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GLOBAL EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE FOR A REACTION-DIFFUSION EQUATION WITH MEMORY

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Abstract

Global existence, uniqueness and continuous dependence on initial data are established for a quasilinear functional reaction-diffusion equation which arises from a two-dimensional energy balance climate model. Our approach relies heavily on the so-called stability estimates for linear evolution equations of parabolic type (cf. [6]).

§1. Introduction

This paper is concerned with the initial value problem

$$\begin{cases} c\Big(x, \displaystyle\int_{-T}^{0} \beta(s)u(t+s,x)\,ds\Big)\partial_t u(t,x) - \operatorname{div}\,(k\,\operatorname{grad} u(t,\cdot))(x) \\ = R\Big(t,x,u(t,x), \displaystyle\int_{-T}^{0} \beta(s)u(t+s,x)\,ds\Big) \quad x \in M,\ t > 0 \\ u(s,x) = \vartheta(s,x) \quad s \in [-T,0],\ x \in M, \end{cases} \tag{1.1}$$

which arises in the context of energy balance climate models where one accounts for the long response time of the huge continental ice-sheets. We refer to [11] for more about the climatological background (cf. in particular Section 11 there) and mention here only that (1.1) models the evolution of, say, a ten-year mean of atmospheric temperature u at sea-level in Kelvin –M is then equal to the Euclidean unit sphere $\mathbf{S^2}$, which stands for the earth's surface. The right hand side represents the correspondingly averaged net radiation flux; $R(t,x,u,v) = \mu Q(t,x)[1-\alpha(x,u,v)] - g(u)$ for $t \geq 0$, $x \in M$ and u, $v \in \mathbf{R}_+$ with μQ the incoming solar radiation flux, α the albedo, and g the outgoing terrestrial radiation flux. The variable v serves here as an entry for $\int_{-T}^{0} \beta(s)u(t+s,\cdot)\,ds$, a weighted long-term mean of u, say $T = 10^4$ years. Such a mean is a more appropriate indicator than u when modeling the extent of perennially ice-covered regions or the amount of continental ice accumulated at time t. The second term on the left hand side arises from a diffusive approximation of the ten-year mean of the horizontal heat flux, and the thermal inertia c depends on $\int_{-T}^{0} \beta(s)u(t+s,x)\,ds$ because significantly more continental ice is accumulated during colder climate regimes than during warmer periods. In earlier work ([8, 10, 13]) this last effect was neglected in favor of having the well established basic dynamic theory for semilinear functional reaction-diffusion equations at hand. At issue

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is, then, how classes of climatologically relevant solutions depend on the parameter $\mu \in \mathbf{R}_+$, the so-called solar constant. For Example, considering the case of a model with seasonal forcing $Q(t,\cdot)$, 1-periodic in t, the possible climate regimes of the earth are identified with the stable 1-periodic solutions of the functional reaction-diffusion equation under consideration, and one is interested in the structure of the unbounded branch of solution pairs (μ, w) with w 1-periodic in t. Likewise, eliminating the seasonal forcing leads to problems involving a solar forcing $Q = Q(x)$. The function w should be a stationary solution in that context.

The same program will ultimately guide the study of the reaction-diffusion equation in (1.1), but before addressing such structural questions one has of course to deal with some basic mathematical aspects and to establish a dynamic theory for this setting, that is to say global existence, uniqueness, continuous dependence and boundedness of solution trajectories for the initial value problem (1.1) have to be derived first. These questions will be the subject of this paper.

Existence and uniqueness results for certain classes of quasilinear functional differential equations were previously obtained in the literature, cf. [18] and the references therein, but (1.1) does not fall into the scope of those papers, which mostly focus on problems with time-delays in the highest order spatial derivatives. It should also be noted that the special form of the memory term is, as far as c is concerned, crucial for obtaining the rather sharp results in this paper.

It turns out that Amann's approach [6] to linear evolution equations of parabolic type provides an appropriate frame for our purposes, and we will rely heavily on some of his so-called *Stability Estimates*. Moreover, we will follow the line of reasoning in [1] when establishing maximal solvability, uniqueness and continuous dependence in Section 2. Of course, the delay term $\int_{-T}^{0} \beta(s)u(t+s,x)\,ds$ requires special attention, and we shall frequently utilize its smoothing action in time, which is one of the reasons for focusing on (1.1) rather than investigating general quasilinear reaction-diffusion systems with delays. On the other hand, since we are employing tools from [6] that were developed for dealing with systems of quasilinear parabolic differential equations, our results promise to be extendable to problems arising from multi-layer energy balance models as considered in [12] for example, when delays of the above form are added.

Section 2 is devoted to the study of local aspects, maximal unique solvability and continuous dependence; global existence on $\mathbf{R}_+$ and boundedness of the solutions are treated in Section 3, where the special form of the delay term is crucial for obtaining boundedness for mild solutions in [6]. This is the reason, why L_∞-estimates translate here so much more easily into estimates with respect to Sobolev norms than it is usually the case in a quasilinear parabolic setting (cf. e.g. [2, 3, 6] for the effort necessary in case of parabolic systems without delays).

§2. Local Existence, Uniqueness and Continuous Dependence

Throughout we are going to employ the following hypotheses:

(H1) M is a connected, 2-dimensional, compact, oriented Riemannian manifold without boundary;

(H2) $k \in C^2(M)$ is positive, $c \in C^2(M \times \mathbf{R})$ is bounded, with $\inf c > 0$, $\partial_2 c$ is bounded, $T \in (0,\infty)$, $\beta \in C^\infty([-T,0])$, $\beta(-T) = 0$, $\beta(s) > 0$ for $s \in (-T,0]$, $\int_{-T}^0 \beta(s)\,ds = 1$;

(H3) $R \in C^3(\mathbf{R}_+ \times M \times \mathbf{R^2})$.

Clearly, we later have to be more specific about the net radiation flux term R when addressing global existence. It should also be noted that it is more convenient to deal with the solvability of (1.1) allowing arbitrary initial conditions rather than only the climatologically relevant nonnegative ones. It will not be too hard to see that solutions with nonnegative initial data stay nonnegative under those hypotheses which R fulfills in the climatological context. For the moment, we could think of R and c as being appropriately extended to the non-physical range of "negative absolute temperature". Fixing $a \in (0,\infty)$ and $\vartheta \in C([-T,0], C(M))$ e.g. we are going to deal with the initial value problem

$$\begin{cases} c\Big(x, \int_{-T}^0 \beta(s)u(t+s,x)\,ds\Big)\partial_t u(t,x) - \operatorname{div}(k \operatorname{grad} u(t,\cdot))(x) \\ = R\Big(t, x, u(t,x), \int_{-T}^0 \beta(s)u(t+s,x)\,ds\Big) \quad x \in M,\ t > a \\ u(a+s,x) = \vartheta(s,x) \quad s \in [-T,0],\ x \in M. \end{cases} \tag{2.1}$$

In order to reformulate (2.1) as a functional evolution equation we select $p \in (4,\infty)$ and set $E_0 := L_p(M)$ and $E_1 := W^{2,p}(M)$. Moreover, $\mathcal{L}(E_1,E_0)$ denotes the Banach space of bounded linear operators from E_1 into E_0. Define $A \in C^1(C(M), \mathcal{L}(E_1,E_0))$ by

$$A(\psi)(\varphi)(x) := -\frac{\operatorname{div}(k \operatorname{grad}\varphi)(x)}{c(x,\psi(x))}$$

for $x \in M$, $\varphi \in E_1$ and $\psi \in C(M)$. It is easy to derive that

$$\|A(\psi_1) - A(\psi_2)\|_{\mathcal{L}(E_1,E_0)} \le C_{\text{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2} \|\psi_1 - \psi_2\|_\infty \tag{2.2}$$

for all ψ_1, $\psi_2 \in C(M)$, where $\|\cdot\|_{\mathcal{L}(E_1,E_0)}$ denotes the operator norm on $\mathcal{L}(E_1,E_0)$ and $C_{\text{diff}} := \|\varphi \mapsto -\operatorname{div}(k \operatorname{grad}\varphi)\|_{\mathcal{L}(E_1,E_0)}$.

