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FIBER PRESERVING TRANSFORMATIONS AND THE EQUATION $y''' = f(x, y, y', y'')$

RAOUF DRIDI, SYLVAIN NEUT

Abstract. In this paper, we give simple explicit conditions for a third order ordinary differential equation to be equivalent to the equations $y''' = 0$ and $y''' + y = 0$ under fiber preserving transformations.

1. Introduction

First introduced by Cartan [1], the method of equivalence can decide the equivalence of geometric objects (G-structures) under the action of a given pseudogroup of diffeomorphisms. Cartan in his classic paper solved many complicated problems. Chern [2, 3], following Cartan, proved that the third order equation

$$y''' = f(x, y, y', y'') \tag{1.1}$$

is equivalent to the equations $y''' = 0$ and $y''' + y = 0$ under contact transformations if and only if some explicit conditions are satisfied.

Due to its extremely complicated computations, the method of equivalence fell into forgets until the works of Gardner [4], Kamran [7], Fels [5] and Olver [6]. More recently, the modern advent of algebraic computers has allowed the completion of many of problems and opened the door to variety of new problems arising in control theory, Riemannian geometry, CR geometry, general relativity, object recognizing, etc.

This work drives explicit conditions that the equation (1.1) be equivalent to $y''' = 0$ and $y''' + y = 0$ under the pseudogroup of fiber preserving transformations. The computations here made great use of the symbolic package *cartan*, implemented by the second author. The paper is organized as follows. In section 2, we formulate the equivalence problem and give the explicit conditions for the equation (1.1) to be equivalent to $y''' = 0$. In section 3, the equivalence with $y''' + y = 0$ is studied and explicit conditions are given.

2. First linearization

Let Φ be the group of fiber preserving transformations from J^2 to J^2 and $\mathrm{x} := (x, y, p = y', q = y'') \in \mathbb{R}^4$ a local coordinates in J^2.

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The equivalence of the equations $y''' = f(x,y,y',y'')$ and $\bar{y}''' = \bar{f}(\bar{x},\bar{y},\bar{y}',\bar{y}'')$ under a fiber preserving transformation reads

$$\phi^* \underbrace{\begin{pmatrix} d\bar{q} - \bar{f}(\bar{x},\bar{y},\bar{p},\bar{q})d\bar{x} \\ d\bar{y} - \bar{p}d\bar{x} \\ d\bar{p} - \bar{q}d\bar{x} \\ d\bar{x} \end{pmatrix}}_{\bar{\omega}(\bar{\mathrm{x}})} = \underbrace{\begin{pmatrix} a_1(\mathrm{x}) & a_2(\mathrm{x}) & a_3(\mathrm{x}) & 0 \\ 0 & a_4(\mathrm{x}) & 0 & 0 \\ 0 & a_5(\mathrm{x}) & a_6(\mathrm{x}) & 0 \\ 0 & 0 & 0 & a_7(\mathrm{x}) \end{pmatrix}}_{g(a)\in G} \underbrace{\begin{pmatrix} dq - f(x,y,p,q)dx \\ dy - pdx \\ dp - qdx \\ dx \end{pmatrix}}_{\omega(\mathrm{x})}$$

In accordance with Cartan, we take the lifted coframes $\theta = g\omega$ and $\bar{\theta} = \bar{g}\bar{\omega}$.

After four normalizations, the structure equations for the lifted coframe have the form

$$\begin{aligned} d\theta^1 &= -\pi^1 \wedge \theta^3 + \pi^2 \wedge \theta^1 - \pi^3 \wedge \theta^1 + I\theta^2 \wedge \theta^4 \\ d\theta^2 &= \pi^2 \wedge \theta^2 + \pi^3 \wedge \theta^2 + \theta^3 \wedge \theta^4 \\ d\theta^3 &= \pi^1 \wedge \theta^2 + \pi^2 \wedge \theta^3 - \theta^1 \wedge \theta^4 \\ d\theta^4 &= \pi^3 \wedge \theta^4 + I_1\theta^3 \wedge \theta^4 \end{aligned} \tag{2.1}$$

involving the two relative invariants (The numerator of I is the well known Wünschmann relative invariant [3])

$$\begin{gathered} I = \frac{1}{54{a_7}^3}(4{f_q}^3 + 18f_pf_q - 18D_xf_qf_q + 54f_y - 27D_xf_p + 9D_x^2f_q) \\ I_1 = \frac{1}{6}\frac{f_{qq}}{a_1a_7} \end{gathered} \tag{2.2}$$

where

$$D_x := \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f(x,y,p,q)\frac{\partial}{\partial q}.$$

