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EXISTENCE AND ATTRACTIVITY OF PERIODIC SOLUTIONS TO COHEN-GROSSBERG NEURAL NETWORK WITH DISTRIBUTED DELAYS

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Abstract. We study the existence and exponential attractivity of periodic solutions to Cohen-Grossberg neural network with distributed delays. Our results are obtained by applying the continuation theorem of coincidence degree theory and a general Halanay inequality.

1. Introduction

Since 1983, when Cohen and Grossberg [1] proposed a class of neural networks, their model has received increasing interest due to its promising potential for applications in classification, parallel computing, associative memory, and specially in solving some optimization problems. Such applications rely not only on the existence of equilibrium points, or on the unique equilibrium point and the qualitative properties of stability, but on the dynamic behavior, such as periodic oscillatory behavior, almost periodic oscillatory properties, chaos, and bifurcation [6]. Thus, the qualitative analysis of the dynamic behavior is a prerequisite step for the practical design and application of neural networks.

Li [4] used the continuation theorem of coincidence degree theory and Liapunov functions to study the existence and stability of periodic solutions for the following Cohen-Grossberg neural network with multiple delays

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i(t, x_i(t))\Big[b_i(t, x_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^k(t) s_j(x_j(t-\tau_k)) + J_i(t)\Big],$$

for $i = 1, 2, \dots, n$, where the $n \times n$ matrixes $T_k = (t_{ij}^k(t))$ represent the interconnections which are associated with delay τ_k and the delays τ_k, $k = 0, 1, \dots, K$, J_i, $i = 1, 2, \dots, n$ denote the inputs at time t from outside the system.

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In this paper, we are concerned with the Cohen-Grossberg neural networks with distributed delays:

$$\frac{dx_i}{dt} = -a_i(t, x_i(t))\Big[b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) g_j\Big(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s) x_j(s) ds\Big) + I_i(t)\Big], \tag{1.1}$$

for $i = 1, 2, \dots, n$, where the delay kernel functions l_{ij} are piecewise continuous and satisfy

$$l_{ij}(t) \geq 0, \int_0^\infty l_{ij}(t) dt = 1, \quad \int_0^\infty t l_{ij}(t) dt < \infty.$$

Throughout this paper, we assume that

- (H1) c_{ij}, I_i, $i = 1, 2, \dots, n$, are continuous ω-periodic functions on R, and $\tau = \max\{\tau_{ij} \geq 0,\ i, j = 1, 2, \dots, n, \}$.
- (H2) For each $i = 1, 2, \dots, n$, $|g_i(x)| \leq M_i$, $x \in R$ for some constant $M_i > 0$.
- (H3) For each $i = 1, 2, \dots, n$, $a_i \in C(R^2, (0, +\infty))$ is ω-periodic with respect to its first argument.
- (H4) For each $i = 1, 2, \dots, n$, $b_i \in C(R^2, R)$ is ω-periodic with respect to its first argument, $\lim_{u \to +\infty} b_i(t, u) = +\infty$ and $\lim_{u \to -\infty} b_i(t, u) = -\infty$ are uniformly in t, respectively.

The purpose of this paper is to investigate the existence and exponential attactivity of solutions to (1.1). This paper is organized as follows. In Sections 2, we shall use Mawhin's continuation theorem [2] to establish the existence of periodic solutions of (1.1). In Sections 3, by using a general Halanay inequality we shall derive sufficient conditions to ensure that the periodic solution of (1.1) is exponential attractivity. In Sections 4, we give an example to illustrate that the conditions of our results are feasible.

2. Existence of positive periodic solutions

In this section, based on the Mawhin's continuation theorem, we study the existence of at least one positive periodic solution of (1.1). First, we shall make some preparations.

Let X, Y be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim Ker L=codim Im $L < +\infty$ and Im L is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that Im P=Ker L, Ker Q=Im L=Im $(I - Q)$, it follows that mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to$Im L is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since Im Q is isomorphic to Ker L, there exists an isomorphism $J : \operatorname{Im} Q \to$ Ker L.