Now, choose $\kappa \in (\frac{1}{4}, \frac{1}{2})$, $\kappa^* \in (0, 2\kappa - \frac{2}{p})$ and $\overline{\kappa} \in (\kappa, 1)$ and denote by E_k the real interpolation space $[E_0,E_1]_{k,p}$ for $k \in \{\kappa, \overline{\kappa}, \kappa^*\}$. We refer to [17] for function spaces on manifolds and mention only that E_k is norm-isomorph to $W^{2k,p}(M)$ for $k \in (0,1)\backslash\{\frac{1}{2}\}$. This fact is well-known if M is a bounded domain in $\mathbf{R^n}$ (cf. [5, 14, 16]) and carries easily over to the situation in (H1) thanks to the existence of a finite oriented atlas for M with subordinated partition of unity. Define $F \in C^1(\mathbf{R}_+ \times E_\kappa \times E_\kappa, E_{\kappa^*})$ by

$$F(t,\varphi,\psi) = \frac{R(t,\varphi(\cdot),\psi(\cdot))}{c(\cdot,\psi(\cdot))} \qquad \varphi\ ,\psi \in E_\kappa.$$

One has

Lemma 2.1. *There exists a function $C_F : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ with*

$$\begin{aligned} ||F(t_1,\varphi_1,\psi_1)-&F(t_2,\varphi_2,\psi_2)||_{E_{\kappa*}} \\ &\le C_F(r)[|t_1-t_2| + \|\varphi_1-\varphi_2\|_{E_\kappa} + \|\psi_1-\psi_2\|_{E_\kappa}] \end{aligned} \tag{2.3}$$

for all $r \in \mathbf{R}_+$, $t_1,\ t_2 \in \mathbf{R}_+$ and $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2 \in \overline{B}_{E_\kappa}(0,r)$, the closed ball with radius r and center 0.

A well-known procedure for deriving such results consists in proving mapping properties of

$$(t,\varphi,\psi) \mapsto \frac{R(t,\varphi(\cdot),\psi(\cdot))}{c(\cdot,\psi(\cdot))}$$

in a suitable Hölder space setting and then employing the embeddings $W^{2\kappa,p}(M) \hookrightarrow C^\eta(M)$ for $2\kappa - \frac{2}{p} > \eta$ and $C^{\tilde\eta}(M) \hookrightarrow W^{2\kappa^*,p}(M)$ for $\tilde\eta > 2\kappa^*$. We refer to [9] for a proof of similar results and note only that some extra care is necessary, since the function spaces under consideration are based over a manifold. The C^3-regularity of R required in (H3) is a convenient sufficient condition in this context and can actually be relaxed, e.g. to R being C^2 and $Q \in C^3(\mathbf{R}_+ \times M)$ supposing the special form $R(t,x,u,v) = \mu Q(t,x)[1-\alpha(x,u,v)] - g(u)$ mentioned in the introduction.

For $b > 0$ define $I \in \mathcal{L}(C([a-T,a+b],C(M)),C([a,a+b],C(M)))$ by $Iw(t,x) := \int_{-T}^0 \beta(s)w(t+s,x)\,ds$ for $w \in C([a-T,a+b],C(M))$, $t \in [a,a+b]$ and $x \in M$. It turns out that $I \in \mathcal{L}(C([a-T,a+b],E_\kappa),C([a,a+b],E_\kappa))$ $\Big($note $E_\kappa \hookrightarrow C(M)$ compactly$\Big)$, and we shall write $|||I|||$ for $\|I\|_{\mathcal{L}(C([a-T,a+b],E_\kappa),C([a,a+b],E_\kappa))}$ throughout.

We can now reformulate (2.1) as a quasilinear functional evolution equation

$$\begin{cases} \dot u + (A \circ I\ u)u = F(t,u,I\ u) & t > a \\ \qquad u(a+s) = \vartheta(s,\cdot) & s \in [-T,0] \end{cases} \tag{2.4}$$

and call u a *local solution* of (2.4), iff there exists a $\overline{b} > 0$ and a $u \in C([a-T,a+\overline{b}],E_0) \cap C^1((a,a+\overline{b}),E_0)$ with $\mathrm{dom}(u(t)) \in E_1$ for $t \in (a,a+\overline{b})$ satisfying (2.4) on $(a,a+\overline{b})$.

A standard method for dealing with quasilinear problems consists in freezing the "nonlinearities" and applying a fixed point argument to the solution operator generated by the family of associated linear problems. In our situation this takes also care of the delay terms, and thus we can employ the theory of linear parabolic evolution equations as developed in [6]. We adapt the following

Notations. Let $\mathcal{H}(E_1,E_0)$ denote the set of all $B \in \mathcal{L}(E_1,E_0)$ such that $-B$ considered as a mapping in E_0 is the infinitesimal generator of a strongly continuous analytic semigroup on E_0. Moreover, given $\varsigma \in [1,\infty)$ and $\omega \in (0,\infty)$ we mean by $\mathcal{H}(E_1,E_0,\varsigma,\omega)$ the subset of all $B \in \mathcal{H}(E_1,E_0)$ such that $B+\omega\mathrm{Id}$ is a homeomorphism and

$$\varsigma^{-1} \le \frac{\|\lambda\phi + B_{\mathbf{C}}\phi\|_{L_p(M,\mathbf{C})}}{|\lambda|\,\|\phi\|_{L_p(M,\mathbf{C})} + \|\phi\|_{W^{2,p}(M,\mathbf{C})}} \le \varsigma$$

for $\phi \in W^{2,p}(M,\mathbf{C}) \setminus \{0\}$ and $\lambda \in \mathbf{C}$ with $\Re\lambda \ge \omega$. Here, $B_{\mathbf{C}}$ denotes the complexification of B. Since it will mostly be clear from the context that the complexifications of the space or operator are meant, we will sometimes just use E_0 and B for $L_p(M,\mathbf{C})$ and $B_{\mathbf{C}}$, respectively.

Setting $A_w(t) := A \circ Iw(t)$ for $w \in C([a-T, a+b], C(M))$ and $t \in [a, a+b]$ we have:

Lemma 2.2. *Let $b > 0$ and $w,\ w_1,\ w_2 \in C([a-T, a+b], C(M))$. Then*

1. $A_w \in C^1([a, a+b], \mathcal{L}(E_1, E_0))$;
2. $\|A_w(t_1) - A_w(t_2)\|_{\mathcal{L}(E_1,E_0)} \le C_{\mathrm{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2} (\beta(0) + \|\beta'\|_{L^1}) \|w\|_\infty |t_1 - t_2|$ *for all* $t_1,\ t_2 \in [a, a+b]$;
3. $\|A_{w_1} - A_{w_2}\|_{C([a,a+b],\mathcal{L}(E_1,E_0))} \le C_{\mathrm{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2} \|w_1 - w_2\|_\infty$;
4. *There exist* $\varsigma \in [1, \infty)$ *and* $\omega \in (0, \infty)$ *with* $A_{w^*} \in C([a, a+b], \mathcal{H}(E_1, E_0, \varsigma, \omega))$ *for* $w^* \in C([a-T, a+b], C(M))$.

Proof of 1. Since $A \in C^1(C(M), \mathcal{L}(E_1, E_0))$, it suffices to observe that $\check{w} : t \mapsto \int_{-T}^0 \beta(s) w(t+s, \cdot)\, ds \in C^1([a, a+b], C(M))$ with $\check{w}'(t) = \beta(0)w(t, \cdot) - \int_{-T}^0 \beta'(s) w(t+s, \cdot) ds$ for $t \in [a, a+b]$.

