{gathered} u'(t) + Au(t) = g(u(t)), \quad t \geq 0, \\ u(0) = \phi, \end{gathered} \tag{1.2}$$

have been considered by Segal [19], Murakami [17], Heinz and von Wahl [13]. Bazley [6, 7] has considered the following semilinear wave equation

$$\begin{gathered} u''(t) + Au(t) = g(u(t)), \quad t \geq 0, \\ u(0) = \phi, \quad u'(0) = \psi, \end{gathered} \tag{1.3}$$

and has established the uniform convergence of approximations of solutions to (1.3) using the existence results of Heinz and von Wahl [13]. Goethel [12] has proved the convergence of approximations of solutions to (1.2) but assumed g to be defined on the whole of H. Based on the ideas of Bazley [6, 7], Miletta [16] has proved the convergence of approximations to solutions of (1.2). In the present work, we use the ideas of Miletta [16] and Bahuguna [3, 4] to establish the convergence of finite dimensional approximations of the solutions to (1.1).

2. Preliminaries and Assumptions

Existence of a solution to (1.1) is closely associated with the existence of a function $u \in \mathcal{C}_{\tilde{T}}$, $0 < \tilde{T} \leq T$ satisfying

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[Bu(s) & \\ +Cu(s-\tau) + \int_{-\tau}^0 a(\theta)Lu(s+\theta)d\theta]ds, & t \in [0, \tilde{T}] \end{cases}$$

and such a function u is called a *mild solution* of (1.1) on $[-\tau, \tilde{T}]$. A function $u \in \mathcal{C}_{\tilde{T}}$ is called a *classical solution* of (1.1) on $[-\tau, \tilde{T}]$ if $u \in C^1((0, \tilde{T}]; H)$ and u satisfies (1.1) on $[-\tau, \tilde{T}]$.

We assume in (1.1), that the linear operator A satisfies the following.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H, A has the pure point spectrum
$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$
and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,
$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad (\phi_i, \phi_j) = \delta_{ij},$$
where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$ in H (cf. Pazy [18, pp. 60-69]). It follows that the fractional powers A^α of A for $0 \leq \alpha \leq 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf. Pazy [18, pp. 69-75]). Hence for convenience, we suppose that

$$\|e^{-tA}\| \leq M \quad \text{for all} \quad t \geq 0$$

and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$.

$D(A^\alpha)$ is a Banach space endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$.

For $t \in [0,T]$, we denote by $\mathcal{C}_t^\alpha := C([-\tau,t];D(A^\alpha))$ endowed with the norm

$$\|\psi\|_{t,\alpha} := \sup_{-\tau\le\nu\le t} \|\psi(\nu)\|_\alpha.$$

Further, we assume the following.

- (H2) $h \in \mathcal{C}_0^\alpha$ and h is locally hölder continuous on $[-\tau,0]$.
- (H3) We shall assume that the map $B : D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: for each $\eta > 0$ and some $0<\alpha<1$ there exists a constant $K_1(\eta)$ such that
  (i) $\|B(\psi)\| \le K_1(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha\psi\| \le \eta$,
  (ii) $\|B(\psi_1) - B(\psi_2)\| \le K_1(\eta)\|A^\alpha(\psi_1-\psi_2)\|$ for $\psi_1,\psi_2 \in D(A^\alpha)$ with $\|A^\alpha\psi_i\| \le \eta$ for $i=1,2$.
- (H4) The map $C : D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: For each $\eta > 0$ and some $0<\alpha<1$ there exists a constant $K_2(\eta)$ such that
  (iii) $\|C(\psi)\| \le K_2(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha\psi\| \le \eta$,
  (iv) $\|C(\psi_1) - C(\psi_2)\| \le K_2(\eta)\|A^\alpha(\psi_1-\psi_2)\|$ for $\psi_1,\psi_2 \in D(A^\alpha)$ with $\|A^\alpha\psi_i\| \le \eta$ for $i=1,2$.
- (H5) The map $L : D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: For each $\eta > 0$ and some $0<\alpha<1$ there exists a constant $K_3(\eta)$ such that
  (v) $\|L(\psi)\| \le K_3(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha\psi\| \le \eta$,
  (vi) $\|L(\psi_1) - L(\psi_2)\| \le K_3(\eta)\|A^\alpha(\psi_1-\psi_2)\|$ for $\psi_1,\psi_2 \in D(A^\alpha)$ with $\|A^\alpha\psi_i\| \le \eta$ for $i=1,2$.
- (H6) $a \in L^p_{\rm loc}(-\tau,0)$ for some $1<p<\infty$ and $a_T = \int_{-\tau}^0 |a(\theta)|\,d\theta$.