Now, we introduce Mawhin's continuation theorem [2, p.40] as follows.

Lemma 2.1. *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

- (a) *For each $\lambda \in (0, 1), x \in \partial\Omega \cap \operatorname{Dom} L, Lx \neq \lambda N x$*
- (b) *For each $x \in \partial\Omega \cap$ Ker $L, QNx \neq 0$*
- (c) $\deg(JNQ, \Omega \cap \text{Ker } L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Theorem 2.2. *Assume that (H1)-(H4) hold. Then the system* (1.1) *has at least one positive ω-periodic solution.*

Proof. To apply the continuation theorem of coincidence degree theory and establish the existence of an ω-periodic solution of (1.1), we take

$$X = Y = \{x \in C(R, R^n) : x(t + \omega) = x(t), t \in R\}$$

and denote

$$\|x\| = \sup_{t\in[0,\omega]} \{|x_i(t)|, i = 1, 2, \dots, n\}.$$

Then X is a Banach space. Set

$$L : \text{Dom } L \cap X, \quad Lx = \dot{x}(t), \quad x \in X$$

where $\text{Dom}\, L = \{x \in C^1(R, R^n)\}$ and $N : X \to X$

$$Nx_i = -a_i(t, x_i(t))\Big[b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) g_i\Big(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s)x_j(s)ds\Big) + I_i(t)\Big],$$

$i = 1, 2, \dots, n$. Define two projectors P and Q as

$$Qx = Px = \frac{1}{\omega}\int_0^\omega x(s)ds, \quad x \in X.$$

Clearly, $\text{Ker } L = R^n$,

$$\text{Im}\, L = \{(x_1, x_2, \dots, x_n)^T \in X : \int_0^\omega x_i(t)dt = 0,\ i = 1, 2, \dots, n\}$$

is closed in X and $\dim \text{Ker } L = \text{codim}\, \text{Im}\, L = n$. Hence, L is a Fredholm mapping of index 0. Furthermore, similar to the proof in [6, Theorem 1], one can easily show that N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$. Corresponding to operator equation $Lx = \lambda Nx, \lambda \in (0, 1)$, we have

$$\frac{dx_i}{dt} = -\lambda a_i(t, x_i(t))\Big[b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) g_i\Big(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s)x_j(s)ds\Big) + I_i(t)\Big], \tag{2.1}$$

$i = 1, 2, \dots, n$. Suppose that $x = (x_1, x_2, \dots, x_n) \in X$ is a solution of (2.1) for some $\lambda \in (0, 1)$. Let $\xi_i \in [0, \omega]$ such that $x_i(\xi_i) = \max_{t\in[0,\omega]} x_i(t),\ i = 1, 2, \dots, n$, then

$$-\lambda a_i(\xi_i, x_i(\xi_i))\Big[b_i(\xi_i, x_i(\xi_i)) - \sum_{j=1}^{n} c_{ij}(\xi_i) g_i\Big(\int_{\xi_i-\tau_{ij}}^{\xi_i} l_{ij}(\xi_i - s)x_j(s)ds\Big) + I_i(\xi_i)\Big] = 0,$$

$i = 1, 2, \dots, n$. In view of (H3), we have

$$\begin{aligned} b_i(\xi_i, x_i(\xi_i)) &\le \sum_{j=1}^{n} |c_{ij}(\xi_i)| \Big| g_i\Big(\int_{\xi_i-\tau_{ij}}^{\xi_i} l_{ij}(\xi_i - s)x_j(s)ds\Big)\Big| + |I_i(\xi_i)| \\ &\le n\|c\|\overline{M} + \|I\|, \quad i = 1, 2, \dots, n. \end{aligned} \tag{2.2}$$

where $\overline{M} = \sup\{M_i, i = 1, 2, \dots, n\}$. According to (H4), we know that there exists a constant $A_1 > 0$ such that