Let us only consider the differentiability from the right at $t \in [a, a+b)$. Let $\tau \in (0, a+b-t)$ with $2\tau < T$. We get for $x \in M$:

$$
\begin{aligned}
&\Big|\check{w}(t+\tau)(x) - \check{w}(t)(x) - \tau\beta(0)w(t,x) + \tau \int_{-T}^0 \beta'(s) w(t+s,x) ds\Big| \\
&\quad \le \Big| \int_{-T}^0 \beta(s) w(t+\tau+s,x)\, ds - \int_{-T}^0 \beta(s) w(t+s,x)\, ds - \tau\beta(0) w(t,x) \\
&\qquad + \tau \int_{-T}^0 \beta'(s) w(t+s,x) ds\Big| \\
&\quad \le \Big| \int_{\tau-T}^0 [\beta(s-\tau) - \beta(s) + \tau\beta'(s)] w(t+s,x) ds\Big| \\
&\qquad + \Big| \int_0^\tau \beta(s-\tau) w(t+s,x) ds - \tau\beta(0) w(t,x)\Big| \\
&\qquad + \Big| \int_{-T}^{\tau-T} [\beta(s) - \tau\beta'(s)] w(t+s,x) ds\Big| \\
&\quad \le \|w\|_\infty\, \tau \Big[T \sup_{s\in[-T,0]} \sup_{\sigma\in[s-\tau,s]} |\beta'(s) - \beta'(\sigma)| + \sup_{s\in[0,\tau]} |\beta(s-\tau) - \beta(0)| \\
&\qquad + \sup_{s\in[-T,\tau-T]} |\beta(s) - \tau\beta'(s)| \Big] + \beta(0)\tau \sup_{s\in[0,\tau]} \sup_{x\in M} |w(t+s,x) - w(t,x)|\,.
\end{aligned}
$$

The first two terms under the last bracket tend to 0 as $\tau \longrightarrow 0+$ thanks to the uniform continuity of β' and the continuity of β at 0, respectively. Also, $\sup_{s\in[-T,\tau-T]} |\beta(s) - \tau\beta'(s)| \le \sup_{s\in[0,\tau]} |\beta(s-T)| + 2\tau \|\beta'\|_\infty \longrightarrow 0$ as $\tau \longrightarrow 0+$ in view of $\beta(-T) = 0$, and the last term divided by τ converges to 0 thanks to the uniform continuity of w.

Proof of 2. Let $t,\ \tau \in [a, a+b)$ with $\tau < t$, then

$$\begin{aligned}
&\|A_w(t) - A_w(\tau)\|_{\mathcal{L}(E_1,E_0)} \\
&\quad \le C_{\text{diff}} \left\| \frac{1}{c(\cdot, \int_{-T}^0 \beta(s) w(t+s,\cdot)\, ds)} - \frac{1}{c(\cdot, \int_{-T}^0 \beta(s) w(\tau+s,\cdot)\, ds)} \right\|_\infty \\
&\quad \le C_{\text{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2} \left\| \int_{-T}^0 \beta(s)[w(t+s,\cdot) - w(\tau+s,\cdot)]\, ds \right\|_\infty \\
&\quad \le C_{\text{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2} (\beta(0) + \|\beta'\|_{L^1})\, \|w\|_\infty\, |t-\tau|
\end{aligned}$$

Proof of 3. We have

$$\begin{aligned}
&\|A_{w_1} - A_{w_2}\|_{C([a,a+b],\mathcal{L}(E_1,E_0))} \\
&\quad \le C_{\text{diff}} \sup_{t\in[a,a+b]} \left\| \frac{1}{c(\cdot, \int_{-T}^0 \beta(s) w_1(t+s,\cdot)\, ds)} - \frac{1}{c(\cdot, \int_{-T}^0 \beta(s) w_2(t+s,\cdot)\, ds)} \right\|_\infty \\
&\quad \le C_{\text{diff}} \frac{\|\partial_2 c\|_\infty}{(\inf c)^2}\, \|w_1 - w_2\|_\infty
\end{aligned}$$

Proof of 4. This can be derived from general results in [7], cf. in particular Theorem 10.1 there. A direct argument would utilize the local L_p-estimates (cf. [15; 3.1.5, p. 76] e.g.) and an appropriate choice of a finite atlas for M to conclude that for each $0 < \underline{\gamma} < \bar{\gamma}$ there exist $\zeta \in (1,\infty)$ and $\omega > 0$ with $|\lambda|\, \|\phi\|_{L_p} \le \zeta\, \|\lambda\phi - \gamma \text{div}\,(k \text{ grad}\,\phi)\|_{L_p}$ for all $\lambda \in \mathbf{C}$, $\Re\lambda \ge \omega$, $\phi \in W^{2,p}(M, \mathbf{C})$ and all $\gamma \in C(M)$ with $\text{ran}(\gamma) \subset [\underline{\gamma}, \bar{\gamma}]$. [6; I.1.2.1.(a)] shows that this is equivalent to the estimate

$$\varsigma^{-1} \le \frac{\|\lambda\phi - \gamma \text{div}\,(k \text{ grad}\,\phi)\|_{L_p}}{|\lambda|\, \|\phi\|_{L_p} + \|\phi\|_{W^{2,p}}} \le \varsigma$$

for all $\lambda \in \mathbf{C}$, $\Re\lambda \ge \omega$, $\phi \in W^{2,p}(M, \mathbf{C})$ and all $\gamma \in C(M)$ with $\text{ran}(\gamma) \subset [\underline{\gamma}, \bar{\gamma}]$. The claim follows then in view of c bounded and $\inf c > 0$.

Moreover, defining $F_w \in C([a, a+b], E_{\kappa^*})$ by $F_w(t) := F(t, w(t), I(w(t))$ for $t \in [a, a+b]$ and $w \in C([a-T, a+b], C(M))$ we get

Lemma 2.3. *Let $b,\ r \in (0,\infty)$, $w_1,\ w_2 \in C([a-T, a+b], E_\kappa)$ with $\|w_j\|_{E_\kappa} \le r$ for $j = 1,\ 2$, $\rho \in (0,1]$ and $w \in C([a-T, a+b], E_\kappa) \cap C^\rho([a, a+b], E_\kappa)$. Then*

1. $$\begin{aligned}&\|F_{w_1} - F_{w_2}\|_{C([a,a+b],E_{\kappa^*})} \\ &\qquad \le C_F(r \max\{1, |||I|||\})\big(1 + |||I|||\big)\, \|w_1 - w_2\|_{C([a-T,a+b],E_\kappa)}\end{aligned}$$
2. $F_w \in C^\rho([a, a+b], E_{\kappa^*})$.

Proof. Statement 1 follows from (2.3). This inequality and $Iw \in C^{1-}([a, a+b], E_\kappa)$ yield Statement 2, hence the latter remains to be shown. Let $t_1,\ t_2 \in [a, a+b]$ with

$t_1 < t_2$, we have

$$\begin{aligned}
&\|Iw(t_2) - Iw(t_1)\|_{E_\kappa} \\
&= \left\| \int_{-T}^{0} \beta(s) w(t_2+s)\, ds - \int_{-T}^{0} \beta(s) w(t_1+s)\, ds \right\|_{E_\kappa} \\
&= \left\| \int_{t_2-T}^{t_2} \beta(s-t_2) w(s) ds \; - \int_{t_1-T}^{t_1} \beta(s-t_1) w(s) ds \right\|_{E_\kappa} \\
&\le \left\| \int_{t_1-T}^{t_2-T} \beta(s-t_1) w(s) ds \right\|_{E_\kappa} + \left\| \int_{t_2-T}^{t_1} [\beta(s-t_2) - \beta(s-t_1)] w(s) ds \right\|_{E_\kappa} \\
&\quad + \left\| \int_{t_1}^{t_2} \beta(s-t_2) w(s) ds \right\|_{E_\kappa} \\
&\le \text{const.} \Big[2 \|\beta\|_\infty \sup_{s\in[a-T,a+b]} \|w(s)\|_{E_\kappa} + T \sup_{s\in[a-T,a+b]} \|w(s)\|_{E_\kappa} \|\beta'\|_\infty \Big] |t_2 - t_1|
\end{aligned}$$