3. Approximate Solutions and Convergence

Let H_n denote the finite dimensional subspace of H spanned by $\{\phi_0,\phi_1,\cdots,\phi_n\}$ and let $P^n : H \longrightarrow H_n$ be the corresponding projection operator for $n=0,1,2,\cdots$. Let $0 < T_0 \le T$ be such that

$$\sup_{0\le t\le T_0} \|(e^{-tA}-I)A^\alpha h(0)\| \le \frac{R}{2}, \tag{3.1}$$

where $R>0$ be a fixed quantity.

Let us define

$$\bar h(t) = \begin{cases} h(t), & \text{if } t\in[-\tau,0],\\ h(0), & \text{if } t \in [0,T]. \end{cases}$$

We set

$$T_0 < \min\big[\{\frac{R}{2}(1-\alpha)(K(\eta_0)C_\alpha)^{-1}\}^{\frac{1}{1-\alpha}},\ \{\frac{1}{2}(1-\alpha)(K(\eta_0)C_\alpha)^{-1}\}^{\frac{1}{1-\alpha}}\big], \tag{3.2}$$

where

$$K(\eta_0) = [K_1(\eta_0) + K_2(\eta_0) + K_3(\eta_0)a_T] \tag{3.3}$$

and C_α is a positive constant such that $\|A^\alpha e^{-tA}\| \le C_\alpha t^{-\alpha}$ for $t>0$. We define $B_n : H \longrightarrow H$ such that

$$B_n x = BP^n x,\ x\in H.$$

Similarly C_n and L_n are given by

$$C_n x = CP^n x,\ x \in H, \quad L_n x = LP^n x,\ x\in H.$$

Let $A^\alpha : \mathcal{C}_t^\alpha \to \mathcal{C}_t$ be given by $(A^\alpha\psi)(s) = A^\alpha(\psi(s))$, $s \in [-\tau, t]$, $t \in [0, T]$. We define a map F_n on $B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ as follows

$$(F_n u)(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[B_n u(s) \\ +C_n u(s-\tau) + \int_{-\tau}^0 a(\theta)L_n u(s+\theta)d\theta]ds, & t \in [0, T_0], \end{cases}$$

for $u \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$.

Theorem 3.1. *Suppose that the conditions (H1)-(H6) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then there exists a unique $u_n \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ such that $F_n u_n = u_n$ for each $n = 0, 1, 2, \cdots$, i.e., u_n satisfies the approximate integral equation*

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[B_n u_n(s) \\ +C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta)L_n u_n(s+\theta)d\theta]ds, & t \in [0, T_0]. \end{cases} \tag{3.4}$$

Proof. First we show that $F_n : B_R(\mathcal{C}_{T_0}^\alpha, \bar{h}) \to B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$. For this first we need to show that the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau, T_0]$ into $D(A^\alpha)$ with respect to $\|\cdot\|_\alpha$ norm. Thus for any $u \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$, and $t_1, t_2 \in [-\tau, 0]$, we have

$$(F_n u)(t_1) - (F_n u)(t_2) = h(t_1) - h(t_2). \tag{3.5}$$

Now for t_1 $t_2 \in (0, T_0]$ with $t_1 < t_2$ we have

$$\begin{aligned} &\|(F_n u)(t_2) - (F_n u)(t_1)\|_\alpha \\ &\leq \|(e^{-t_2 A} - e^{-t_1 A})h(0)\|_\alpha + \int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| \\ &\quad \times \big[\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^0 |a(\theta)|\|L_n u(s+\theta)\|d\theta\big]ds \\ &\quad + \int_{t_1}^{t_2} \|(e^{-(t_2-s)A})A^\alpha\|\big[\|B_n u(s)\| + \|C_n u(s-\tau)\| \\ &\quad + \int_{-\tau}^0 |a(\theta)|\|L_n u(s+\theta)\|d\theta\big]ds. \end{aligned} \tag{3.6}$$