$$x_i(\xi_i) \le A_1, \quad i = 1, 2, \dots, n.$$

Similarly, let $\eta_i \in [0,\omega]$ such that $x_i(\eta_i) = \min_{t\in[0,\omega]} x_i(t), i = 1,2,\dots,n$, then

$$-\lambda a_i(\eta_i, x_i(\eta_i))\Big[b_i(\eta_i, x_i(\eta_i)) - \sum_{j=1}^{n} c_{ij}(\eta_i) g_i\Big(\int_{\eta_i-\tau_{ij}}^{\eta_i} l_{ij}(\eta_i - s)x_j(s)ds\Big) + I_i(\eta_i)\Big] = 0,$$

$i = 1,2,\dots,n$. Then,

$$\begin{aligned} b_i(\xi_i, x_i(\xi_i)) &\geq -\sum_{j=1}^{n} |c_{ij}(\xi_i)| \Big| g_i\Big(\int_{\xi_i-\tau_{ij}}^{\xi_i} l_{ij}(\xi_i - s)x_j(s)ds\Big)\Big| - |I_i(\xi_i)| \\ &\geq -n\|c\|\overline{M} - \|I\|, \quad i = 1,2,\dots,n. \end{aligned}$$

where $\overline{M}$ is the same as those in (2.2). Therefore, there exists a constant $A_2 > 0$ such that

$$x_i(\eta_i) \geq -A_2, \quad i = 1,2,\dots,n.$$

Denote $D = \max\{A_1, A_2\}$, clearly, D is independent of λ. Now, we take $\Omega = \{x \in X, \|x\| < D\}$. This Ω satisfies condition (a) in Lemma 2.1.

When $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^n$, x is a constant vector in R^n with $\|x\| = D$. Then

$$\begin{aligned} x^T QNx = \frac{1}{\omega} \sum_{i=1}^{n} x_i \int_0^{\omega} -a_i(t, x_i(t))\Big[b_i(t, x_i(t)) \\ - \sum_{j=1}^{n} c_{ij}(t) g_i\Big(\int_{t-\tau_{ij}}^{t} l_{ij}(t-s)x_j(s)ds\Big) + I_i(t)\Big]dt \\ \leq -\frac{1}{\omega} \sum_{i=1}^{n} x_i \int_0^{\omega} a_i(t, x_i)\Big[b_i(t, x_i) - n\|c\|\overline{M} - \|I\|\Big]dt < 0, \end{aligned}$$

$i = 1,2,\dots,n$. If necessary, we let D be large enough such that

$$-\frac{1}{\omega} \sum_{i=1}^{n} x_i \int_0^{\omega} a_i(t, x_i)\Big[b_i(t, x_i) - n\|c\|\overline{M} - \|I\|\Big]dt < 0.$$

So for any $x \in \partial\Omega \cap \ker\ L, QNx \neq 0$. This prove that condition (b) in Lemma 2.1 is satisfied.

Finally, we prove that condition (c) in Lemma 2.1 is also satisfied. Indeed, let $\psi(\nu; x) = -\nu x + (1-\nu)QNx$, then for any $x \in \partial\Omega \cap \operatorname{Ker} L, x^T\psi(\nu, x) < 0$, we get

$$\deg(JQM, \Omega \cap \operatorname{Ker} L, 0) \neq 0.$$

Thus, by Lemma 2.1, we conclude that $Lx = Nx$ has at least one solution in X, that is, (1.1) has at least one positive ω-periodic solution. The proof is complete. □

3. Attractivity of periodic solution

First, we introduce the general Halanay inequality whose proof can be found in [3].