It should be noted that the Sobolev–Slobodeckii norm $\|\cdot\|_{2\kappa,p}$ is an equivalent norm on E_κ and that, say,

$$\begin{aligned}
\Big[\int_{\underline{t}}^{\overline{t}} \gamma(s) w(s) ds \Big]_{2\kappa,p} &= \Big(\int_{M\times M} \frac{\left| \int_{\underline{t}}^{\overline{t}} \gamma(s) w(s)(x) ds - \int_{\underline{t}}^{\overline{t}} \gamma(s) w(s)(y) ds \right|^p}{|x-y|_M^{2+2\kappa p}} dx\ dy \Big)^{\frac{1}{p}} \\
&\le \Big(\int_{\underline{t}}^{\overline{t}} |\gamma(s)|^{p'}\ ds \Big)^{\frac{1}{p'}} \Big(\int_{M\times M} \frac{\int_{\underline{t}}^{\overline{t}} |w(s)x - w(s)y|^p\, ds}{|x-y|_M^{2+2\kappa p}} dx\ dy \Big)^{\frac{1}{p}} \\
&\le \Big(\int_{\underline{t}}^{\overline{t}} |\gamma(s)|^{p'}\ ds \Big)^{\frac{1}{p'}} \left| \overline{t} - \underline{t} \right|^{\frac{1}{p}} \Big(\sup_{s\in[a-T,a+b]} \|w(s)\|_{2\kappa,p}^p \Big)^{\frac{1}{p}} \\
&\le 2 \|\beta\|_{p'} T^{\frac{1}{p}} \sup_{s\in[a-T,a+b]} \|w(s)\|_{2\kappa,p} ,
\end{aligned}$$

in case that γ stands for one of the expressions $\beta(s-t_1)$, $\beta(s-t_2) - \beta(s-t_1)$ or $\beta(s-t_2)$ and $[\underline{t}, \overline{t}]$ denotes one of the respective integration intervals, which has length $\le T$. Here, dx refers to integration with respect to the volume form induced by the Riemannian metric of M, and $|x-y|_M$ is the distance between x and y on M.

Now, we can apply [6; II.1.2.2, p. 44] for fixed $w \in C([a-T, a+b], E_\kappa) \cap C^\rho([a, a+b], E_\kappa)$ to

$$\begin{cases} \dot{u} + A_w u = F_w & \text{on } (a, a+b] \\ u(a) = w(a) \end{cases} \tag{2.5}$$

and obtain a solution $U = U(t; w) = U(t, x; w)$ with $U(\cdot; w) \in C([a, a+b], E_\kappa) \cap C^1((a, a+b], E_\kappa) \cap C((a, a+b], E_1)$.

Clearly, $U(\cdot, w)$ is a solution of (2.4), iff $w = U(\cdot, w)$. Thus we can derive unique solvability for (2.4) as in the case of a quasilinear parabolic system via the contraction mapping principle by investigating the dependence of the solution operator of a non-homogeneous linear parabolic problem on its coefficients. Note that no delays are involved in the linear equation (2.5), thus the memory effect only enters as "parameter dependence". The line of reasoning is rather similar to that in the proof of [1; Proposition 6.1] (cf. also [4] for corrections of statement and proof of that proposition).

Proposition 2.1. *Fix $b \in (0,\infty)$. Let $r > 0$ and $\sigma \in (0, \overline{\kappa} - \kappa)$. Then there exists $\overline{b} \in (0, b]$ such that (2.4) has a unique solution in $C([a-T, a+\overline{b}], E_\kappa) \cap C^\sigma([a, a+\overline{b}], E_\kappa) \cap C^1((a, a+\overline{b}], E_\kappa) \cap C((a, a+\overline{b}), E_1)$ for each $\vartheta \in C([-T,0], E_\kappa)$ with $\vartheta(0) \in E_{\overline{\kappa}}$ and $\|\vartheta\|_{C([-T,0],E_\kappa)} \le r$. This solution is a Lipschitz function on the above set of initial data under the metric induced by $\|\cdot\|_{C([-T,0],E_\kappa)}$.*

We are going to employ several estimates from chap. II.5. in [6] and begin therefore by deriving hypotheses (5.0.1) there, which we reformulate here for the reader's convenience:

Hypotheses (5.0.1) in [6]. Let $a,\ \eta \in \mathbf{R}_+,\ b,\ \omega \in (0,\infty),\ \rho \in (0,1),\ \varsigma \in (1,\infty)$, $\varrho \in \mathbf{R}$ and $\mathcal{B} \subset C^\rho([a, a+b], \mathcal{L}(E_1, E_0))$ such that

$$[B]_{\rho,[a,a+b]} := \sup_{a \le \tau_1 < \tau_2 \le a+b} \frac{\|B(\tau_1) - B(\tau_2)\|}{|\tau_1 - \tau_2|^\rho} \le \eta \qquad \text{for } B \in \mathcal{B}$$

and

$$\varrho + B \in \mathcal{H}(E_1, E_0, \varsigma, \omega) \qquad \text{for } B \in \mathcal{B}.$$

These assumptions guarantee in particular the existence of a uniform exponential bound ν for the parabolic evolution operator $V_B = V_B(t,\tau)$ of $B \in \mathcal{B}$. More precisely, [6; (5.1.1)] states:

Existence of a uniform Exponential Bound ν. ([6; II.5.1.1]) There exists a constant $c_0(\rho) > 0$ independent of η such that $\nu := c_0(\rho)\eta^{1/\rho} + \varrho + \omega$ fulfills:

$$\|V_B(t,s)\|_{\mathcal{L}(E_j)} + (t-s)\,\|V_B(t,s)\|_{\mathcal{L}(E_1,E_0)} \le Ce^{\nu(t-s)}$$

for $a < s < t < a+b$, $B \in \mathcal{B}$ and $j = 1,2$, where $C \in \mathbf{R}_+$ is independent of $0 < s < t < b$, $B \in \mathcal{B}$ and $j = 1,2$.

Proof of Proposition 2.1. Let $\sigma \in (0, \overline{\kappa} - \kappa)$ and $C_{\kappa,\infty} \in [1,\infty)$ with

$$\|w\|_{C([a-T,a+b],C(M))} \le C_{\kappa,\infty}\,\|w\|_{C([a-T,a+b],E_\kappa)} \qquad \forall w \in C(a-T, a+b], E_\kappa).$$

It follows from Statement 2 in Lemma 2.2 that

$$\begin{aligned}\mathcal{B} :=& \{A_w : w \in C([a-T,a+b],E_\kappa), \|w\|_{C(a-T,a+b],E_\kappa)} \le 5r\} \\ &\subset C^\rho([a,a+b],\mathcal{L}(E_1,E_0))\end{aligned}$$

for every $\rho \in (0,1)$. Fix $\rho \in (\sigma, 1)$ sufficiently large. Statement 4 in Lemma 2.2 shows that there exist $\varsigma \in [1,\infty)$ and $\omega \in (0,\infty)$ such that $\{A_w : w \in C([a-T, a+b], C(M))\} \subset C([a,a+b], \mathcal{H}(E_1,E_0,\varsigma,\omega))$, hence in particular $\mathcal{B} \subset C([a,a+b], \mathcal{H}(E_1,E_0,\varsigma,\omega))$. Thus, (5.0.1) in [6] is fulfilled, and we find a uniform exponential bound $\nu \in \mathbf{R}$ for $\mathcal{B}$ and ρ as stated before.