Since part (d) of Theorem 6.13 in Pazy [18, p. 74] states that for $0 < \beta \leq 1$ and $x \in D(A^\beta)$,

$$\|(e^{-tA} - I)x\| \leq C_\beta t^\beta \|A^\beta x\|.$$

Hence if $0 < \beta < 1$ is such that $0 < \alpha + \beta < 1$ then $A^\alpha y \in D(A^\beta)$. Therefore for $t, s \in (0, T_0]$, we have

$$\|(e^{-tA} - I)A^\alpha e^{-sA}x\| \leq C_\beta t^\beta \|A^{\alpha+\beta}e^{-sA}x\| \leq C_\beta C_{\alpha+\beta} t^\beta s^{-(\alpha+\beta)}\|x\|. \tag{3.7}$$

We use the inequality (3.7) to obtain

$$\begin{aligned}
&\int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| [\|B_n u(s)\| + \|C_n u(s-\tau)\| \\
&+ \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| d\theta] ds \\
&\leq \int_0^{t_1} \|(e^{-(t_2-t_1)A} - I)e^{-(t_1-s)A}A^\alpha\| [\|B_n u(s)\| + \|C_n u(s-\tau)\| \\
&\quad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| d\theta] ds \\
&\leq C_{\alpha,\beta}(t_2-t_1)^\beta,
\end{aligned} \tag{3.8}$$

where

$$C_{\alpha,\beta} = C_\beta C_{\alpha+\beta} K(\eta_0) \frac{T_0^{1-(\alpha+\beta)}}{[1-(\alpha+\beta)]},$$

$K(\eta_0)$ is given by (3.3) and $\eta_0 = R + \|h\|_{0,\alpha}$. We calculate the second part of the integral (3.6) as follows. We have

$$\begin{aligned}
&\int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\| [\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| d\theta] ds \\
&\leq C_\alpha K(\eta_0) \frac{(t_2-t_1)^{1-\alpha}}{(1-\alpha)}.
\end{aligned} \tag{3.9}$$

Hence from (3.5), (3.8) and (3.9) the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau, T_0]$ into $D(A^\alpha)$ with respect to $\|\cdot\|_\alpha$ norm.

Now, for $t \in [-\tau, 0]$, $(F_n u)(t) - \bar{h}(t) = 0$.
For $t \in (0, T_0]$, we have

$$\begin{aligned}
&\|(F_n u)(t) - \bar{h}(t)\|_\alpha \\
&\leq \|(e^{-tA} - I)A^\alpha h(0)\| \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\| \big[\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| d\theta\big] ds \\
&\leq \frac{R}{2} + C_\alpha K(\eta_0) \frac{T_0^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

Hence $\|F_n u - \bar{h}\|_{T_0,\alpha} \leq R$. Thus $F_n : B_R(\mathcal{C}^\alpha_{T_0}, \bar{h}) \to B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$.

Now, for any $u, v \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ and $t \in [-\tau, 0]$ we have $F_n u(t) - F_n v(t) = 0$. For $t \in (0, T_0]$ and $u, v \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ we have

$$\begin{aligned}
&\|F_n u(t) - F_n v(t)\|_\alpha \\
&\leq \int_0^t \|e^{-(t-s)A}A^\alpha\| \Big[\|B_n u(s) - B_n v(s)\| + \|C_n u(s-\tau) - C_n v(s-\tau)\| \\
&\quad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta) - L_n v(s+\theta)\| d\theta\Big] ds \\
&\leq \int_0^t C_\alpha (t-s)^{-\alpha} [K_1(\eta_0)\|u(s) - v(s)\|_\alpha + K_2(\eta_0)\|u(s-\tau) - v(s-\tau)\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau}^{0}|a(\theta)|K_3(\eta_0)\|u(s+\theta)-v(s+\theta)\|_\alpha d\theta]ds\\
&\le \int_0^t C_\alpha(t-s)^{-\alpha}K(\eta_0)\|u-v\|_{T_0,\alpha}ds\\
&\le \frac{1}{2}\|u-v\|_{T_0,\alpha}.
\end{aligned}
$$