Lemma 3.1. *Let $a > b > 0$, and $x(t)$ be a nonnegative continuous function on $[t_0 - \tau, t_0]$, and as $t \geq t_0$, satisfies the following inequality:*

$$D^+x(t) \leq -ax(t) + b\overline{x}(t),$$

where $\overline{x}(t) = \sup_{t-\tau\le s\le t} x(s)$, τ *is a constant, and* $\tau \ge 0$, *then as* $t \ge t_0$, *the following inequality holds*

$$x(t) \le \overline{x}(t_0)e^{-\lambda(t-t_0)},$$

in which λ *is unique positive solution of the equation* $\lambda = a - be^{\lambda\tau}$.

Next, we use this general Halanay inequality to prove that all solution to (1.1) converge exponentially to an ω-periodic solution. In fact, this ω-periodic solution is unique.

Let $x(t) = (x_1(t), \dots, x_n(t))$ be a solution of (1.1) and $x^*(t) = (x_1^*(t), \dots, x_n^*(t))$ be an ω-periodic solution of (1.1). Set $u(t) = x(t) - x^*(t)$. Then

$$\frac{du_i}{dt} = -\alpha_i(u_i(t)) + \beta_i(u_i(t)) + \gamma(u_i(t)) - \delta(u_i(t)), \quad i = 1, 2, \dots, n, \tag{3.1}$$

where

$$\alpha_i(u_i(t)) = a_i(t, x_i(t))b_i(t, x_i(t)) - a_i(t, x_i^*(t))b_i(t, x_i^*(t)),$$

$$\begin{aligned}\beta_i(u_i(t)) = a_i(t, x_i(t)) \sum_{j=1}^n c_{ij}(t)\Big[g_j\Big(\int_{t-\tau_{ij}}^t l_{ij}(t-s)x_j(s)ds\Big) \\ - g_j\Big(\int_{t-\tau_{ij}}^t l_{ij}(t-s)x_j^*(s)ds\Big)\Big],\end{aligned}$$

$$\gamma(u_i(t)) = [a_i(t, x_i(t)) - a_i(t, x_i^*(t))] \sum_{j=1}^n c_{ij}(t)g_j\Big(\int_{t-\tau_{ij}}^t l_{ij}(t-s)x_j^*(s)ds\Big),$$

$$\delta(u_i(t)) = [a_i(t, x_i(t)) - a_i(t, x_i^*(t))]I_i(t), \quad i = 1, 2, \dots, n.$$

In the sequel, we use the notation

$$\overline{a} = \sup_{t\in[0,\omega],\ x\in R} \{|a_i(t, x)|, i = 1, 2, \dots, n\}, \quad \overline{A} = \max\{A_i, i = 1, 2, \dots, n\}.$$

For the next theorem, we assume:

(H5) For each $i = 1, 2, \dots, n, g_i : R \to R$ is globally Lipschitz continuous with a Lipschitz constant L_i.

(H6) For each $i = 1, 2, \dots, n$, a_i is bounded on $\mathbb{R}^2$ and there exists a constant $A_i \ge 0$ such that

$$|a_i(t, x) - a_i(t, y)| \le A_i|x - y|, \quad x, y \in R,\ t \in [0, \omega].$$

Moreover, $a_ib_i \in C^1(R^2, R)$ and there exists a positive constant E^{ab} such that

$$[a_i(t, u)b_i(t, u)]'_u \ge E^{ab}, \quad t \in [0, \omega], \quad u \in \mathbb{R}.$$

(H7)

$$E^{ab} - n\|c\|\overline{AM} - \overline{A}\|I\| > n\tau\overline{a}\|c\|\overline{L}.$$

Theorem 3.2. *Assume that (H1)-(H7) hold. Then all solutions of* (1.1) *converge exponentially to the unique* ω*-periodic solution.*