Employing the "Hölder estimate" for mild solutions of (2.5) (cf. [6; (5.3.2)]) one finds a constant C_1 with

$$\|U(t_1;w)-U(t_2;w)\|_{E_\kappa} \le C_1\,|t_1-t_2|^{\overline{\kappa}-\kappa}\,e^{\nu t_2}\Big[\|w(a)\|_{E_{\overline{\kappa}}}+\|F_w\|_{L_\infty([a,a+t_2],E_0)}\Big]$$

for all $a\le t_1\le t_2\le a+b$ and all $w\in C([a-T,a+b],E_\kappa)\cap C^\sigma([a,a+b],E_\kappa)$ with $w(a)\in E_{\overline{\kappa}}$ and $\|w\|_{C([a-T,a+b],E_\kappa)}\le 2r$. Note that A_w belongs to $\mathcal{B}$ for each such w. Moreover,

$$\|F_w\|_{L_\infty([a,a+t],E_0)} \le \operatorname{area}(M)^{1/p}\left\|R|\mathbf{R}_+\times M\times[-2rC_{\kappa,\infty},2rC_{\kappa,\infty}]^2\right\|_\infty$$

for all $t\in[a,a+b]$) and w as above, hence selecting $b_1\in(0,b]$ with

$$C_1\ \max\{1,e^{\nu b}\}b_1^{\overline{\kappa}-\kappa-\sigma}\big[r+\operatorname{area}(M)^{1/p}\left\|R|\mathbf{R}_+\times M\times[-2rC_{\kappa,\infty},2rC_{\kappa,\infty}]^2\right\|_\infty\big]\le 1$$

one obtains

$$\|U(t_1;w)-U(t_2;w)\|_{E_\kappa}\le|t_1-t_2|^\sigma\qquad\forall t_1,t_2\in[a,a+b_1] \tag{2.6}$$

for all $w\in C([a-T,a+b_1],E_\kappa)\cap C^\sigma([a,a+b_1],E_\kappa)$ with $w(a)\in E_{\overline{\kappa}}$, $\|w(a)\|_{\overline{\kappa}}\le r$ and $\|w\|_{C([a-T,a+b_1],E_\kappa)}\le 2r$.

Finally, let us consider solutions $u_1=U(\cdot;w_1)$ and $u_2:=U(\cdot;w_2)$ to (2.5) for given $w_1,\ w_2\in C([a-T,a+b],E_\kappa)\cap C^\rho([a,a+b],E_\kappa)$ with $w_j(a)\in E_{\overline{\kappa}}$, $\|w_j(a)\|_{E_{\overline{\kappa}}}\le r$ and $\|w_j\|_{C([a-T,a+b],E_\kappa)}\le 2r$ for $j=1,2$. [6; II.5.2.1, p. 71] shows the existence of a $C_2\in(0,\infty)$ such that

$$\begin{aligned}&\|u_1(t)-u_2(t)\|_{E_\kappa}\\ &\le C_2\max\{1,e^{\nu b}\}\Big\{t^{\overline{\kappa}-\kappa}\,\|A_{w_1}-A_{w_2}\|_{C([a,a+t],\mathcal{L}(E_1,E_0))}\,\big[\|w_1(a)\|_{E_{\overline{\kappa}}}\\ &\qquad+t^{1-\overline{\kappa}+\kappa^*}\,\|F_{w_1}\|_{L_\infty([0,t],E_{\kappa^*})}\big]+\|w_1(a)-w_2(a)\|_{E_\kappa}\\ &\qquad+t^{1-\kappa}\,\|F_{w_1}-F_{w_2}\|_{L_\infty([0,t],E_0)}\Big\}\end{aligned} \tag{2.7}$$

for $t\in[a,a+b]$. Statement 3 in Lemma 2.2 and the choice of $C_{\kappa,\infty}$ imply

$$\|A_{w_1}-A_{w_2}\|_{C([a,a+t],\mathcal{L}(E_1,E_0))}\le C_{\text{diff}}\frac{\|\partial_2c\|_\infty}{(\inf c)^2}C_{\kappa,\infty}\,\|w_1-w_2\|_{C([a-T,a+t],E_\kappa)}\,. \tag{2.8}$$

Statement 1 in Lemma 2.3 yields

$$\|F_{w_1}\|_{L_\infty([0,t],E_{\kappa^*})}\le\sup_{a\le t\le a+b}\|F(t,0,0)\|_{E_{\kappa^*}}+C_F\big(2r\max\{1,|||I|||\}\big)2r(1+|||I|||)\,. \tag{2.9}$$

Observing that $\max_{a\le t\le a+b}\left|\int_{-T}^0\beta(s)w(t+s,x)\,ds\right|\le\|w\|_\infty$ for $w\in C([a-T,a+b],C(M))$ and setting $C^3_{R,2r}:=\sup\left\|\partial_3R|\mathbf{R}_+\times M\times[-2rC_{\kappa,\infty},2rC_{\kappa,\infty}]^2\right\|_\infty$ and $C^4_{R,2r}:=\sup\left\|\partial_4R|\mathbf{R}_+\times M\times[-2rC_{\kappa,\infty},2rC_{\kappa,\infty}]^2\right\|_\infty$ one obtains

$$\begin{aligned}\|F_{w_1}-F_{w_2}\|_{L_\infty([0,t],E_0)}&\le[C^3_{R,2r}+C^4_{R,2r}\,\|\beta'\|_{L_{p'}}\,T]\,\|w_1-w_2\|_{C([a-T,a+t],E_0)}\\ &\le C_{\kappa,0}[C^3_{R,2r}+C^4_{R,2r}\,\|\beta'\|_{L_{p'}}\,T]\,\|w_1-w_2\|_{C([a-T,a+t],E_\kappa)}\end{aligned} \tag{2.10}$$

for $a \le t \le a+b$ with $C_{\kappa,0}$ the operator norm of the embedding from $C([a-T,a+b],E_\kappa)$ into $C([a-T,a+t],E_0)$. Inserting (2.8), (2.9) and (2.10) into (2.7) we get

$$
\begin{aligned}
&\|u_1-u_2\|_{C([a,a+t],E_\kappa)}\\
&\quad\le C_2\max\{1,e^{\nu b}\}\Big\{t^{\overline{\kappa}-\kappa}C_{\text{diff}}\frac{\|\partial_2 c\|_\infty}{(\inf c)^2}C_{\kappa,\infty}\|w_1-w_2\|_{C([a-T,a+t],E_\kappa)}\\
&\quad\times\big[r+t^{1-\overline{\kappa}+\kappa^*}\sup_{a\le t\le a+b}\|F(t,0,0)\|_{E_{\kappa^*}}+C_F\big(2r\max\{1,|||I|||\}\big)2r(1+|||I|||)\big]\\
&\quad+\|w_1(a)-w_2(a)\|_{E_\kappa}\\
&\quad+t^{1-\kappa}C_{\kappa,0}[C^3_{R,2r}+C^4_{R,2r}\|\beta'\|_{L_{p'}}T]\,\|w_1-w_2\|_{C([a-T,a+t],E_\kappa)}\Big\}.
\end{aligned}
\tag{2.11}
$$

If $w_1(a)=w_2(a)$, $\overline{b}\in(0,b_1]$ can be chosen in view of (2.11) such that

$$
\|u_1-u_2\|_{C([a,a+\overline{b}],E_\kappa)}\le\frac{1}{2}\|w_1-w_2\|_{C([a-T,a+\overline{b}],E_\kappa)}
\tag{2.12}
$$

holds for all $w_1,\ w_2\in C([a-T,a+\overline{b}],E_\kappa)\cap C^\sigma([a,a+\overline{b}],E_\kappa)$ with $\|w_j\|_{C([a-T,a+\overline{b}],E_\kappa)}\le 2r$ for $j=1,\ 2$, $w_1(a)\in E_{\overline{\kappa}}$ and $\|w_1(a)\|_{E_{\overline{\kappa}}}\le r$.