Taking the supremum on t over $[-\tau, T_0]$ we get

$$
\|F_n u - F_n v\|_{T_0,\alpha} \le \frac{1}{2}\|u-v\|_{T_0,\alpha}.
$$

Hence there exists a unique $u_n \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ such that $F_n u_n = u_n$, which satisfies the approximate integral equation (3.4). This completes the proof of Theorem 3.1. □

Corollary 3.2. *If all the hypotheses of the Theorem 3.1 are satisfied then $u_n(t) \in D(A^\beta)$ for all $t \in [-\tau, T_0]$ where $0 \le \beta < 1$.*

Proof. From Theorem 3.1 there exists a unique $u_n \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ satisfying (3.4). From [18, Theorem 1.2.4] we have that $e^{-tA}x \in D(A)$ for $x \in D(A)$. Also from Part (a) of [18, Theorem 2.6.13] we have $e^{-tA} : H \mapsto D(A^\beta)$ for $t > 0$ and $0 \le \beta < 1$. Hölder continuity of u_n follows from the similar arguments as used in (3.8) and (3.9). From [18, Theorem 4.3.2], for $0 < t < T$, we have

$$
\int_0^t e^{-(t-s)A} f(s)ds \in D(A).
$$

Since $D(A) \subseteq D(A^\beta)$ for $0 \le \beta \le 1$, the result of Corollary 3.2 thus follows. □

Corollary 3.3. *If $h(0) \in D(A^\alpha)$, where $0 < \alpha < 1$ and $t_0 \in (0, T_0]$ then there exists a constant M_{t_0}, independent of n, such that*

$$
\|A^\beta u_n(t)\| \le M_{t_0}
$$

for all $t_0 \le t \le T_0$ and $0 \le \beta < 1$. Furthermore if $h(t) \in D(A)$ for all $t \in [-\tau, 0]$ then there exist a constant M_0, independent on n, such that

$$
\|A^\beta u_n(t)\| \le M_0
$$

for all $-\tau \le t \le T_0$ and $0 \le \beta < 1$.

Proof. For any $t_0 \in (0, T_0]$, we have,

$$
\|u_n(t)\|_\beta \le C_\beta t_0^{-\beta}\|h(0)\| + C_\beta K(\eta_0)\frac{T_0^{1-\beta}}{1-\beta} \le M_{t_0}.
$$

Now as $h(t) \in D(A)$ for all $t \in [-\tau, 0]$ hence $h(t) \in D(A^\beta)$ for all $t \in [-\tau, 0]$ so for any $t \in [-\tau, 0]$, we have

$$
\|u_n(t)\|_\beta = \|A^\beta h(t)\| \le \|h\|_{0,\beta} \quad \text{for all } t \in [-\tau, 0].
$$

Now again for any $t \in (0, T_0]$ we have

$$
\|u_n(t)\|_\beta \le M\|h\|_{0,\beta} + C_\beta K(\eta_0)\frac{T_0^{1-\beta}}{1-\beta}. \tag{3.10}
$$

This completes the proof of the Corollary 3.3. □

Theorem 3.4. *Suppose that the conditions (H1)-(H6) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then the sequence $\{u_n\} \subset \mathcal{C}^\alpha_{T_0}$ is a Cauchy sequence and therefore converges to a function $u \in \mathcal{C}^\alpha_{T_0}$.*

Proof. For $n \geq m \geq n_0$, where n_0 is large enough, $n, m, n_0 \in \mathbb{N}$, $t \in [-\tau, 0]$ we have

$$\|u_n(t) - u_m(t)\|_\alpha = \|h(t) - h(t)\|_\alpha = 0. \tag{3.11}$$

For $t \in (0, T_0]$ and n, m and n_0 as above we have

$$\begin{aligned}\|u_n(t) - u_m(t)\|_\alpha \leq \int_0^t &\|e^{-(t-s)A}A^\alpha\|[\|B_n u_n(s) - B_m u_m(s)\| \\ &+ \|C_n u_n(s-\tau) - C_m u_m(s-\tau)\| \\ &+ \int_{-\tau}^0 |a(\theta)|\|L_n u_n(s+\theta) - L_m u_m(s+\theta)\|d\theta]ds.\end{aligned}$$