Suppose that

$$I_1 = 0 \tag{2.3}$$

and let us consider the meaning of the vanishing of I. At this stage no more normalization is possible and we have to prolong. This gives the following structure equation on certain 7-dimensional manifold [8]

$$\begin{aligned} d\theta^1 &= -\theta^1 \wedge \theta^6 + \theta^1 \wedge \theta^7 + \theta^3 \wedge \theta^5 \\ d\theta^2 &= -\theta^2 \wedge \theta^6 - \theta^2 \wedge \theta^7 + \theta^3 \wedge \theta^4 \\ d\theta^3 &= -\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^5 - \theta^3 \wedge \theta^6 \\ d\theta^4 &= -\theta^4 \wedge \theta^7 \\ d\theta^5 &= T^5_{2,3}\theta^2 \wedge \theta^3 + \theta^5 \wedge \theta^7 \\ d\theta^6 &= T^6_{2,3}\theta^2 \wedge \theta^3 + T^6_{2,4}\theta^2 \wedge \theta^4 \\ d\theta^7 &= T^7_{2,4}\theta^2 \wedge \theta^4 - \theta^4 \wedge \theta^5 \end{aligned} \tag{2.4}$$

By simple calculation we can see that (2.4) is in involution.

Lemma 2.1. *The invariants in* (2.4) *vanish if and only if*

$$T^5_{3,4} = \frac{1}{3}\frac{-f_{yq} + D_xf_{pq}}{a_1{a_7}^3} = 0.$$

In the case of the equation $y''' = 0$ all the invariants in (2.4) vanish.

Theorem 2.2. *The following statements are equivalent*

(i) *Equation* (1.1) *is equivalent to the equation* $y''' = 0$ *under a fiber preserving transformation.*
(ii) *Equation* (1.1) *admits a 7-dimensional Lie group of fiber preserving point symmetries.*
(iii) *The function* f *satisfies*

$$\begin{gathered} f_{qq} = 0, \\ 4f_q^3 + 18f_pf_q - 18D_xf_pf_q + 54f_y - 27D_xf_p + 9D_x^2f_q = 0, \\ f_{yq} - D_xf_{pq} = 0. \end{gathered} \tag{2.5}$$

Proof. We need to prove only (ii) $\Leftrightarrow$ (iii). Let S be the Lie group of fiber preserving symmetries of the equation (1.1). If $\dim S = 7$ then all the invariants are constant and we have to prove that they vanish. Using Poincaré identity (applied to the structure equations (2.4)) we find

$$\frac{\partial T^5_{3,4}}{\partial \theta^6} + T^5_{3,4} = 0\,;$$

therefore $T^5_{3,4}$ vanish and according to Lemma 2.1 all the invariants in (2.4) vanish.

Conversely, the symmetry group is finite-dimensional Lie group of dimension $7-r$ where r is the rank of the coframe given by (2.4). On the other hand suppose that f satisfy (2.5) then $r = 0$ and the theorem follows. $\square$

3. The equation $y''' + y = 0$

If $I \neq 0$ (I_1 is always null), we can normalize I to 1 by setting $a_7 = J = \sqrt[3]{\text{numerator}(I)/54}$ in (2.1). This leads to the following involutive structure equations