In order to apply the contraction mapping principle, let

$$
\begin{aligned}
Y:=\big\{\eta\in C([a,a+\overline{b}],E_\kappa):\eta(a)=0,\ \|\eta\|_{C([a,a+\overline{b}],E_\kappa)}\le r,\\
|\eta(t_1)-\eta(t_2)|\le|t_1-t_2|^\sigma\ \ \forall t_1,t_2\in[a,a+\overline{b}]\big\}
\end{aligned}
$$

and

$$
Z:=\big\{\vartheta\in C([-T,0],E_\kappa):\|\vartheta\|_{C([-T,0],E_\kappa)}\le r,\ \vartheta(0)\in E_{\overline{\kappa}},\ \|\vartheta(0)\|_{E_{\overline{\kappa}}}\le r\big\}.
$$

It is easy to see that Y is a closed subset of $C([a,a+\overline{b}],E_\kappa)$. Then one defines the mapping $w:Z\times Y\longrightarrow C([a-T,a+\overline{b}],E_\kappa)$ by

$$
w(\vartheta,\eta)(t):=\begin{cases}\vartheta(t-a) & a-T\le t\le a\\ \eta(t)+\vartheta(0) & a\le t\le a+\overline{b}.\end{cases}
$$

Note that $\|w(\vartheta,\eta)(t)\|_{C([a-T,a+\overline{b}],E_\kappa)}\le 2r$ for all $(\vartheta,\eta)\in Z\times Y$, hence estimates (2.7)-(2.12) can be applied in the sequel. Finally, let $\Gamma(\vartheta,\eta)(t):=U(\cdot,w(\vartheta,\eta))(t)-\vartheta(0)$ for $a\le t\le a+\overline{b}$.

Now it is easy to derive that $\Gamma(\vartheta,\cdot)$ *is a* $\frac{1}{2}$*-contraction in* Y *for each* $\vartheta\in Z$. In fact, $\Gamma(\vartheta,\eta)$ belongs to $C([a,a+\overline{b}],E_\kappa)$ for $\eta\in Y$, since $U(\cdot,w(\vartheta,\eta))$ is a solution of (2.5); $\Gamma(\vartheta,\eta)(0)=U(0,w(\vartheta,\eta))-\vartheta(0)=0$ and (2.6) and the choice of $\overline{b}$ ($\le b_1$) show $\|\Gamma(\eta,\vartheta)\|_{C([a,a+\overline{b}],E_\kappa)}\le r$ and $|\Gamma(\eta,\vartheta)(t_1)-\Gamma(\eta,\vartheta)(t_2)|\le|t_1-t_2|^\sigma$ $\forall t_1,\ t_2\in[a,a+\overline{b}]$. Moreover, (2.12) yields the contraction property.

Thus the contraction mapping principle ensures the existence of a unique fixed point $\eta(\vartheta)\in Y$ for each $\vartheta\in Z$. Furthermore, given $\vartheta_1,\ \vartheta_2\in Z$ we have

$$
\begin{aligned}
\|\eta(\vartheta_1)-\eta(\vartheta_2)\|_{C([a,a+\overline{b}],E_\kappa)}\le&\frac{1}{2}\|\vartheta_1-\vartheta_2\|_{C([-T,0],E_\kappa)}\\
&+\|\Gamma(\vartheta_1,\eta(\vartheta_2))-\Gamma(\vartheta_2,\eta(\vartheta_2))\|_{C([a,a+\overline{b}],E_\kappa)}\ ,
\end{aligned}
$$

hence because of (2.11) and the choice of $\overline{b}$

$$\begin{aligned}
&\|\eta(\vartheta_1)-\eta(\vartheta_2)\|_{C([a,a+\overline{b}],E_\kappa)}\\
&\quad\le 2\|U(\cdot,w(\vartheta_1,\eta(\vartheta_2))-\vartheta_1(0)-U(\cdot,w(\vartheta_2,\eta(\vartheta_2))+\vartheta_2(0)\|_{C([a,a+\overline{b}],E_\kappa)}\\
&\quad\le \|w(\vartheta_1,\eta(\vartheta_2))-w(\vartheta_2,\eta(\vartheta_2))\|_{C([a-T,a+\overline{b}],E_\kappa)}\\
&\qquad + 2(1+C_2\max\{1,e^{\nu b}\})\,\|\vartheta_1(0)-\vartheta_2(0)\|_{E_\kappa}\\
&\quad\le 2(2+C_2\max\{1,e^{\nu b}\})\,\|\vartheta_1-\vartheta_2\|_{C([-T,0],E_\kappa)}\,.
\end{aligned}\tag{2.13}$$

It is clear that the fixed point $\eta(\vartheta)$ provides a solution of (2.4) via $U(\cdot;w(\vartheta,\eta(\vartheta)))$. This is the only solution of (2.4) within $C([a-T,a+b],E_\kappa)\cap C^\rho([a,a+b],E_\kappa)$ and a Lipschitz function of the initial data as (2.13) shows.

Sometimes, a setting involving only one intermediate space is more desirable. Again, following Amann's approach one notes that the estimates in [6; 5.2.1 and 5.3.1] actually apply to mild solutions of (2.5) and with "$\kappa=\overline{\kappa}$". Though the resulting inequalities are insufficient as far as contraction properties are concerned, they allow to derive continuous dependence on initial data in the following framework.

Lemma 2.4. *Let $\vartheta_0\in C([-T,0],E_{\overline{\kappa}})$ and $u(\cdot;\vartheta_0)\in C([a-T,a+\overline{b}],E_\kappa)$ be the solution of (2.4) with $\vartheta=\vartheta_0$. Then there exists a neighborhood Θ of ϑ_0 in $C([-T,0],E_{\overline{\kappa}})$ such that a solution $u(\cdot;\vartheta)\in C([a-T,a+\overline{b}],E_{\overline{\kappa}})$ of (2.4) exist for each $\vartheta\in\Theta$. Moreover, the mapping $(t,\vartheta)\mapsto u(t+\cdot;\vartheta)$ is continuous from $[a,a+\overline{b}]\times C([-T,0],E_{\overline{\kappa}})$ into $C([-T,0],E_{\overline{\kappa}})$.*

Proof. Choose $r\in(\|\vartheta_0\|_{C([a-T,a],E_{\overline{\kappa}})},\infty)$ with $\|u(\cdot;\vartheta_0)\|_{C([a-T,a+\overline{b}],E_\kappa)}<r$ and set

$$\mathcal{B}:=\{A_w : w\in C([a-T,a+b],E_\kappa),\|w\|_{C(a-T,a+b],E_\kappa)}\le 5r\},$$

then (2.13) and $C([a-T,a+\overline{b}],E_{\overline{\kappa}})\hookrightarrow C([a-T,a+\overline{b}],E_\kappa)$ imply that there exists a $\delta>0$ such that $\|u(\cdot;\vartheta)\|_{C([a-T,a],E_\kappa)}\le r$ for $\vartheta\in C([-T,0],E_{\overline{\kappa}})$ satisfying $\|\vartheta-\vartheta_0\|_{C([-T,0],E_{\overline{\kappa}})}<\delta$. Noting that $u(\cdot;\vartheta_0)\in C([a-T,a+\overline{b}],E_\kappa)$ implies $F_{u(\cdot;\vartheta_0)}\in L_\infty([a,a+\overline{b}],E_0)$ one can utilize once more [6; II.5.3.1] –this time with $\alpha=\beta=\overline{\kappa}$– and conclude that $u(\cdot;\vartheta_0)\in C([a-T,a+\overline{b}],E_{\overline{\kappa}})$. Moreover, this choice in [6; II.5.2.1] yields the existence of a $\tilde{C}$ and a $\nu\in\mathbf{R}$ with

$$\begin{aligned}
&\|u(t;\vartheta)-u(t;\vartheta_0)\|_{E_{\overline{\kappa}}}\\
&\quad\le\tilde{C}\max\{1,e^{\nu\overline{b}}\}\Big\{\left\|A_{u(\cdot;\vartheta)}-A_{u(\cdot;\vartheta_0)}\right\|_{C([a,a+t],\mathcal{L}(E_1,E_0))}\,[\|\vartheta_0(0)\|_{E_{\overline{\kappa}}}\\
&\qquad\qquad + t^{1-\overline{\kappa}+\kappa^*}\left\|F_{u(t;\vartheta_0)}\right\|_{L_\infty([0,t],E_{\kappa^*})}]+\|\vartheta(0)-\vartheta_0(0)\|_{E_{\overline{\kappa}}}\\
&\qquad\qquad + t^{1-\overline{\kappa}}\left\|F_{u(\cdot;\vartheta)}-F_{u(\cdot;\vartheta_0)}\right\|_{L_\infty([0,t],E_0)}\Big\}
\end{aligned}\tag{2.14}$$