For $0 < t_0' < t_0$, we have

$$\begin{aligned}\|u_n(t) - u_m(t)\|_\alpha \leq (\int_0^{t_0'} + \int_{t_0'}^t)&\|e^{-(t-s)A}A^\alpha\|[\|B_n u_n(s) - B_m u_m(s)\| \\ &+ \|C_n u_n(s-\tau) - C_m u_m(s-\tau)\| \\ &+ \int_{-\tau}^0 |a(\theta)|\|L_n u_n(s+\theta) - L_m u_m(s+\theta)\|d\theta]ds.\end{aligned} \tag{3.12}$$

Now for $0 < \alpha < \beta < 1$, we have

$$\begin{aligned}&\|[B_n(u_n(s)) - B_m(u_m(s))]\| \\ &\leq \|B_n(u_n(s)) - B_n(u_m(s))\| + \|B_n(u_m(s)) - B_m(u_m(s))\| \\ &\leq K_1(\eta_0)\|A^\alpha[u_n(s) - u_m(s)]\| + K_1(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s)\| \\ &\leq K_1(\eta_0)\|A^\alpha[u_n(s) - u_m(s)]\| + \frac{K_1(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s)\|.\end{aligned} \tag{3.13}$$

Similarly

$$\begin{aligned}&\|[C_n(u_n(s-\tau)) - C_m(u_m(s-\tau))]\| \\ &\leq \|C_n(u_n(s-\tau)) - C_n(u_m(s-\tau))\| + \|C_n(u_m(s-\tau)) - C_m(u_m(s))\| \\ &\leq K_2(\eta_0)\|A^\alpha[u_n(s-\tau) - u_m(s-\tau)]\| \\ &\quad + K_2(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s-\tau)\| \\ &\leq K_2(\eta_0)\|A^\alpha[u_n(s-\tau) - u_m(s-\tau)]\| + \frac{K_2(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s-\tau)\|\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\|[L_n(u_n(s+\theta)) - L_m(u_m(s+\theta))]\| \\ &\leq \|L_n(u_n(s+\theta)) - L_n(u_m(s+\theta))\| \\ &\quad + \|L_n(u_m(s+\theta)) - L_m(u_m(s+\theta))\| \\ &\leq K_3(\eta_0)\|A^\alpha[u_n(s+\theta) - u_m(s+\theta)]\| \\ &\quad + K_3(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s+\theta)\| \\ &\leq K_3(\eta_0)\|A^\alpha[u_n(s+\theta) - u_m(s+\theta)]\| + \frac{K_3(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s+\theta)\|.\end{aligned} \tag{3.15}$$

From inequalities (3.13), (3.14) and (3.15), inequality (3.12) becomes

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha \\
&\le (\int_0^{t_0'} + \int_{t_0'}^t)\|e^{-(t-s)A}A^\alpha\|[K_1(\eta_0)\|A^\alpha[u_n(s) - u_m(s)]\| \\
&\quad + \frac{K_1(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s)\| + K_2(\eta_0)\|A^\alpha[u_n(s-\tau) - u_m(s-\tau)]\| \\
&\quad + \frac{K_2(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s-\tau)\| + \int_{-\tau}^0 |a(\theta)|K_3(\eta_0)\|A^\alpha[u_n(s+\theta) - u_m(s+\theta)]\| \\
&\quad + \frac{K_3(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s+\theta)\|d\theta]ds.
\end{aligned} \tag{3.16}$$

¿From Corollaries 3.2 and 3.3, inequality (3.16) becomes

$$\|u_n(t) - u_m(t)\|_\alpha \le C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'}^t (t-s)^{-\alpha}\|u_n - u_m\|_{s,\alpha}ds, \tag{3.17}$$

where $C_1 = 2C_\alpha(t_0 - t_0')^{-\alpha}CK(\eta_0)$ and $C_2 = \frac{2K(\eta_0)C_\alpha T^{1-\alpha}}{(1-\alpha)}$. Now we replace t by $t+\theta$ in inequality (3.17) where $\theta \in [t_0' - t, 0]$, we get

$$\begin{aligned}
&\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\
&\le C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'}^{t+\theta} (t+\theta-s)^{-\alpha}\|u_n - u_m\|_{s,\alpha}ds.
\end{aligned} \tag{3.18}$$