$$\begin{aligned} d\theta^1 &= -\theta^1\wedge\theta^5 + T^1_{2,3}\theta^2\wedge\theta^3 + \theta^2\wedge\theta^4 + T^1_{3,4}\theta^3\wedge\theta^4, \\ d\theta^2 &= T^2_{2,3}\theta^2\wedge\theta^3 - \theta^2\wedge\theta^5 - \theta^3\wedge\theta^4, \\ d\theta^3 &= T^3_{1,2}\theta^1\wedge\theta^2 + T^3_{1,3}\theta^1\wedge\theta^3 - \theta^1\wedge\theta^4 + T^3_{2,3}\theta^2\wedge\theta^3 + T^3_{2,4}\theta^2\wedge\theta^4 - \theta^3\wedge\theta^5, \\ d\theta^4 &= T^4_{1,4}\theta^1\wedge\theta^4 + T^4_{2,4}\theta^2\wedge\theta^4 + T^4_{3,4}\theta^3\wedge\theta^4, \\ d\theta^5 &= T^5_{1,2}\theta^1\wedge\theta^2 + T^5_{1,3}\theta^1\wedge\theta^3 + T^5_{1,4}\theta^1\wedge\theta^4 + T^5_{2,3}\theta^2\wedge\theta^3 + T^5_{2,4}\theta^2\wedge\theta^4 + T^5_{3,4}\theta^3\wedge\theta^4. \end{aligned} \tag{3.1}$$

We give the following lemma

Lemma 3.1. *The invariants in* (3.1) *vanish if and only if*

$$\begin{aligned} T^1_{3,4} &= \frac{1}{6}(-J^2{f_q}^2 - 3f_pJ^2 + 9(D_xJ)^2 + 3J^2D_xf_q - 6J{D_x}^2J)/J^4, \\ T^4_{1,4} &= \frac{J_q}{a_1J}, \\ T^4_{3,4} &= \frac{1}{3}\frac{3JJ_p + 2JJ_qf_q - 3J_qD_xJ}{a_1J^3}, \\ T^5_{1,4} &= \frac{1}{3}(4JJ_qf_q - 6J_qD_xJ + 6JJ_p - f_{qq}J^2)/(a_1J^3), \end{aligned}$$

vanish.

Proof. By applying Poincaré identity to (3.1) we prove that the set $\{T^1_{3,4}, T^4_{1,4}, T^4_{3,4}, T^5_{1,4}\}$ forms a basis of the *differential ideal* generated by the invariants of (3.1). $\square$

In the case of $y''' + y = 0$ all the invariants vanish. It follows,

Theorem 3.2. *The following statements are equivalent.*

(i) *Equation* (1.1) *is equivalent to the equation $y''' + y = 0$ under a fiber preserving transformation.*
(ii) $d\theta^4 = d\theta^5 = 0$ *and* $-J^2 f_q^2 - 3f_p J^2 + 9(D_x J)^2 + 3J^2 D_x f_q - 6J D_x^2 J = 0$
(iii) *J and f satisfy*

$$\begin{gathered} J_q = 0, \quad J_p = 0, \quad f_{qq} = 0, \\ -J^2 f_q^2 - 3f_p J^2 + 9(D_x J)^2 + 3J^2 D_x f_q - 6J D_x^2 J = 0. \end{gathered} \tag{3.2}$$

where

$$J^3 = \frac{1}{54}(4f_q^3 + 18 f_p f_q - 18 D_x f_q f_q + 54 f_y - 27 D_x f_p + 9 D_x^2 f_q)$$

and

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f(x,y,p,q)\frac{\partial}{\partial q}$$

Proof. Here again we only prove (ii) $\Leftrightarrow$ (iii).
(ii) $\Rightarrow$ (iii): If $d\theta^4 = d\theta^5 = 0$ then (in particular) $T^4_{1,4}$,$T^4_{3,4}$ and $T^5_{1,4}$ vanish ($\{\theta^i \wedge \theta^j\}$ are linearly independent).
(iii) $\Rightarrow$ (ii): If J and f satisfy (3.2) then $T^1_{3,4} = T^4_{1,4} = T^4_{3,4} = T^5_{1,4} = 0$ and according to Lemma 3.1 all the invariants vanish, hence $d\theta^4 = d\theta^5 = 0$. $\square$

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LIFL Laboratory, Campus Box 59655, Lille1 University, Villeneuve D'Ascq, France
E-mail address, R. Dridi: `dridi@lifl.fr`
E-mail address, S. Neut: `neut@lifl.fr`