for $t\in[a,a+\overline{b}]$ and $\vartheta\in C([-T,0],E_{\overline{\kappa}})$ with $\|\vartheta-\vartheta_0\|_{C([-T,0],E_{\overline{\kappa}})}<\delta$. By adapting estimates (2.8) and (2.10) to the present situation one finds $\check{C}\in\mathbf{R}_+$ with

$$\left\|A_{u(\cdot;\vartheta)}-A_{u(\cdot;\vartheta_0)}\right\|_{C([a,a+t],\mathcal{L}(E_1,E_0))}\le\check{C}\,\|u(\cdot;\vartheta)-u(\cdot;\vartheta_0)\|_{C([a-T,a+\overline{b}],E_\kappa)}$$

and

$$\left\|F_{u(\cdot;\vartheta)} - F_{u(\cdot;\vartheta_0)}\right\|_{L_\infty([0,t],E_0)} \le \check{C}\,\|u(\cdot;\vartheta) - u(\cdot;\vartheta_0)\|_{C([a-T,a+\overline{b}],E_\kappa)},$$

hence (2.13) and (2.14) provide for a $C \in (0,\infty)$ with

$$\|u(t;\vartheta) - u(t;\vartheta_0)\|_{E_{\overline{\kappa}}} \le C\,\|\vartheta - \vartheta_0\|_{C([-T,0],E_{\overline{\kappa}})} \tag{2.15}$$

for all $\vartheta \in C([-T,0],E_{\overline{\kappa}})$ with $\|\vartheta - \vartheta_0\|_{C([-T,0],E_{\overline{\kappa}})} < \delta$ and $t \in [a, a+\overline{b}]$. Now,

$$\begin{aligned}\|u(t+s;\vartheta) - u(t_0+s;\vartheta_0)\|_{E_{\overline{\kappa}}} \le& \|u(t+s;\vartheta) - u(t+s;\vartheta_0)\|_{E_{\overline{\kappa}}}\\ &+ \|u(t+s;\vartheta_0) - u(t_0+s;\vartheta_0)\|_{E_{\overline{\kappa}}}\end{aligned}$$

for $s \in [-T,0]$. Equation (2.15) and the uniform continuity of $u(\cdot;\vartheta_0)$ yield the second statement of the lemma.

In view of Proposition 2.1 and Lemma 2.4, it is a matter of technique to derive a maximal existence, uniqueness and continuous dependence result.

Theorem 2.1. *Let $\sigma \in (0,\overline{\kappa}-\kappa)$ and $\vartheta \in C([-T,0],E_\kappa)$ with $\vartheta(0) \in E_{\overline{\kappa}}$. Then there exists a unique maximal solution $u = u(\cdot;a,\vartheta)$ of (2.4), which has a domain of the form $[a-T, a+t_+(a,\vartheta))$ (maximal interval of existence) with $t_+(a,\vartheta) \in (a,\infty]$. Also,*

$$\begin{aligned}u(\cdot;a,\vartheta) \in& C([a-T,a+t_+(a,\vartheta)),E_\kappa) \cap C^\sigma([a,a+t_+(a,\vartheta)),E_\kappa)\\ &\cap C^1((a,a+t_+(a,\vartheta)),E_\kappa) \cap C((a,a+t_+(a,\vartheta)),E_1)\end{aligned}$$

and is unbounded at $t_+(a,\vartheta)$, if $t_+(a,\vartheta) < \infty$. Moreover, let $t \in (a, a+t_+(a,\vartheta))$, then $\vartheta \mapsto u(t+\cdot;a,\vartheta)$ is Lipschitz continuous from $\{\theta \in C([-T,0],E_\kappa) : \theta(0) \in E_{\overline{\kappa}}\}$ into $C([-T,0],E_\kappa)$. Finally, $\{(t,\vartheta) : \vartheta \in C([-T,0],E_{\overline{\kappa}}), a \le t < t_+(a,\vartheta)\}$ is open in $[a,\infty) \times C([-T,0],E_{\overline{\kappa}})$ and $(t,\vartheta) \mapsto u(t+\cdot;a,\vartheta)$ is continuous from that set into $C([-T,0],E_{\overline{\kappa}})$.

Of course, $v \in C^\sigma([a,a+t_+(a,\vartheta)),E_\kappa)$ means that $v|[a,a+b] \in C^\sigma([a,a+b],E_\kappa)$ for all $b \in (0, a+t_+(a,\vartheta))$.

It is easy to see that Theorem 2.1 yields in fact a classical solution of (2.1), since $E_\kappa \hookrightarrow C^{\kappa^*}(M)$. Indeed, $u(\cdot;a,\vartheta) \in C([a-T,a+t_+(a,\vartheta)),E_\kappa)$ immediately implies $u \in C([a-T,a+t_+(a,\vartheta)) \times M)$. Fixing $t \in (a,a+t_+(a,\vartheta))$ we can use standard elliptic regularity to conclude $u(t;a,\vartheta) \in C^{2+\kappa^*}(M)$, whereas $u(\cdot;a,\vartheta) \in C^1((a,a+t_+(a,\vartheta)),E_\kappa)$ in particular yields $u(\cdot,x;a,\vartheta) \in C^1((a,a+t_+(a,\vartheta))$. To summarize we state the following.

Corollary 2.1. *Given $\vartheta \in C([-T,0],E_\kappa)$ with $\vartheta(0) \in E_{\overline{\kappa}}$, then the unique maximal solution u of (2.4) is a classical solution of (2.1) in the sense that $u \in C([a-T,a+t_+(a,\vartheta)) \times M) \cap C^1((a,a+t_+(a,\vartheta)) \times M)$ with $u(t,\cdot) \in C^2(M)$ for $t \in (a,a+t_+(a,\vartheta))$ and (2.1) is satisfied pointwise in $(a,a+t_+(a,\vartheta)) \times M$.*

§3. Global Existence

Here we are concerned with

$$\begin{cases} c\Big(x, \int_{-T}^{0} \beta(s)u(t+s,x)\,ds\Big)\partial_t u(t,x) - \operatorname{div}(k \operatorname{grad} u(t,\cdot))(x) \\ = R\Big(t,x,u(t,x), \int_{-T}^{0} \beta(s)u(t+s,x)\,ds\Big) \quad x \in M,\ t>0 \\ u(s,x) = \vartheta(s,x) \qquad s \in [-T,0],\ x \in M \end{cases} \tag{3.1}$$

under the hypotheses (H1)-(H3) stated at the beginning of Section 2 and the additional hypothesis

(H4) $R(t,x,y_1,y_2) = \mu Q(t,x)[1-\alpha(x,y_1,y_2)] - g(y_1)$ for $t \geq 0$, $x \in M$ and $y_1, y_2 \in \mathbf{R}$, where $Q \geq 0$ is bounded; $\alpha \in C^2(M \times \mathbf{R}_+ \times \mathbf{R}_+)$, with $\inf \alpha > 0$ and $\sup \alpha < 1$, and where $g \in C^2(\mathbf{R}_+)$, $g(0)=0$, with $g \in C^2(\mathbf{R})$ strictly increasing and odd, and $\lim_{y\to\infty} g(y) = \infty$

Throughout we assume that $\vartheta \in C([-T,0], E_{\overline{\kappa}})$ for some $\overline{\kappa} \in (\frac{1}{4}, \frac{1}{2})$. Choosing $\kappa \in (\frac{1}{4}, \overline{\kappa})$ we can apply Theorem 2.1 and obtain a maximal solution $u = u(\cdot;\vartheta)$ of (3.1), actually, of the associated evolution equation (2.4) with $a=0$, which is a classical solution of (3.1). Writing $t_+(\vartheta)$ for $t_+(0,\vartheta)$ we have:

Theorem 3.1. *$t_+(\vartheta) = \infty$, and $u(\cdot;\vartheta)$ is bounded with respect to $\|\cdot\|_{E_{\overline{\kappa}}}$ on $[-T,\infty]$.*