We put $s - \theta = \gamma$ in (3.18) to get

$$\begin{aligned}
&\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\
&\le C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'-\theta}^t (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\alpha}ds \\
&\le C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'}^t (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\alpha}ds.
\end{aligned}$$

Now

$$\begin{aligned}
&\sup_{t_0'-t\le\theta\le 0}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\
&\le C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'}^t (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\alpha}ds.
\end{aligned} \tag{3.19}$$

We have

$$\begin{aligned}
&\sup_{-\tau-t\le\theta\le 0}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\
&\le \sup_{0\le\theta+t\le t_0'}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha + \sup_{t_0'-t\le\theta\le 0}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha.
\end{aligned}$$

Using inequalities (3.19) and (3.16) in the above inequality, we get

$$\begin{aligned}
&\sup_{-\tau\le t+\theta\le t}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\
&\le (C_1 + C_3)t_0' + \frac{(C_2 + C_4)}{\lambda_m^{\beta-\alpha}} + C_\alpha K(\eta_0)\int_{t_0'}^t (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\alpha}ds,
\end{aligned}$$

where C_3 and C_4 are constants. An application of Gronwall's inequality to the above inequality gives the required result. This completes the proof of the Theorem 3.4. $\square$

With the help of Theorems 3.1 and 3.4, we may state the following existence, uniqueness and convergence result.

Theorem 3.5. *Suppose that the conditions (H1)-(H6) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$ hold. Then there exist a function $u_n \in C([-\tau, T_0]; H)$ and $u \in C([-\tau, T_0]; H)$ satisfying*

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[B_n u_n(s) & \\ +C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta) d\theta] ds, & t \in [0, T_0] \end{cases} \tag{3.20}$$

and

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[Bu(s) & \\ +Cu(s-\tau) + \int_{-\tau}^0 a(\theta) Lu(s+\theta) d\theta] ds, & t \in [0, \tilde{T}] \end{cases} \tag{3.21}$$

such that $u_n \to u$ in $C([-\tau, T_0]; H)$ as $n \to \infty$, where B_n, C_n and L_n are as defined earlier.

4. Faedo-Galerkin Approximations

We know from the previous sections that for any $-\tau \le T_0 \le T$, we have a unique $u \in C_{T_0}^{\alpha}$ satisfying the integral equation

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[Bu(s) & \\ +Cu(s-\tau) + \int_{-\tau}^0 a(\theta) Lu(s+\theta) d\theta] ds, & t \in [0, \tilde{T}]. \end{cases} \tag{4.1}$$

Also, there is a unique solution $u \in C_{T_0}^{\alpha}$ of the approximate integral equation

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[B_n u_n(s) & \\ +C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta) d\theta] ds, & t \in [0, T_0]. \end{cases} \tag{4.2}$$

Faedo-Galerkin approximation $\bar{u}_n = P^n u_n$ is given by

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ e^{-tA}P^n h(0) + \int_0^t e^{-(t-s)A}P^n[B_n u_n(s) & \\ +C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta) d\theta] ds, & t \in [0, T_0], \end{cases} \tag{4.3}$$

where B_n, C_n and L_n are as defined earlier.

If the solution $u(t)$ to (4.1) exists on $-\tau \le t \le T_0$ then it has the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) \phi_i, \tag{4.4}$$

where $\alpha_i(t) = (u(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$ and

$$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\phi_i, \tag{4.5}$$

where $\alpha_i^n(t) = (\bar{u}_n(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$.

As a consequence of Theorem 3.1 and Theorem 3.4, we have the following result.