Proof. Set $\|u(t)\|_0 = \max_{i\in\{1,2,\dots,n\}} |u_i(t)| = |u_{i_0}(t)|$. Then

$$\begin{aligned}&\frac{d\|u(t)\|_0}{dt} = \frac{d|u_{i_0}(t)|}{dt} \\ &= \operatorname{sign}(u_{i_0}(t))\{-\alpha_{i_0}(u_{i_0}(t)) + \beta_{i_0}(u_{i_0}(t)) + \gamma(u_{i_0}(t)) - \delta(u_{i_0}(t))\} \\ &\le -E^{ab}|u_{i_0}(t)| + |\beta_{i_0}(u_{i_0}(t))| + |\gamma(u_{i_0}(t))| + |\delta(u_{i_0}(t))|\end{aligned}$$

$$
\begin{aligned}
&\leq -E^{ab}|u_{i_0}(t)| + |a_{i_0}(t,x_{i_0}(t))| \sum_{j=1}^{n} |c_{i_0 j}(t)| \Big| g_j\Big(\int_{t-\tau_{i_0 j}}^{t} l_{i_0 j}(t-s)x_j(s)ds \Big) \\
&\quad - g_j\Big(\int_{t-\tau_{i_0 j}}^{t} l_{i_0 j}(t-s)x_j^*(s)ds \Big) \Big| + |a_{i_0}(t,x_{i_0}(t)) - a_{i_0}(t,x_{i_0}^*(t))| \\
&\quad \sum_{j=1}^{n} |c_{i_0 j}(t)| \Big| g_j\Big(\int_{t-\tau_{i_0 j}}^{t} l_{i_0 j}(t-s)x_j^*(s)ds \Big) \Big| \\
&\quad + |a_{i_0}(t,x_{i_0}(t)) - a_{i_0}(t,x_{i_0}^*(t))||I_{i_0}(t)| \\
&\leq -E^{ab}|u_{i_0}(t)| + n\overline{a}\|c\|\overline{L} \sup_{t-\tau\leq s\leq t} \|u(s)\|_0 + n\|c\|\overline{AM}|u_{i_0}(t)| + \overline{A}\|I\||u_{i_0}(t)| \\
&\leq -(E^{ab} - n\|c\|\overline{AM} - \overline{A}\|I\|)\|u(t)\|_0 + n\overline{a}\|c\|\overline{L} \sup_{t-\tau\leq s\leq t} \|u(s)\|_0 .
\end{aligned}
$$

By the general Halanay inequality, we derive that there exist constant $\lambda > 0, K > 0$ such that

$$\|u(t)\|_0 \leq K e^{-\lambda(t-t_0)},$$

where $K = \sup_{t_0-\tau\leq s\leq t_0} \|u(s)\|_0$. This implies that all solution of (1.1) converge exponentially to the unique periodic solution. This completes the proof. $\square$

Example. Consider the Cohen-Grossberg neural network with distributed delays

$$
\begin{aligned}
\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 2+\cos t & 0 \\ 0 & 3+\sin t \end{pmatrix} \Big[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6x_1 + \sin t \\ 3x_2 - \frac{1}{6}\cos x_2 \end{pmatrix} \\
- \begin{pmatrix} \sin t & \frac{1}{7}\cos t \\ \cos t & \frac{1}{5}\sin t \end{pmatrix} \begin{pmatrix} \sin\big(\int_{t-1}^{t} x_1(s)ds\big) \\ \cos\big(\int_{t-1}^{t} x_2(s)ds\big) \end{pmatrix} + \begin{pmatrix} \frac{2}{9}\sin t \\ \cos t \end{pmatrix} \Big].
\end{aligned}
\tag{3.2}
$$

Clearly, $n = 2$, $\omega = 2\pi$, $\overline{M} = 1$, $\|c\| = \frac{1}{5}$, $\|I\| = \frac{2}{9}$, $\overline{L} = 1$, $\overline{A} = 4$, $\overline{a} = 4$, $E^{ab} = 5$, and

$$E^{ab} - n\|c\|\overline{AM} - \overline{A}\|I\| = \frac{13}{3} > n\overline{a}\|c\|\overline{L} = \frac{8}{5}.$$

By Theorems 2.2 and 3.2, the system (3.2) has a periodic solution and all other solution of the system (3.2) converge exponentially to the unique periodic solution.

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