Proof. *We first establish a priori bounds w.r.t. $\|\cdot\|_\infty$, which are easy to obtain by recalling that $R(t,x,y,z) = \mu Q(t,x)[1-\alpha(x,y,z)] - g(y)$. In fact, assume for $b \in (0, t_+(\vartheta))$ that $u|[-T,b] \times M$ takes on a positive maximum $\overline{u}$ in $(\overline{t},\overline{x}) \in (0,b] \times M$. The left hand side of (3.1) is ≥ 0 at $(\overline{t},\overline{x})$, hence $0 \leq \mu\|Q\|_\infty [1-\inf\alpha] - g(\overline{u})$, which shows $\sup u \leq \max\{\|\vartheta\|_\infty, g^{-1}(\mu\|Q\|_\infty[1-\inf\alpha])\}$. Likewise, $\inf u \geq \min\{-\|\vartheta\|_\infty, g^{-1}(\mu \inf Q[1-\|\alpha\|_\infty])\}$, thus $\|u\|_\infty \leq \max\{\|\vartheta\|_\infty, g^{-1}(\mu\|Q\|_\infty[1-\inf\alpha])\}$.*

Now, assume that $t_+(\vartheta) < \infty$. We set $\check{u}(t) := \int_{-T}^{0} \beta(s)u(t+s,\cdot)\,ds$ for $t \in [0, t_+(\vartheta))$ and observe that $\check{u}$ can be extended continuously to $[0,t_+(\vartheta)]$ as a function into $C(M)$. In fact, $\check{u} \in C([0,t_+(\vartheta)), C(M)) \cap C^1((0,t_+(\vartheta)), C(M))$ with $\check{u}'(t) = \beta(0)u(t,\cdot) - \int_{-T}^{0} \beta'(s)u(t+s,\cdot)\,ds$ in view of $\beta(-T)=0$. Thus, $\|\check{u}'(t)\|_\infty \leq (\beta(0) + \|\beta'\|_{L^1([-T,0])})\|u\|_\infty$ for $t \in (0,t_+(\vartheta))$, which implies $\|\check{u}(t) - \check{u}(\tau)\| \leq (\beta(0) + \|\beta'\|_{L^1([-T,0])})\|u\|_\infty |t-\tau|$ for $t,\tau \in [0,t_+(\vartheta))$. We denote the continuous extension of $\check{u}$ into $t_+(\vartheta))$ again by $\check{u}$.

In order to employ [6; II.5.4.1], we introduce $A : [0,t_+(\vartheta)] \longrightarrow \mathcal{L}(E_1,E_0)$ by setting

$$A(t)\varphi(x) := \frac{-\operatorname{div}(k \operatorname{grad}\varphi)(x)}{c(x,\check{u}(t)(x))} \qquad \forall\, t \in [0,t_+(\vartheta)],\ x \in M \text{ and } \varphi \in E_1$$

and establish that the mapping A fulfills hypotheses (5.0.1) in [6] stated behind Proposition 2.1 here.

- It follows from Lemma 2.2.2 that $A \in C^{1-}([0,t_+(\vartheta)],\mathcal{L}(E_1,E_0))$ and that $\eta(\rho) := C_{\text{diff}}\frac{\|\partial_2 c\|_\infty}{(\inf c)^2}(\beta(0)+\|\beta'\|_{L^1})\,\|u\|_\infty\, t_+(\vartheta)^{1-\rho}$ is an appropriate choice for any $\rho \in (0,1)$.
- Also one obtains in quite the same way as described in the proof of Lemma 2.2.4 that there exist $\varsigma \in [1,\infty)$ and $\omega \in (0,\infty)$ with $A \in C([0,t_+(\vartheta)],\mathcal{H}(E_1,E_0,\varsigma,\omega))$.

Set $f(t) := \frac{R(t,\cdot,u(t,\cdot),\breve{u}(t))}{c(\cdot,\breve{u}(t))}$ for $t \in [0,t_+(\vartheta))$, then $f \in L_\infty([0,t_+(\vartheta)),L_p(M))$ and

$$\|f\|_{L_\infty([0,t_+(\vartheta)),L_p(M))} \le (\inf c)^{-1}\Big[\|Q\|_\infty\,[1-\inf\alpha]+g(\|u\|_\infty)\Big](\text{meas}(M))^{1/p}\,.$$

Noting that u is a mild solution of

$$\dot{v}+A(t)v = f(t) \qquad 0<t<t_+(\vartheta) \tag{3.2}$$

and choosing ν according to [6; (5.1.1)] –observe $\nu>0$, here– and setting $\alpha=\beta=\overline{\kappa}$ and $\beta_- = \gamma = 0$ in [6;II.5.4.1], one concludes that there is a $C\in(0,\infty)$ with

$$\|u(t,\cdot)\|_{E_{\overline{\kappa}}} \le C\Big(t^{-\overline{\kappa}}e^{\nu t}\,\|\vartheta(0)\|_{E_{\overline{\kappa}}} + B_{\overline{\kappa}}(t,\nu)\,\|f\|_{L_\infty([0,t_+(\vartheta)),L_p(M))}\Big), \tag{3.3}$$

for $t\in[0,t_+(\vartheta))$, where $B_{\overline{\kappa}}(t,\nu) := \nu^{\overline{\kappa}-1}\int_0^{\nu t}\xi^{-\overline{\kappa}}e^{\xi}\,d\xi$. Consequently, $\|u(t,\cdot)\|_{E_{\overline{\kappa}}} < \infty$, which contradicts $t_+(\vartheta)<\infty$ in view of Theorem 2.1, hence $t_+(\vartheta)=\infty$ is derived.

The boundedness of u as a curve in $E_{\overline{\kappa}}$ follows by refining the previous argument somewhat. Roughly speaking, we pass to

$$\dot{v}+(A(t)-\sigma)\,v = f(t)-\sigma u(t,\cdot) \qquad t\in(0,\infty) \tag{3.4}$$

and observe that the right hand side of (3.4) is still in $L_\infty(\mathbf{R}_+,L_p(M))$. Moreover, writing $UC^\rho(\mathbf{R}_+,C(M))$ for the Banach space of uniformly Hölder bounded functions on $\mathbf{R}_+$ we have $\breve{u}\in UC^\rho(\mathbf{R}_+,C(M))$ for every $\rho\in(0,1)$, since $\breve{u}\in UC^{1-}(\mathbf{R}_+,C(M))$ (same argument as before) and $\|\breve{u}\|_\infty \le \|u\|_\infty$. In fact, $\|\breve{u}\|_{UC^\rho(\mathbf{R}_+,C(M))} \le (2+\beta(0)+\|\beta'\|_{L_1([-T,0])})\,\|u\|_\infty$. Thus, fixing $\rho\in(0,1)$ and selecting $c_0(\rho)$ according to [6; II.5.1.1] we can find $\varrho\in(-\infty,0)$ with $\nu := c_0(\rho)\eta^{\frac{1}{\rho}}+\varrho+\omega<0$, where ω has the same meaning as in the first part of this proof and $\eta := \|A(\cdot)\|_{C^\rho(\mathbf{R}_+,\mathcal{L}(E_1,E_0))}$, which is finite in view of the previous observation and Lemma 2.2.3. Thus, one can employ [6; II.5.4.2] (rather than [6; II.5.4.1]) and obtains

$$\|u(t,\cdot)\|_{E_{\overline{\kappa}}} \le C\Big(t^{-\overline{\kappa}}e^{\nu t}\,\|\vartheta(0)\|_{E_{\overline{\kappa}}} + \|f\|_{L_\infty([0,t_+(\vartheta)),L_p(M))}\Big) \qquad t\in(0,\infty),$$

which yields the second part of this theorem.

Remark. It is of interest to note that the bound for u depends only on $\|u\|_\infty$ and $\|\vartheta(0)\|_{E_{\overline{\kappa}}}$. Moreover, the proof shows that the statement of Theorem 3.1 remains true under hypotheses (H1)-(H3), whenever L_∞-boundedness of u can be established otherwise.

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