Theorem 4.1. *Suppose that the conditions (H1)-(H6) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then there exist unique functions $\bar{u}_n \in C([-\tau, T_0]; H_n)$ and $u \in C([-\tau, T_0]; H)$ satisfying*

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ e^{-tA}P^n h(0) + \int_0^t e^{-(t-s)A}P^n[B_n u_n(s) & \\ +C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta)L_n u_n(s+\theta)d\theta]ds, & t \in [0, T_0] \end{cases}$$

and

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A}[Bu(s) & \\ +Cu(s-\tau) + \int_{-\tau}^0 a(\theta)Lu(s+\theta)d\theta]ds, & t \in [0, \tilde{T}], \end{cases}$$

such that $\bar{u}_n \to u$ in $C([-\tau, T_0]; H)$ as $n \to \infty$, where B_n, C_n and L_n are as defined earlier.

Theorem 4.2. *Let (H1)-(H6) hold. If $h(t) \in D(A)$ for all $t \in [-\tau, 0]$ then for any $-\tau \le t \le T_0 \le T$,*

$$\lim_{n\to\infty} \sup_{-\tau\le t\le T_0} \big[\sum_{i=0}^{n} \lambda_i^{2\alpha}\{\alpha_i(t) - \alpha_i^n(t)\}^2\big] = 0.$$

Proof. Let $\alpha_i^n(t) = 0$ for $i > n$. We have

$$A^\alpha[u(t) - \bar{u}_n(t)] = A^\alpha\big[\sum_{i=0}^{\infty}\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i\big] = \sum_{i=0}^{\infty} \lambda_i^\alpha\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i.$$

Thus we have

$$\|A^\alpha[u(t) - \bar{u}_n(t)\|^2 \ge \sum_{i=0}^{n} \lambda_i^{2\alpha}|\alpha_i(t) - \alpha_i^n(t)|^2. \tag{4.6}$$

Hence as a consequence of Theorem 3.5 we have the required result. $\square$

5. Example

Consider the following partial differential equation with delay,

$$\begin{gathered} w_t(t,x) = w_{xx}(t,x) + b(w(t,x))w_x(t,x) + c(w(t-\tau,x))w_x(t-\tau,x) \\ + \int_{-\tau}^0 a(s)l(w(t+s,x))w_x(t+s,x)ds, \quad t \ge 0,\ x \in (0,1), \\ w(t,x) = \tilde{h}(t,x), \quad t \in [-\tau, 0],\ x \in (0,1), \\ w(t,0) = w(t,1) = 0, \quad t \ge 0, \end{gathered} \tag{5.1}$$

where the kernel $a \in L^p_{\rm loc}(-\tau, 0)$, b, c, l are smooth functions from $\mathbb{R}$ into $\mathbb{R}$, $\tilde{h}$ is a given continuous function and $\tau > 0$ is a given number.

We define an operator A, as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = H_0^1(0,1) \cap H^2(0,1). \tag{5.2}$$

Here clearly the operator A satisfies the hypothesis (H1) and is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$.

For $0 \leq \alpha < 1$, and $t \in [0,T]$, we denote $C_t^\alpha := C([-\tau,t]; D(A^\alpha))$, which is the Banach space endowed with the sup norm

$$\|\psi\|_{t,\alpha} := \sup_{-\tau \leq \eta \leq t} \|\psi(\eta)\|_\alpha.$$

We observe some properties of the operators A and A^α defined by (5.2) (cf. [3] for more details). For $\phi \in D(A)$ and $\lambda \in \mathbb{R}$, with $A\phi = -\phi'' = \lambda u$, we have $\langle A\phi, \phi\rangle = \langle \lambda\phi, \phi\rangle$; that is,

$$\langle -\phi'', \phi\rangle = |u'|_{L^2}^2 = \lambda|\phi|_{L^2}^2,$$

so $\lambda > 0$. A solution ϕ of $A\phi = \lambda\phi$ is of the form

$$\phi(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $\phi(0) = \phi(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is

$$\phi_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle \phi_n, \phi_m\rangle = 0$, for $n \neq m$ and $\langle \phi_n, \phi_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n\phi_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\ \alpha_n\ \phi_n(x)$$

with $u \in D(A^{1/2}) = H_0^1(0,1)$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$.

Then equation(5.1) can be reformulated as the following abstract equation in a separable Hilbert space $H = L^2(0,1)$:

$$u'(t) + Au(t) = Bu(t) + Cu(t-\tau) + \int_{-\tau}^0 a(\theta)Lu(t+\theta)d\theta,\ 0 < t \leq T < \infty, \tau > 0,$$
$$u(t) = h(t), \quad t \in [-\tau, 0],$$

where $u(t) = w(t,.)$ that is $u(t)(x) = w(t,x)$, $u_t(\theta)(x) = w(t+\theta,x)$, $t \in [0,T]$, $\theta \in [-\tau,0]$, $x \in (0,1)$, the operator A is as define in equation (5.2) and $h(\theta)(x) = \tilde{h}(\theta,x)$ for all $\theta \in [-\tau,0]$ and $x \in (0,1)$. The operators B, C and L are given by as follows:
$B : D(A^{1/2}) \mapsto H$, where $Bu(t)(x) = b(w(t,x))w_x(t,x)$,
$C : D(A^{1/2}) \mapsto H$, where $Cu(t-\tau)(x) = c(w(t-\tau,x))w_x(t-\tau,x)$, and
$L : D(A^{1/2}) \mapsto H$, where $Lu(t+s)(x) = l(w(t+s,x))w_x(t+s,x)$, where $s \in [-\tau,0]$ and $x \in (0,1)$.

Let α be such that $3/4 < \alpha < 1$. For $u, v \in D(A^\alpha)$ with $\|A^\alpha u\| \leq \eta$ and $\|A^\alpha v\| \leq \eta$, we have

$$\begin{aligned} &|b(u(x))u_x(x) - b(v(x))v_x(x)| \\ &\leq |b(u(x)) - b(v(x))||u_x(x)| + |b(v(x))||u_x(x) - v_x(x)| \end{aligned}$$

$$\leq L_b|u(x)-v(x)||u_x(x)|+b_1|u_x(x)-v_x(x)|,$$

where L_b is the Lipschitz constant for b and $b_1 = L_b\frac{\eta}{\lambda_0^{1/2}} + |b(0)|$. For $u, v \in D(A^\alpha) \subset D(A^{1/2})$, we have

$$\|B(u)-B(v)\|^2 \leq \int_0^1 |[L_b|u(x)-v(x)||u_x(x)|+b_1|u_x(x)-v_x(x)|]|^2 dx.$$

Thus, from [18, Lemma 8.3.3], we get

$$\begin{aligned}
&\|B(u)-B(v)\|^2\\
&\leq 2{L_b}^2\int_0^1 |u(x)-v(x)|^2|u_x(x)|^2dx + 2{b_1}^2\int_0^1 |u_x(x)-v_x(x)||^2dx\\
&\leq 2{L_b}^2\|u-v\|_\infty^2\int_0^1 |u_x(x)|^2dx + 2{b_1}^2\int_0^1 |u_x(x)-v_x(x)|^2dx\\
&\leq 2{L_b}^2\|u-v\|_\infty^2\|A^{1/2}u\|^2 + 2{b_1}^2\|A^{1/2}(u-v)\|^2\\
&\leq 2{L_b}^2c^2\eta^2\|A^\alpha(u-v)\|^2 + 2{b_1}^2\|A^\alpha(u-v)\|^2\\
&\leq M_b(\eta)^2\|A^\alpha(u-v)\|^2,
\end{aligned}$$

where $3/4 < \alpha < 1$, $\|A^\alpha u\| \leq \eta$, $\|A^\alpha v\| \leq \eta$, $M_b(\eta) = \sqrt{2}[L_b c\eta + b_1]$, $\|u\|_\infty = \sup_{0\leq x\leq 1}|u(x)|$ and $\|u\|_\infty \leq c\|A^\alpha u\|$ for any $u \in D(A^\alpha)$. Hence the operator B restricted to $D(A^\alpha)$ satisfies the hypothesis (H3) for $K_1(\eta) = M_b(\eta)$. Similarly we can show that the operators C and L satisfies the hypothesis (H4) and (H5) respectively.

These kinds of nonlinear operators appear in the theory of shock waves, turbulence and continuous stochastic processes (cf. [9] for more details).

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